J Geom Anal (2011) 21:665–682
DOI 10.1007/s12220-010-9163-y

Univalent Functions, *VMOA* and Related Spaces

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Received: 25 May 2009 / Published online: 5 August 2010

Abstract This paper is concerned mainly with the logarithmic Bloch space $\mathcal{B}_{\log}$ which consists of those functions f which are analytic in the unit disc $\mathbb{D}$ and satisfy $\sup_{|z|<1}(1-|z|)\log\frac{1}{1-|z|}|f'(z)|<\infty$, and the analytic Besov spaces B^p, $1\le p<\infty$. They are all subspaces of the space *VMOA*. We study the relation between these spaces, paying special attention to the membership of univalent functions in them. We give explicit examples of:

- A bounded univalent function in $\bigcup_{p>1} B^p$ but not in the logarithmic Bloch space.
- A bounded univalent function in $\mathcal{B}_{\log}$ but not in any of the Besov spaces B^p with $p<2$.

We also prove that the situation changes for certain subclasses of univalent functions. Namely, we prove that the convex univalent functions in $\mathbb{D}$ which belong to any of the spaces $\mathcal{B}_0$, *VMOA*, B^p ($1\le p<\infty$), $\mathcal{B}_{\log}$, or some other related spaces are the same, the bounded ones.

Communicated by Michael Lacey.

This research is partially supported by grants from “el Ministerio de Ciencia e Innovación, Spain” (MTM2007-60854, MTM2007-30904-E, MTM2008-0289-E and ‘Ingenio Mathematica (i-MATH)’ No. CSD2006-00032); from “La Junta de Andalucía” (FQM210, P06-FQM01504, and P09-FQM4468); from the European Networking Programme “HCAA” of the European Science Foundation; and by FONDECYT Grant ♯11070055, Chile.

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We also consider the question of when the logarithm of the derivative, $\log g'$, of a univalent function g belongs to Besov spaces. We prove that no condition on the growth of the Schwarzian derivative Sg of g guarantees $\log g' \in B^p$. On the other hand, we prove that the condition $\int_{\mathbb{D}}(1-|z|^2)^{2p-2}|Sg(z)|^p\, dA(z) < \infty$ implies that $\log g' \in B^p$ and that this condition is sharp. We also study the question of finding geometric conditions on the image domain $g(\mathbb{D})$ which imply that $\log g'$ lies in B^p. First, we observe that the condition of $g(\mathbb{D})$ being a convex Jordan domain does not imply this. On the other hand, we extend results of Pommerenke and Warschawski, obtaining for every $p \in (1, \infty)$, a sharp condition on the smoothness of a Jordan curve Γ which implies that if g is a conformal mapping from $\mathbb{D}$ onto the inner domain of Γ, then $\log g' \in B^p$.

Keywords Univalent functions · *VMOA* · Bloch function · Besov spaces · Logarithmic Bloch spaces · Logarithmic derivative · Schwarzian derivative · Smooth Jordan curve

Mathematics Subject Classification (2000) 30C35 · 30A10 · 30D45 · 30H05

1 Introduction

Throughout the paper, the letter Ω will be used to denote a domain in the complex plane $\mathbb{C}$ and $\partial\Omega$ will be its boundary. For $w \in \Omega$, $d_\Omega(w)$ will stand for the distance from w to the boundary of Ω. The unit disc in $\mathbb{C}$ will be denoted by $\mathbb{D}$. If $a \in \mathbb{C}$ and $r > 0$, $D(a, r)$ will denote the open disc of radius r centered at a, and $dA(z) = r\, dr\, d\theta = dx\, dy$ will be the Lebesgue area measure in $\mathbb{C}$. Also, $\mathcal{H}ol(\mathbb{D})$ will denote the space of all analytic functions in $\mathbb{D}$.

A complex-valued function defined in $\mathbb{D}$ is said to be *univalent* if it is analytic and one-to-one there. We refer to [18, 36], and [39] for the theory of these functions. The class of all univalent functions in $\mathbb{D}$ will be denoted by $\mathcal{U}$. Sometimes it is useful to consider certain normalized subclasses of $\mathcal{U}$ such as the class S and the class S_0:

$$S = \{f \in \mathcal{U} : f(0) = 0,\ f'(0) = 1\},$$
$$S_0 = \{f \in \mathcal{U} : f \text{ is zero-free in } \mathbb{D},\ f(0) = 1\}.$$

If $f \in \mathcal{U}$ and $\Omega = f(\mathbb{D})$ then f is said to be a *conformal mapping from* $\mathbb{D}$ *onto* Ω. Note that in such a case Ω is a simply connected domain strictly contained in $\mathbb{C}$.

One of the principal aims in geometric function theory is to find relations between analytic properties of a function $f \in \mathcal{U}$ and geometric properties of the image domain $\Omega = f(\mathbb{D})$. Many results of this kind are known. Let us list some examples:

- A well-known theorem of Carathéodory [39, Theorem 2.6] asserts that Ω is a Jordan domain if and only if f has a continuous and injective extension to $\overline{\mathbb{D}}$.
- Ω is convex if and only if $\mathrm{Re}[1 + z\frac{f''(z)}{f'(z)}] > 0$, for all $z \in \mathbb{D}$ [36, Theorem 2.7].
- For $f \in S$, Ω is starlike with respect to the origin if and only if $\mathrm{Re}[z\frac{f'(z)}{f(z)}] > 0$, for all $z \in \mathbb{D}$ [36, Theorem 2.5].

For $0 < p \leq \infty$, the classical Hardy space H^p is defined as the set of all $f \in \mathcal{H}ol(\mathbb{D})$ for which

$$\|f\|_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r, f) < \infty,$$

where, for $0 < r < 1$ and $f \in \mathcal{H}ol(\mathbb{D})$,

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p} \qquad (0 < p < \infty);$$

$$M_\infty(r, f) = \sup_{\theta\in\mathbb{R}} |f(re^{i\theta})|.$$

We mention [19] as a general reference for the theory of Hardy spaces.

A function $f \in \mathcal{H}ol(\mathbb{D})$ is a Bloch function if

$$\|f\|_{\mathcal{B}} \stackrel{\text{def}}{=} |f(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty.$$

The space of all Bloch functions is denoted by $\mathcal{B}$. The *little Bloch space* $\mathcal{B}_0$ consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\lim_{|z|\to 1}(1 - |z|^2)|f'(z)| = 0.$$

Alternatively, $\mathcal{B}_0$ is the closure of the polynomials in the Bloch norm. A classical source for Bloch functions is [2].

The space *BMOA* consists of those functions f in H^1 whose boundary values have bounded mean oscillation on the unit circle $\partial\mathbb{D}$ as defined by F. John and L. Nirenberg [26].

The space *VMOA* [41] is the closure of the polynomials in the *BMOA*-norm. We mention [10, 24, 25], and [42] as general references for these spaces. Let us recall that

$$H^\infty \subset BMOA \subset \bigcap_{0<p<\infty} H^p, \qquad H^\infty \subset BMOA \subset \mathcal{B}, \qquad VMOA \subset \mathcal{B}_0.$$

The well-known *Koebe*-1/4-*Theorem* [36, p. 22] asserts that if $f \in S$ then $D(0, 1/4) \subset f(\mathbb{D})$. This and Schwarz's lemma easily yield the following:

Theorem A *If* $f \in \mathcal{U}$ *and* $\Omega = f(\mathbb{D})$ *then*:

(i) $\frac{1}{4}(1 - |z|^2)|f'(z)| \leq \text{dist}(f(z), \partial\Omega) \leq (1 - |z|^2)|f'(z)|$.
(ii) $f \in \mathcal{B} \Leftrightarrow \Omega$ *does not contain discs of arbitrarily large radius.*
(iii) $f \in \mathcal{B}_0 \Leftrightarrow \lim_{w\in\Omega, |w|\to\infty} d_\Omega(w) = 0$.

Even though the inclusion $BMOA \subset \mathcal{B}$ is strict, a result of Pommerenke [37] implies that

$$\mathcal{U} \cap \mathcal{B} = \mathcal{U} \cap BMOA. \tag{1.1}$$

We also have (see, e.g., [25, p. 146])

$$\mathcal{U} \cap \mathcal{B}_0 = \mathcal{U} \cap VMOA. \tag{1.2}$$

For $0 < p < \infty$, we set

$$Q_p = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g(z,a)^p \, dA(z) < \infty \right\}$$

and

$$Q_{p,0} = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \lim_{|a| \to 1} \int_{\mathbb{D}} |f'(z)|^2 g(z,a)^p \, dA(z) = 0 \right\},$$

where $g(z,a) = \log |\frac{1-\overline{a}z}{z-a}|$ is the Green function of $\mathbb{D}$. These spaces were introduced by Aulaskari and Lappan in [7] while looking for new characterizations of Bloch functions, and we mention the books [49] and [50] as general references for the spaces Q_p and $Q_{p,0}$. Let us just mention here that $Q_p = \mathcal{B}$ and $Q_{p,0} = \mathcal{B}_0$ for all $p > 1$; $Q_1 = BMOA$ and $Q_{1,0} = VMOA$; and that whenever $0 < p < 1$, Q_p is a proper subspace of $BMOA$ and $Q_{p,0}$ is a proper subspace of $VMOA$. We recall also that the classical Dirichlet space $\mathcal{D}$ of those $f \in \mathcal{H}ol(\mathbb{D})$ satisfying $\int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < \infty$ is contained in all the $Q_{p,0}$ spaces. Aulaskari, Lappan, Xiao, and Zhao [8] extended (1.1) and (1.2) showing that

$$\mathcal{U} \cap Q_p = \mathcal{U} \cap \mathcal{B} \quad \text{and} \quad \mathcal{U} \cap Q_{p,0} = \mathcal{U} \cap \mathcal{B}_0, \quad 0 < p < \infty, \tag{1.3}$$

a result which has been further improved in [34, Theorems 4 and 9].

In this paper we shall turn our attention to other subspaces of $VMOA$. Namely, the logarithmic Bloch space $\mathcal{B}_{\log}$ and Besov spaces, but before doing so we shall introduce some special classes of univalent functions.

We shall let $\mathcal{K}$ denote the class of all functions $f \in \mathcal{U}$ with $f(0) = 0$ for which $f(\mathbb{D})$ is a convex domain, and $S^\star$ is the class of those $f \in \mathcal{U}$ with $f(0) = 0$ such that $f(\mathbb{D})$ is a domain which is starlike with respect to the origin. It is well-known that if $f \in \mathcal{H}ol(\mathbb{D})$ and $f(0) = 0$ then $f \in S^\star$ if and only if $\mathrm{Re}(\frac{zf'(z)}{f(z)}) > 0$, for all $z \in \mathbb{D}$. Finally, we shall let $\mathcal{C}$ denote the class of those functions $f \in \mathcal{H}ol(\mathbb{D})$ with $f(0) = 0$ for which there exists a function $g \in S^\star$ such that $\mathrm{Re}(\frac{zf'(z)}{g(z)}) > 0$, for all $z \in \mathbb{D}$. Trivially, $\mathcal{K} \subset S^\star \subset \mathcal{C}$, and it is well-known that $\mathcal{C} \subset \mathcal{U}$. The elements of $\mathcal{C}$ are called close-to-convex functions. We refer to [18, Chap. 2] and [36, Chap. 2] for the basic facts about these classes of univalent functions.

We close this section remarking that, as usual, we shall be using the convention that $C, C_1, C_2, M, M_1, M_2, \ldots$ will denote positive constants (which depend upon certain parameters which often will be omitted), but not necessarily the same at different occurrences.

2 Logarithmic Bloch Spaces and Besov Spaces

For $\alpha > 0$, *the* α*-logarithmic Bloch space* $\mathcal{B}_{\log,\alpha}$ is the Banach space of those functions $f \in \mathcal{H}ol(\mathbb{D})$ which satisfy

$$\|f\|_{\log,\alpha} \stackrel{\text{def}}{=} |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)\left(\log\frac{2}{1-|z|^2}\right)^{\alpha}|f'(z)| < \infty. \tag{2.1}$$

For simplicity, the space $\mathcal{B}_{\log,1}$ will be denoted by $\mathcal{B}_{\log}$.

It is clear that $B_{\log,\alpha} \subset \mathcal{B}_0$, for all $\alpha > 0$. Using the characterization of *VMOA* in terms of Carleson measures [25, p. 102], it follows easily that

$$B_{\log,\alpha} \subset VMOA, \quad \text{for all } \alpha > 1/2.$$

In particular,

$$\mathcal{B}_{\log} \subset VMOA. \tag{2.2}$$

The functions in the spaces $B_{\log,\alpha}$ show up naturally as pointwise multipliers of certain subspaces of $\mathcal{H}ol(\mathbb{D})$. If X is a space of analytic functions in $\mathbb{D}$, we let $M(X)$ denote the space of pointwise multipliers of X, that is,

$$M(X) = \{f : \text{analytic in } \mathbb{D} \text{ such that } fg \in X \text{ for all } g \in X\}.$$

Arazy [3] and Brown and Shields [13] showed that

$$M(\mathcal{B}) = \mathcal{B}_{\log} \cap H^{\infty}. \tag{2.3}$$

On the other hand, Axler and Shields [9, Theorem 9] proved that

$$M(\mathcal{D}) \subset B_{\log,1/2}. \tag{2.4}$$

Let us mention also that the logarithmic Bloch spaces play a basic role studying the boundedness of Hankel and Toeplitz operators in Bergman and Bloch-type spaces (see, e.g., [6] and [48]).

For $1 < p < \infty$, the *analytic Besov space* B^p is defined as the set of all $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{B^p} \stackrel{\text{def}}{=} |f(0)| + \left((p-1)\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^{p-2}dA(z)\right)^{1/p} < \infty.$$

When $p = 2$ the Besov space B^2 is the Dirichlet space $\mathcal{D}$. The minimal Besov space B^1 is defined as the space of all functions $f \in \mathcal{H}ol(\mathbb{D})$ for which $f'' \in A^1$. We recall here that for $0 < p < \infty$ the Bergman space A^p consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that $\int_{\mathbb{D}}|f(z)|^p\,dA(z) < \infty$ [20]. It is well-known [4] that

$$B^1 \subset \Lambda(1,1) \subset \bigcap_{1<p<\infty} B^p,$$

where $\Lambda(1,1)$ denotes the Lipschitz space of all $f \in \mathcal{H}ol(\mathbb{D})$ such that $f' \in H^1$. We also have (see [16, p. 52])

$$B^p \subset VMOA, \quad \text{for all } p. \tag{2.5}$$

The univalent functions in Besov spaces have been extensively studied in [14, 16, 17], and [47].

It seems natural to look for possible relations between the space $\mathcal{B}_{\log}$ and Besov spaces. A typical example of an unbounded univalent function in the space $\mathcal{B}_{\log}$ is the function

$$f(z) = \log\log\frac{2}{1-z}.$$

We note that this function belongs to B^p for all $p > 1$ but not to $\Lambda(1,1)$ [16, p. 46].

2.1 Univalent Functions in Besov Spaces but not in the Logarithmic Bloch Spaces

Using (2.5), it follows that:

If $1 \le p < \infty$ and $f \in B^p$, then $M_\infty(r, f') = \mathrm{o}((1-r)^{-1})$, as $r \to 1$.

Our next result asserts that this is the best possible even for univalent functions, at least for $p > 1$. We recall that a domain Ω in $\mathbb{C}$ is said to be *Steiner symmetric* if, for every $x \in \mathbb{R}$; $\Omega \cap \{\mathrm{Re}\, z = x\}$ is either empty, is the whole line $\{\mathrm{Re}\, z = x\}$, or is an interval symmetric with respect to the real axis.

Theorem 1 *Let ϕ be a positive function defined in $(0,1)$ with $\phi(r) \to 0$, as $r \to 1$. Then there exists a function $f \in \mathcal{U} \cap \Lambda(1,1)$ with $f(0) = 0$ whose image is a Steiner symmetric Jordan domain, and such that $f(r)$ is real and positive for all positive r, and satisfies*

$$\limsup_{r\to 1} \frac{(1-r)f'(r)}{\phi(r)} = \infty. \tag{2.6}$$

Since $\Lambda(1,1) \subset B^p$, for all $p > 1$, Theorem 1 in particular implies that

$$B^p \cap \mathcal{U} \not\subset \mathcal{B}_{\log,\alpha}, \quad p > 1, \alpha > 0. \tag{2.7}$$

Proof of Theorem 1 Set $a_0 = b_0 = 0$. Let $\{a_n\}_{n=1}^\infty$, and $\{b_n\}_{n=1}^\infty$ be two sequences of positive numbers with $\sum_{n=1}^\infty a_n < \infty$ and $\sum_{n=1}^\infty b_n < \infty$, and let $\{\epsilon_n\}_{n=1}^\infty$ be a sequence of positive numbers with $\epsilon_n < \min(a_n, a_{n-1})$, for all n, to be determined later. For each $n \ge 1$ we let Q_n be the square and R_n the rectangle defined as follows

$$Q_n = \left\{ z = x + iy : \sum_{k=0}^{n-1}(a_k + b_k) < x < a_n + \sum_{k=0}^{n-1}(a_k + b_k), |y| < a_n \right\},$$

$$R_n = \left\{ z = x + iy : a_n + \sum_{k=0}^{n-1}(a_k + b_k) \le x \le a_n + \sum_{k=0}^{n}(a_k + b_k), |y| < \epsilon_n \right\}.$$

We set also $Q_0 = \{z = x + iy : -a_1 < x \leq 0, |y| < a_1\}$. Finally, we set $\Omega = Q_0 \cup (\bigcup_{n=1}^{\infty}(Q_n \cup R_n))$. Clearly, Ω is a Jordan domain which is Steiner symmetric with $0 \in \Omega$, and $\partial\Omega$ is rectifiable. Let f be the conformal mapping from $\mathbb{D}$ onto Ω with $f(0) = 1$ and $f'(0) > 0$. Then it is clear that $f(r)$ is real and positive for all positive r and, since $\partial\Omega$ is rectifiable, $f \in \Lambda(1,1)$. Now, arguing as in pp. 147–148 of [43], we see that a theorem due to Lindelöf [29] allows us to conclude that it is possible to choose the sequence $\{\epsilon_n\}$ in such a way that (2.6) holds. □

We remark that Theorem 1 shows that the estimate

$$|f'(z)| = \mathrm{o}\left((1-|z|)^{-1}\right), \quad \text{as } |z| \to 1,$$

is sharp in any of the Besov spaces B^p with $p > 1$.

The function f constructed in the proof of Theorem 1 is close-to-convex. Hence, Theorem 1 implies that

$$\mathcal{C} \cap H^\infty \not\subset \mathcal{B}_{\log,\alpha}, \quad \text{for any } \alpha > 0. \tag{2.8}$$

Using (2.4), Theorem 1 yields also the following result.

Corollary 1 *There exists a bounded Steiner symmetric domain Ω with rectifiable boundary and $0 \in \Omega$, such that if f is a conformal mapping from $\mathbb{D}$ onto Ω with $f(0) = 0$, then f is not a multiplier of the Dirichlet space $\mathcal{D}$.*

Hence, we obtain a new proof of Theorem 10 of [9].

The question of whether or not there exists $f \in \mathcal{U} \cap B^1$ satisfying (2.6) remains open.

2.2 Univalent Functions in Logarithmic Bloch Spaces

Using Theorem D of [16] and Proposition 2.3 of [23] we see that if $f(z) = \sum_{k=1}^{\infty} \frac{1}{k} z^{2^k}$ then $f \in \mathcal{B}_{\log}$ but $f \notin B^p$ for any $p \geq 1$. More generally, using the results of [45], we see that if $f(z) = \sum_{k=1}^{\infty} \frac{1}{k^\alpha} z^{2^k}$ then $f \in \mathcal{B}_{\log,\alpha}$, but $f \notin B^p$ for any $p \geq 1$. Thus

$$B_{\log,\alpha} \not\subset B^p, \quad p \geq 1, \ \alpha > 0.$$

The functions we have used to prove these results are not univalent. Let us now turn to the question of finding univalent functions in $B_{\log,\alpha} \setminus B^p$. We shall prove the following result.

Theorem 2 *There exists a bounded univalent function* (*in fact*, *a bounded starlike function*) $f \in \mathcal{B}_{\log} \setminus (\bigcup_{1\leq p<2} B^p)$.

Proof A result of Twomey [46] asserts that if $f \in \mathcal{U}$ and $f(\mathbb{D})$ is a bounded starlike domain then $f \in \mathcal{B}_{\log}$, in other words,

$$S^\star \cap H^\infty \subset \mathcal{B}_{\log}. \tag{2.9}$$

We recall here that (2.8) shows that this is not true with $S^\star$ in the place of $\mathcal{C}$. Now Donaire, Girela, and Vukotić have proved in [17] that there exists $f \in S^\star \cap H^\infty$ which does not belong to B^p, if $p < 2$. Putting these results together we obtain our result. □

Any bounded univalent function belongs to the Dirichlet space. Hence the starlike function f used in the proof of Theorem 2 belongs to $\mathcal{D} = B^2$. It is natural to ask whether or not there exists a (necessarily unbounded) univalent function in $\mathcal{B}_{\log} \setminus \mathcal{D}$. More generally, it would be interesting to study relations between $\mathcal{B}_{\log} \cap \mathcal{U}$ and $\mathcal{D} \cap \mathcal{U}$.

It is clear that there exist unbounded starlike functions in the Dirichlet space. Indeed, let $\phi : [1, \infty) \to (0, 1]$ be a decreasing function with $\phi(1) = 1$, and $\int_1^\infty \phi(x)\, dx < \infty$, and let

$$\Omega = \{z = x + iy \in \mathbb{C} : |x| < 1, |y| < 1\} \cup \{z = x + iy \in \mathbb{C} : x \geq 1, |y| < \phi(x)\}.$$

Then Ω is an unbounded domain which is starlike with respect to the origin. Let f be the conformal mapping from $\mathbb{D}$ onto Ω with $f(0) = 0$ and $f'(0) > 0$. Notice that Ω has finite area. Then it follows that $f \in S^\star \cap \mathcal{D}$.

However, we cannot find unbounded convex functions in $\mathcal{D}$. In fact, we have the following stronger result.

Theorem 3 *Let Ω be an unbounded convex domain in $\mathbb{C}$ and let f be a conformal mapping from $\mathbb{D}$ onto Ω. Then $f \notin VMOA$.*

In other words,

$$\mathcal{K} \cap VMOA \subset H^\infty. \tag{2.10}$$

Proof Suppose that Ω is an unbounded convex domain in $\mathbb{C}$ and that a conformal mapping f from $\mathbb{D}$ onto Ω is in *VMOA*. For every $\theta \in [-\pi, \pi)$ set $R_\theta = \{re^{i\theta} : r > 0\}$. We claim that there exists $\theta \in [-\pi, \pi)$ such that $\text{dist}(z, R_\theta) \to 0$, as $z \to \infty$ in Ω. Otherwise, there would exist $\theta_1, \theta_2 \in [-\pi, \pi)$ with $\theta_1 \neq \theta_2$ and two sequences $\{z_n\}_{n=1}^\infty, \{w_n\}_{n=1}^\infty$ in Ω tending to ∞ such that

$$\lim_{n\to\infty} \text{dist}(z_n, R_{\theta_1}) = \lim_{n\to\infty} \text{dist}(w_n, R_{\theta_2}) = 0.$$

This and the convexity of Ω implies that there exists $r_0 > 0$ such that Ω contains all the points in a certain sector with absolute value bigger than r_0. This is in contradiction with the fact that $\lim_{w\in\Omega, |w|\to\infty} d_\Omega(w) = 0$. Hence our claim is proved.

We may assume without loss of generality that $\text{dist}(z, R_0) \to 0$, as $z \to \infty$ in Ω. Then there exists a sequence $\{x_n + iy_n\}_{n=1}^\infty$ in Ω with $x_n \uparrow \infty$, $y_n \neq 0$ for all n and $|y_n| \downarrow 0$. Now, passing to a subsequence, we may assume that all the y_n are either positive or negative. For the sake of simplicity, assume without loss of generality that $y_n > 0$, for all n. Since Ω is convex, for all $n \geq 3$, Ω contains all the points in the interior of the triangle of vertices $x_1 + iy_1, x_2 + iy_2, x_n + iy_n$. Then it follows that the strip $\{z = x + iy \in \mathbb{C} : x > x_2, y_2 < y < y_1\}$ is contained in Ω. Again, this is in contradiction with the fact that $f \in VMOA$. □

We remark that Donaire, Girela, and Vukotić [17] have proved that $\mathcal{K} \cap H^\infty \subset B^1$. Using this result, recalling that $B^1 \subset B^p \subset VMOA$, and using also Twomey's result (2.9), the inclusion $\mathcal{K} \subset S^\star$, (2.10), (2.2), (1.2), (1.3), and Theorem 3, we obtain the following.

Theorem 4 *Let f be a conformal mapping from $\mathbb{D}$ onto a convex domain. Then the following conditions are equivalent*:

(i) $f \in H^\infty$.
(ii) $f \in VMOA$.
(iii) $f \in \mathcal{B}_0$.
(iv) $f \in Q_{p,0}$, *for some* $p > 0$.
(v) $f \in Q_{p,0}$, *for all* $p > 0$.
(vi) $f \in \mathcal{B}_{\log}$.
(vii) $f \in \mathcal{B}_{\log,\alpha}$, *for some* $\alpha > 0$.
(viii) $f \in \mathcal{B}_{\log,\alpha}$, *for all* $\alpha > 0$.
(ix) $f \in B^1$.
(x) $f \in B^p$, *for some* $p \geq 1$.
(xi) $f \in B^p$, *for all* $p \geq 1$.

We close this section by recalling that there are other results relating geometric conditions on the image domain of a bounded univalent function and its membership in logarithmic Bloch spaces. Let us just mention that Mejía and Pommerenke [30] proved that a hyperbolically convex conformal mapping of $\mathbb{D}$ into itself belongs to $\mathcal{B}_{\log,2}$.

3 On the Logarithm of the Derivative of a Univalent Function

The Becker univalence criterion [11] (see also [21], [36, Theorem 6.7], or [39, Theorem 1.11]) asserts that if f is a locally univalent analytic function in $\mathbb{D}$ and

$$(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \leq 1, \quad z \in \mathbb{D}, \tag{3.1}$$

then f is univalent in $\mathbb{D}$. This yields a close connection between Bloch functions and univalent functions:

A locally univalent analytic function f in $\mathbb{D}$ is a Bloch function if and only if there exist a constant $A \in \mathbb{C}$ and $g \in \mathcal{U}$ such that $f = A \log g'$.

Subsequently, a number of results have been obtained relating geometric conditions of the image domain $g(\mathbb{D})$ of a function $g \in \mathcal{U}$ and the membership of $\log g'$ in certain spaces of analytic functions such as *BMOA* [37], *VMOA*, or $\mathcal{B}_0$ [38]. In general, for a given subspace X of $\mathcal{H}ol(\mathbb{D})$, it is interesting to obtain simple characterizations of the univalent functions g such that $\log g' \in X$, and to investigate whether this condition implies some nice geometric conditions on the image domain $g(\mathbb{D})$.

A number of these characterizations have been obtained in terms of the Schwarzian derivative. We recall that if g is a locally univalent analytic function in $\mathbb{D}$ then the

Schwarzian derivative Sg of g is defined by

$$Sg = \left(\frac{g''}{g'}\right)' - \frac{1}{2}\left(\frac{g''}{g'}\right)^2.$$

If $g \in \mathcal{U}$ then

$$|Sg(z)| \le 6(1-|z|^2)^{-2}, \quad z \in \mathbb{D}. \tag{3.2}$$

The Nehari univalence criterion [31] asserts that if g is locally univalent in $\mathbb{D}$ and

$$|Sg(z)| \le 2(1-|z|^2)^{-2}, \quad z \in \mathbb{D}, \tag{3.3}$$

then $g \in \mathcal{U}$. Furthermore, if the stronger inequality $|Sg(z)| \le 2\kappa(1-|z|^2)^{-2}$ $(z \in \mathbb{D})$ holds for some $\kappa \in [0,1)$, then g maps $\mathbb{D}$ conformally onto a quasidisc [1]. We refer to [27, 36], and [39] for a discussion of these results and for the properties of the Schwarzian derivative that we shall need. Let us remark here that the class of univalent functions which satisfy (3.3) is quite large; for instance, Nehari [32] proved that it contains the class $\mathcal{K}$ of convex functions.

Astala and Zinsmeister [5, 51] proved that if $g \in \mathcal{U}$, then $\log g' \in BMOA$ if and only if $|Sg(z)|^2(1-|z|^2)^3\,dx\,dy$ is a Carleson measure. Pau and Peláez [33] have recently extended this result showing that, for any $p > 0$,

$$\log g' \in Q_p \quad \text{if and only if}$$
$$|Sg(z)|^2(1-|z|^2)^{2+p}\,dx\,dy \quad \text{is a } p\text{-Carleson measure.}$$

It is natural to ask whether we can obtain results similar to these for Besov spaces and for the logarithmic Bloch space. More precisely, we can raise the following question:

Question 1 Let $g \in \mathcal{U}$. Does there exist some restriction on the growth of Sg or some integrability condition on Sg which implies that $\log g'$ belongs to B^p, for some $p \ge 1$, or to the logarithmic Bloch space $\mathcal{B}_{\log}$?

The answer to this question is negative. Indeed, recall that the Schwarzian derivative of any Möbius transformation is identically 0. Then take

$$g(z) = \frac{1}{1-z}, \quad z \in \mathbb{D}.$$

We have $g \in \mathcal{U}$ and $Sg \equiv 0$. On the other hand,

$$\log g'(z) = \frac{1}{2}\log\frac{1}{1-z}, \quad z \in \mathbb{D},$$

and then it follows readily that $\log g' \notin (\bigcup_{p\ge 1} B^p) \cup (\bigcup_{\alpha>0} \mathcal{B}_{\log,\alpha})$.

We shall prove the following result in the other direction.

Theorem 5 *Let $g \in \mathcal{U}$. If $1 \le p < \infty$ and $\log g' \in B^p$, then*

$$\int_{\mathbb{D}} (1-|z|^2)^{2p-2}|Sg(z)|^p\,dA(z) < \infty.$$

Proof For simplicity, set $h = \log g'$. With this notation $Sg = h'' - \frac{1}{2}(h')^2$.

Suppose first that $p > 1$ and $h = \log g' \in B^p$. Using a well-known result of Flett [22, Theorem 6] (see also [52, Theorem 4.28]), we deduce that

$$\int_{\mathbb{D}} (1 - |z|^2)^{2p-2} |h''(z)|^p \, dA(z) < \infty. \tag{3.4}$$

On the other hand, recalling that $h \in \mathcal{B}$, we obtain

$$\begin{aligned}\int_{\mathbb{D}} &(1 - |z|^2)^{2p-2} |h'(z)|^{2p} \, dA(z) \\ &= \int_{\mathbb{D}} (1 - |z|^2)^p |h'(z)|^p (1 - |z|^2)^{p-2} |h'(z)|^p \, dA(z) \\ &\le C \|h\|_{\mathcal{B}}^p \|h\|_{B^p}^p < \infty.\end{aligned}$$

This and (3.4) yield that $\int_{\mathbb{D}} (1 - |z|^2)^{2p-2} |Sg(z)|^p \, dA(z) < \infty$.

Suppose now that $h = \log g' \in B^1$. This is equivalent to saying that $\int_{\mathbb{D}} |h''(z)| \, dA(z) < \infty$. Since $h \in \mathcal{B}$, we have $|h''(z)| = \mathrm{O}((1 - |z|)^{-2})$, as $|z| \to 1$, which implies that

$$\int_{\mathbb{D}} (1 - |z|)^2 |h''(z)|^2 \, dA(z) \le C \int_{\mathbb{D}} |h''(z)| \, dA(z) < \infty,$$

which, using again Theorem 6 of [22], implies that $\int_{\mathbb{D}} |h'(z)|^2 \, dA(z) < \infty$. Then it follows that $\int_D |h''(z) - \frac{1}{2} h'(z)^2| \, dA(z) < \infty$ as claimed. □

Our next result shows that the exponent $2p - 2$ cannot be substituted by a smaller one in Theorem 5.

Theorem 6 *Suppose that* $p \ge 1$ *and* $\alpha < 2p - 2$, *then there exists* $g \in \mathcal{U}$ *with* $\log g' \in B^1$ *such that* $\int_{\mathbb{D}} (1 - |z|^2)^\alpha |Sg(z)|^p \, dA(z) = \infty$.

Proof Let δ be a real number with $0 < \delta < \min(1, (2p - 2 - \alpha) p^{-1})$, and let g be a locally univalent function defined by

$$\frac{g''(z)}{g'(z)} = \frac{1}{(1 - z^2)^{1-\delta}}, \quad z \in \mathbb{D}.$$

Since $(1 - |z|^2) \le |1 - z^2|^{1-\delta}$, for all $z \in \mathbb{D}$, the above-mentioned Becker's univalence criterion shows that $g \in \mathcal{U}$. Simple computations show that $\log g' \in B^1$ and $\int_{\mathbb{D}} (1 - |z|^2)^\alpha |Sg(z)|^p \, dA(z) = \infty$. □

In view of Theorem 4, it is natural to ask whether the condition $g \in \mathcal{K} \cap H^\infty$ implies that $\log g'$ lies in some of the subspaces of the Bloch space that we are considering. The answer is negative. Indeed, for any $t \in (0, 1)$ let A_t be the function defined in [15, p. 289], that is,

$$A_t(z) = \frac{1}{\sqrt{1-t}} \frac{(1+z)^{\sqrt{1-t}} - (1-z)^{\sqrt{1-t}}}{(1+z)^{\sqrt{1-t}} + (1-z)^{\sqrt{1-t}}}, \quad z \in \mathbb{D}. \tag{3.5}$$

Then (see [15, p. 290]) A_t is a conformal mapping from $\mathbb{D}$ onto the intersection of two discs (hence a convex Jordan domain) and

$$SA_t(z) = \frac{2t}{(1-z^2)^2}, \quad z \in \mathbb{D}.$$

Using (3.5) and Theorem 5 we readily see that $\log A_t'$ does not belong to any of the Besov spaces. Even more is true: Since $(1-r^2)^2|SA_t(r)| \to 2t \neq 0$, as $r \to 1$, using [12, Satz, 1] (see also [39, Theorem 11.1]), we deduce that $\log A_t' \notin \mathcal{B}_0$. We remark that this can also be deduced from Theorem 1 of [38], with an obvious geometric argument using that $A_t(\mathbb{D})$ is the intersection of two discs.

It turns out that the question of the membership of $\log g'$ in a certain subspace of the Bloch space is very delicate. As we shall see, it has to do with the smoothness of the boundary of the image domain.

From now on we shall be concerned with univalent functions g which map $\mathbb{D}$ conformally onto the inner domain of a Jordan curve Γ in $\mathbb{C}$.

Definition 1 Let Γ be a Jordan curve in $\mathbb{C}$, and let g be a conformal mapping from $\mathbb{D}$ onto the inner domain of Γ. For $\omega_1, \omega_2 \in \Gamma$, let $\Gamma(\omega_1, \omega_2)$ be the arc of the smaller diameter of Γ joining ω_1 and ω_2. For $\delta > 0$, we define

$$\eta(\delta) = \eta_\Gamma(\delta) = \sup_{|\omega_1-\omega_2|<\delta} \sup_{\omega\in\Gamma(\omega_1,\omega_2)} \left(\frac{|\omega_2-\omega| + |\omega-\omega_1|}{|\omega_2-\omega_1|} - 1 \right)^{1/2},$$

$$\beta(\delta) = \beta_g(\delta) = \sup_{1-\delta\le\xi<1} (1-|\xi|) \left| \frac{g''(\xi)}{g'(\xi)} \right|.$$

If Γ is a circle of radius r, then $\eta(t) \sim \frac{1}{r}t$. It is also easy to see that if Γ is an ellipse with semiaxes a and b, then $\eta(t) \sim \frac{1}{\sqrt{ab}}t$. More generally, one can prove that if Γ is of class C^2 and its inner domain Ω is convex then

$$\eta(t) \sim Ct.$$

Indeed, let $\alpha(s)$ be the parameterization of Γ with arc length s as a parameter. Since Ω is convex, the curvature $\kappa(s)$ is positive. Let $M = \kappa(s_0)$ be the maximum of the curvature. Also, for each s let C_s be the osculating circle of Γ at $\alpha(s)$. Then it is well-known that C_s has contact of at least second order with Γ. Consequently, for each point $\alpha(s) = \omega \in \Gamma$, we have

$$\frac{|\omega_1-\omega| + |\omega_2-\omega|}{t} - 1 \sim \kappa^2(s)t^2, \quad \text{as } t = |\omega_1-\omega_2| \to 0.$$

Then it follows that $\eta(t) \sim \kappa(s_0)t$, as $t \to 0$.

We note that $\beta(\delta) > 0$ for all $\delta > 0$, unless g is of the form $g(z) = Az + B$ ($z \in \mathbb{D}$) with A and B constants. The maximum principle implies that β is increasing and subadditive [40, p. 109]. Since $\log g'$ is a Bloch function, the function β is bounded. Clearly,

$$\log g' \in \mathcal{B}_0 \quad \Leftrightarrow \quad \beta(\delta) \to 0, \ \text{ as } \delta \to 0.$$

It turns out that there is a close relation between the functions η and β, and this yields a number of results relating geometric properties of Γ and the membership of $\log g'$ in a certain subspace of the Bloch space $\mathcal{B}$. A key connection between η and β is the following (see [40]):

Suppose that g is not of the form $g(z) = Az + B$ ($z \in \mathbb{D}$) with A and B constants and that $\int_0^1 \frac{\eta(s)}{s}\, ds < \infty$. Then there exist $C_1, C_2 > 0$ such that

$$C_1 \eta(s) \le \beta(s) \le C_2 \left(\int_0^s \frac{\eta(t)}{t}\, dt + s \int_s^1 \frac{\eta(t)}{t^2}\, dt \right), \quad 0 < s < 1. \tag{3.6}$$

Let us recall some of the above-mentioned results:

Theorem B

(i) *The curve Γ is said to be asymptotically conformal if $\eta(\delta) \to 0$, as $\delta \to 0$, and this is equivalent to saying that* $\log g' \in \mathcal{B}_0$ [38].

(ii) *If $\int_0^1 \frac{\eta(s)}{s}\, ds < \infty$, then Γ is smooth and* $\log g'$ *belongs to the disc algebra* $\mathcal{A}$ [28, 40].

(iii) *If $\int_0^1 \frac{\eta^2(s)}{s}\, ds < \infty$, then Γ is asymptotically smooth and* $\log g' \in VMOA$ [38, 40].

(iv) *If* $0 < p < 1$ *and* $\int_0^1 \frac{\eta^2(s)}{s^{2-p}}\, ds < \infty$, *then* $\log g' \in Q_p$ [33].

We shall prove the following result concerning Besov spaces.

Theorem 7 *Suppose that $1 < p < \infty$. Let Γ be a Jordan curve in $\mathbb{C}$ and let g be a conformal mapping from $\mathbb{D}$ onto the inner domain of Γ. If $\int_0^1 \frac{\eta_\Gamma^p(s)}{s^2}\, ds < \infty$ then* $\log g' \in B^p$.

Proof For simplicity, we shall set $\eta_\Gamma = \eta$ and $\log g' = h$. Let us suppose that $p > 1$ and $\int_0^1 \frac{\eta_\Gamma^p(s)}{s^2}\, ds < \infty$.

We shall assume without loss of generality that g is not of the form $g(z) = Az + B$ with A and B constants.

Let q be the exponent conjugate to p, that is, $p^{-1} + q^{-1} = 1$. Using Hölder's inequality, we obtain

$$\int_0^1 \frac{\eta(s)}{s}\, ds = \int_0^1 \frac{\eta(s)}{s^{2/p} s^{1-\frac{2}{p}}}\, ds \le \left(\int_0^1 \frac{\eta^p(s)}{s^2}\, ds \right)^{1/p} \left(\int_0^1 \frac{ds}{s^{1-\frac{q}{p}}} \right)^{1/q} < \infty. \tag{3.7}$$

Hence, (3.6) holds for certain $C_1, C_2 > 0$. Notice that

$$|z| = t \quad \Rightarrow \quad (1-|z|)|h'(z)| \le \beta(1-t), \quad \text{for all } t, \tag{3.8}$$

and then

$$\int_{\mathbb{D}} (1-|z|^2)^{p-2}|h'(z)|^p dA(z) \le C \int_0^1 \int_0^{2\pi} (1-r)^{p-2}|h'(re^{i\theta})|^p \, d\theta dr$$
$$\le C \int_0^1 \frac{\beta^p(1-t)}{(1-t)^2} dt = C \int_0^1 \frac{\beta^p(t)}{t^2} dt.$$

Now, using (3.6) and Hardy's inequalities [44, p. 272] we obtain

$$\int_0^1 \frac{\beta^p(t)}{t^2} dt \le C \int_0^1 \frac{1}{t^2} \left(\int_0^t \frac{\eta(s)}{s} ds \right)^p dt + C \int_0^1 \frac{1}{t^{2-p}} \left(\int_t^1 \frac{\eta(s)}{s^2} ds \right)^p dt$$
$$\le C \int_0^1 \frac{\eta^p(s)}{s^2} ds < \infty.$$

This and (3.7) give that $h = \log g' \in B^p$. □

Remark 1 Using the arguments of the proof of Theorem 7 for $p = 1$, we would obtain the following result:

If $\int_0^1 \frac{\eta(s)}{s^2} ds < \infty$ then $\log g' \in B^1$.

However, this says nothing because $\int_0^1 \frac{\eta(s)}{s^2} ds = \infty$ for any Γ. Indeed, if we had $\int_0^1 \frac{\eta(s)}{s^2} ds < \infty$, then, trivially, $\int_0^1 \frac{\eta(s)}{s} ds < \infty$ and then (3.6) would hold also. Then using Hardy's inequalities as before would yield $\int_0^1 \frac{\beta(s)}{s^2} ds < \infty$. But this is never true: Indeed, since β is subadditive we have $\beta(x) \ge \frac{1}{2}\beta(2x)$ for all $x \in (0, \frac{1}{2})$ and hence

$$\beta(2^{-k}) \ge 2^{-k}\beta(1), \quad k = 1, 2, \dots.$$

Then we have

$$\int_0^1 \frac{\beta(s)}{s^2} ds = \sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} \frac{\beta(s)}{s^2} ds \ge C \sum_{k=0}^{\infty} \beta(2^{-k-1})2^k = \infty.$$

Pommerenke and Warschawski [40, Theorem 4.3] proved that β can be prescribed up to multiplicative bounds, and that the integrability conditions on η cannot be weakened in parts (ii) and (iii) of Theorem B. A similar result regarding part (iv) was obtained in [33].

More precisely, Pommerenke and Warschawski constructed for any increasing and subadditive function $\varphi : (0, 1] \to (0, \infty)$ a univalent function g with the following properties:

(a) g is a conformal mapping from $\mathbb{D}$ onto the inner domain of a Jordan curve Γ.
(b) $\varphi \asymp \beta_g$.

Furthermore, they proved that for such a g one has

$$\int_0^1 \frac{\eta(s)}{s}\,ds < \infty \quad \Leftrightarrow \quad \log g' \in \mathcal{A} \quad \text{and}$$

$$\int_0^1 \frac{\eta^2(s)}{s}\,ds < \infty \quad \Leftrightarrow \quad \log g' \in VMOA.$$

We shall prove the following result regarding Besov spaces.

Theorem 8 *Suppose that φ satisfies also the following condition*:

$$\varphi(x) - \frac{1}{2}\varphi(2x) \ge \alpha\varphi(x), \quad 0 < x < \frac{1}{2}, \textit{ for a certain } \alpha > 0. \tag{3.9}$$

Then for any $p \in (1, \infty)$, we have

$$\int_0^1 \frac{\eta_\Gamma^p(s)}{s^2}\,ds < \infty \quad \Leftrightarrow \quad \log g' \in \mathcal{B}^p. \tag{3.10}$$

Proof The function g constructed by Pommerenke and Warschawski is defined by $g(0) = 0$ and

$$\log g'(z) = \sum_{k=0}^{\infty} b_k z^{2^k}, \quad z \in \mathbb{D},$$

where

$$b_0 = b\varphi(1), \qquad b_k = b\varphi(2^{-k}) - \frac{b}{2}\varphi(2^{-k+1}), \quad k = 1, 2, \ldots,$$

and b is a sufficiently small positive number.

The implication $\int_0^1 \frac{\eta_\Gamma^p(s)}{s^2}\,ds < \infty \Rightarrow \log g' \in \mathcal{B}^p$ follows from Theorem 7.

Let us prove the other implication. Hence, suppose that $\log g' \in \mathcal{B}^p$. Since $\log g'$ is given by a lacunary power series, using Theorem D of [16] we see that

$$\sum_{k=0}^{\infty} 2^k |b_k|^p < \infty. \tag{3.11}$$

Clearly, this implies that $\sum_{k=0}^{\infty} |b_k| < \infty$. Since $B^p \subset \mathcal{B}_0$, using part (i) of Theorem B we deduce that Γ is asymptotically conformal, and then using Theorem 1 of [40] and the fact that $\beta_g \asymp \varphi$ we see that $\eta_\Gamma(t) \le M_1\beta_g(t) \le M_2\varphi(t)$ $(t \in (0, 1))$. Then it follows that

$$\int_0^1 \frac{\eta_\Gamma(t)}{t}\,dt \le M_2 \int_0^1 \frac{\varphi(t^{1/2})}{t}\,dt \le 2M_2 \int_0^1 \frac{\varphi(t)}{t}\,dt$$
$$= 2M_2 \sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} \frac{\varphi(t)}{t}\,dt \le C \sum_{k=0}^{\infty} \varphi(2^{-k}) \le C \sum_{k=0}^{\infty} b_k < \infty.$$

This implies that (3.6) holds. Hence,

$$\eta_\Gamma(t) \le C\beta_g(t), \quad t \in (0, 1]. \tag{3.12}$$

Now, using the fact that φ is increasing, the definition of the sequence $\{b_k\}$, (3.9), and (3.11), we deduce that

$$\int_0^1 \frac{\varphi^p(t)}{t^2}\,dt = \sum_{n=0}^{\infty} \int_{2^{-n-1}}^{2^{-n}} \frac{\varphi^p(t)}{t^2}\,dt \le C \sum_{n=0}^{\infty} 2^n \varphi^p(2^{-n})$$
$$\le C \sum_{n=0}^{\infty} 2^n \left(\varphi(2^{-n}) - \frac{1}{2}\varphi(2^{-n-1})\right)^p = C \sum_{n=0}^{\infty} 2^n |b_n|^p < \infty.$$

Since $\beta_g \asymp \varphi$, this and (3.12) imply

$$\int_0^1 \frac{\eta_\Gamma^p(t)}{t^2}\,dt < \infty. \tag{3.13}$$

□

Remark 2 Since the function β is subadditive, imposing the condition of φ being subadditive as Pommerenke and Warschawski did seems completely natural. However, an examination of their work shows that they only use the weaker condition

$$\varphi(2x) \le 2\varphi(x), \quad 0 < x < \frac{1}{2} \tag{3.14}$$

(to obtain $b_k \ge 0$). Notice that (3.14) is also weaker than our condition (3.9). In the same way our result remains true without assuming the subadditivity, that is, we only have to assume that φ is increasing and satisfies (3.9). Bearing this in mind, we note that we could take

$$\varphi(t) = t^a \left(\log \frac{A}{t}\right)^b, \quad 0 < t < 1,$$

with $0 < a < 1$, $b \in \mathbb{R}$, and A an appropriate positive number.

Acknowledgements The authors would like to thank Hugo Caerols (Universidad Adolfo Ibáñez, Chile) for several conversations on the topic.

Part of this work was done while the third author was visiting the University of Málaga. He wishes to express his gratitude for the hospitality during his visit.

Finally, the authors wish to express their gratitude to the referee for his very useful comments and suggestions that have greatly improved the presentation of this paper.

Note Added in Proof After the submission of this paper we learned that F. Pérez-González and J. Rättyä have shown in [35] that if $1 < p < \infty$ and g is a conformal mapping from $\mathbb{D}$ onto a Jordan domain then

$$\log g' \in B^p \quad \Leftrightarrow \quad \int_{\mathbb{D}} (1 - |z|^2)^{2p-2} |Sg(z)|^p \, dA(z) < \infty.$$

We remark that unbounded g are not considered in [35], and that the example we give just before Theorem 5 deals with an unbounded univalent function and, hence, it is not in contradiction with Theorem 1 of [35].

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