Journal of Cellular Automata, Vol. 0, pp. 1–18
Reprints available directly from the publisher
Photocopying permitted by license only

Number-Conserving Cellular Automata and Communication Complexity: A Numerical Exploration Beyond Elementary CAs

ERIC GOLES[1,2] AND ANDRÉS MOREIRA[2,3]

[1]*Universidad Adolfo Ibáñez, Diagonal Las Torres 2640, Peñalolén, Santiago, Chile.*

[2]*Institute for Complex Systems (ISCV), Subida Artillería 470, Valparaíso, Chile.*
E-mail: andres.moreira@usm.cl

[3]*Departamento de Informática and Centro Tecnológico de Valparaíso (CCTVal), Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile.*

Received: November 22, 2010. Accepted: May 13, 2011.

We perform a numerical exploration of number-conserving cellular automata (NCCA) beyond the class of elementary CAs, in search of examples with high communication complexity. We consider some possible generalizations of the elementary rule 184 (a minimal model of traffic, which is the only non-trivial elementary NCCA), as well as the classes of NCCAs which minimally extend either the radius or the state set (with respect to the 2 states and radius 1 of the elementary case). Both for 3 states and radius 1, and for 2 states and radius 2, NCCA appear that are conjectured to have maximal (exponential) communication complexity. Examples are given also for (conjectured) linear and quadratic behaviour.

Keywords: Number-Conserving, Communication Complexity, One-dimensional Cellular Automata

1 INTRODUCTION

Communication complexity has emerged in the last few years [3, 7] as a promising tool for a better understanding of the dynamics of cellular automata. Between its merits are the direct relation to the view of CAs as parallel computing devices, the explicit decomposition of the complex interactions between cells, and the possibility to disproof (intrinsic) computational universality, something which is otherwise very hard.

In the present text we review some recent results on the application of these ideas to number-conserving and monotone 1D CAs, and explore beyond elementary CAs in search of NCCA with higher communication complexity. The existence of NCCA with exponential communication complexity was already proved, but the study of both NCCAs and monotone CAs among elementary CAs gave only linear instances; the existence of complex cases with few states or small neighbourhoods is not granted. We first consider here the generalizations of the linear case of rule 184 (a well known elementary NCCA which is a minimal model of traffic flow). Then we explore the sets of NCCA rules with radius 2 and 2 states, and radius 1 and 3 states (which correspond to the two natural ways to move outside the class of elementary CAs).

2 NOTATION AND DEFINITIONS

A one-dimensional cellular automaton is a dynamical system acting on $S^{\mathbb{Z}}$, for some finite set of states S; without loss of generality, we will assume that $S \subset \mathbb{Z}$. The dynamics $F : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ is defined through the synchronous application of a local function $f : S^{\ell+1+r} \to S$, $\ell, r \geq 0$, so that $F(x)_i = f(x_{i-\ell}, \ldots, x_i, \ldots, x_{i+r})$. Here ℓ and r are called the left and right radius, respectively; the *neighbourhood size* is $N = \ell + 1 + r$, the number of arguments of f. When $\ell = r$ we just speak of the *radius*; such a symmetrical neighbourhood can always be imposed, by extending f and setting the radius to $\max\{\ell, r\}$. The family of *elementary* cellular automata (ECAs) consists of the CAs with radius 1 and $S = \{0, 1\}$; we number them according to the standard Wolfram coding: *code*(f)=$\sum_{abc} f(abc) 2^{4a+2b+c}$. More generally, the Wolfram code for a rule $f : S^N \to S$, for $q = |S|$, is

$$\text{code}(f) = \sum_{w \in S^N} f(w) q^{\sum_{i=0}^{N-1} w_i q^i}$$

so that code(f) corresponds to the decimal representation for the value of $f(q-1, \ldots, q-1)\, f(q-1, \ldots, q-2) \cdots f(0, \ldots, 1)\, f(0, \ldots, 0)$ when read as a number in base q.

The n-th iteration of f is defined as

$$\begin{aligned} & f^n(x_{-rn}, \ldots, x_{-1}, x_0, x_1, \ldots, x_{rn}) \\ & = f^{n-1}(f(x_{-rn}, \ldots, x_{-rn+2r}), \ldots, f(x_{rn-2r}, \ldots, x_{rn})) \end{aligned}$$

for $n \geq 2$ and $f^1 = f$. It shows how the state of a cell at time $t = n$ depends on the states of its rn left and rn right neighbours at time $t = 0$.

Number-conserving CAs Number-conserving CAs preserve the sum of the states throughout their iteration, for any spatially periodic configuration. More precisely: let S^* be the set of all finite words on S, $L(w)$ be the length of a given word, and for $w \in S^*$, $w = w_1, \ldots, w_{L(w)}$ define $\overline{w} \in S^{\mathbb{Z}}$ as $\overline{w}_i = w_{i \bmod L(w)}$. A *number-conserving* CA (NCCA) is a CA F such that

$$\sum_{i=1}^{L(w)} \overline{w}_i = \sum_{i=1}^{L(w)} F(\overline{w})_i \qquad \forall w \in S^* \tag{1}$$

A natural way to look at NCCAs is in terms of "particles" that can be neither created nor destroyed, and whose distribution along $\mathbb{Z}$ is given by the values in the cells [1, 2]. For one-dimensional NCCAs there is always an alternative description of the dynamics from the particles' point of view: there is a local function which describes the movement of the particles of a cell, and whose parallel application on the whole configuration yields the same image as the CA [12]. This function is unique if we also ask that the particles never jump over each other. A succint way to write these "motion rules", following Boccara & Fukś [1], is through a list of local configurations, which indicate the movement of the particle(s) at the origin; configurations with no movement are not shown, and only the number of particles that leave the origin is given in each case. When no number is stated, the arrow represents the movement of a single particle. If more than one configuration matches a situation, the first one listed is applied. The following are examples of motion rules; each of them corresponds to a different NCCA, with $S = \{0, 1\}$ for M_1 and M_2 and $S = \{0, 1, 2\}$ for M_3 and M_4. The bullet ($\bullet$) in M_4 is a wildcard, representing any state.

$$M_1 = \{\overset{\curvearrowright}{10}\}\ ,\ M_2 = \{\overset{\curvearrowright}{10}, 0\overset{\curvearrowleft}{01}1\}\ ,\ M_3 = \{\overset{\overset{2}{\curvearrowright}}{20}, \overset{\overset{1}{\curvearrowright}}{21}\}\ ,\ M_4 = \{\ \overset{\overset{1}{\curvearrowright}}{2\bullet}\ \}$$

Communication complexity *Communication complexity* was introduced as a tool for understanding the required information exchange in parallel computation, by providing lower bounds for it. Consider a function $F : X \times Y \to \mathbb{Z}$, where X and Y are finite sets. Alice and Bob must compute $F(x, y)$, but x is given only to Alice, and y to Bob. Knowing F in advance, they will apply a previously agreed protocol so that the number of exchanged bits is minimal in the worst case. This worst case cost (in bits) is the *many-round communication complexity*, $cc(F)$. A related notion is *one-round communication complexity*, $cc_1(F)$, where the protocol is only allowed to send information in one direction (from Alice to Bob, or from Bob to Alice) and again the worst case is considered.

The notion of communication complexity has been applied to provide a fresh way to look at the complexity and dynamical behaviour of 1D CAs, whose parallel nature make it a natural conceptual tool [3, 7, 4, 6, 8]. The prefered quantity has been the one-round communication complexity, which is easier to analize and also easier to interpret: we can see it as a measure of the information about the "outside" that a region must include in order to predict the future state of its border. Furthermore, it has been shown [8] that *intrinsic universality* implies maximal communication complexity for a number of problems; finding a subexponential protocol for any of these for a particular CA would thus show its non-universality (which, as other negative results, is otherwise very hard to prove). Unfortunately, the approach has yet to deliver particular interesting results: no subexponential protocol has been found so far for the computationally universal ECA 110, for the suspiciously complex ECA 54, or for any other dubious rule.

The simplest and most natural of the problems that have been studied is PRED (prediction): $F = f^n$ (where f is the local rule of the CA), and X and Y stand for the left and the right side of the initial configuration (the center is shared by both). The growth of $\phi(f, n) = \max_{c \in S} cc_1(f^n(\cdot, c, \cdot))$ is then studied as $n \to \infty$. A useful tool for both computation and visualization are the sequences of matrices

$$M_f^{c,n}(x, y) = f^n(x, c, y) \quad \text{with } x, y \in S^{rn}$$

The value of cc_1 is related to the maximum numbers of different rows and columns in these matrices: if there are N different rows, then Alice must tell Bob which of these N possibilities occurs on her side: otherwise, there will be some case on Bob's side for which he will not be able to compute the answer. In general, if we denote by $d(M)$ the maximum between the number of different rows and the number of different columns in M, we have

$$\phi(f, n) = \lceil \log_2 d^*(f, n) \rceil \quad \text{where} \quad d^*(f, n) = \max_{c \in S} d(M_f^{c,n})$$

We talk of rules with *linear* (or constant, quadratic, exponential, etc.) communication complexity to refer to the way in which the number of rows/columns grows (hence, to the growth of d^*, rather than ϕ). The graphical representation of the matrices, for a certain n and c, gives some intuition of the kind of behavior of the different CAs (see Figure 1).

3 PREVIOUS RESULTS

In [9] we proved the following theorem, which allows to transfer results on communication complexity for the prediction problem from any 1D CA to a NCCA:

Theorem 1. *Let A be a (one-dimensional) CA. Then there exists a (one-dimensional) NCCA $\widetilde{A}$ such that* $|\phi(\widetilde{A}, n) - \phi(A, n)| \leq 2$.

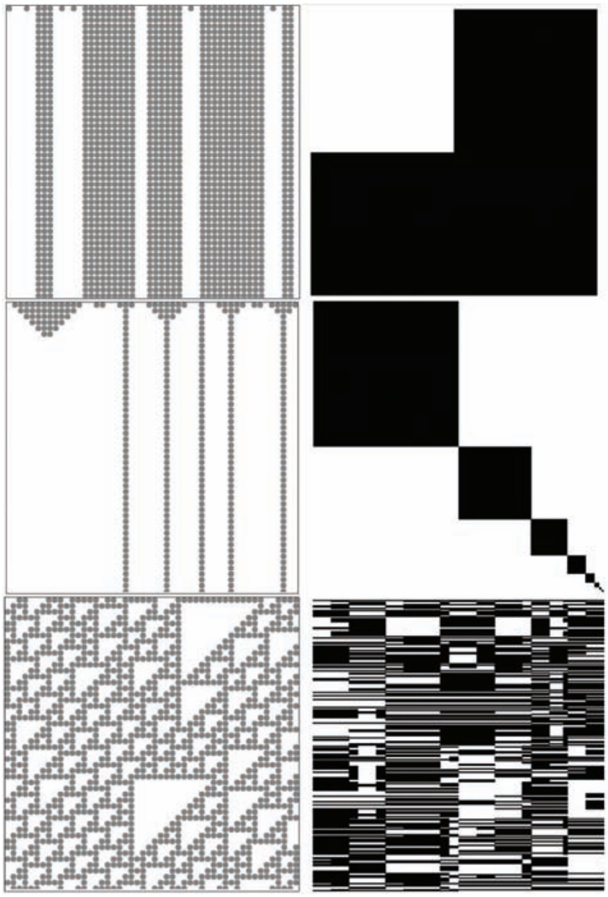

FIGURE 1
Sample iteration (left) and matrices $M^{1,8}$ (right) for three different elementary rules: from top to bottom, elementary rules 200, 132 and 110, exhibiting constant, linear and (conjectured) exponential growth of communication complexity, respectively. White/black stands for 0/1.

Combined with examples constructed in [3], this theorem implies the existence of NCCA for which the communication complexity grows as any polynomial, as well as exponentially.

One limitation of the construction used in the proof of Theorem 1 is that the NCCA is defined with a neighbourhood and a state set which are twice as large as those of the original CA (in fact, $|S'| = 2|S| + 1$). Hence, we cannot deduce from it the existence of any particular complexity for conservative CAs with $|S| < 5$. This limitation also applies to the related construction in [11] which shows the existence of intrinsically universal NCCAs: for $|S|$=2, 3 or 4, their existence is still open.

In [9] we solved the computational complexity of the prediction problem for all elementary NCCAs and all elementary DCAs (*decreasing* cellular automata, where the sum of the states is allowed to decrease between iterations); this was done by giving appropriate protocols (that provide upper bounds) and families of pairwise different rows/columns of the M_f matrices (which give the lower bound). All together (considering also the increasing CAs that are complementary to the DCAs) the rules solved include about 1/3 of all ECAs. Most of them were DCAs: the only elementary NCCA are the identity, the shifts, and rule 184 (along with its reflexion, 226).

All but 5 DCAs turned out to have constant ϕ; the remaining 5 are linear, as is rule 184. The protocols and lower bound configurations for the different linear rules are in general similar, and take advantage of the particle representations for the rules; in some case the same protocol or the same lower bound configurations can be used for two or more rules.

4 RULE 184 AND ITS GENERALIZATIONS

The most sophisticated protocol required for elementary NCCAs or DCAs is the one for the linear NCCA 184; this is no surprise given the intricate form of its matrix (see Figure 2, bottom right). That intricacy follows from the nature of the information given by the protocol: it requires a 2-dimensional analysis, but comes down to a zone of size linear in n. To see why, we first notice that

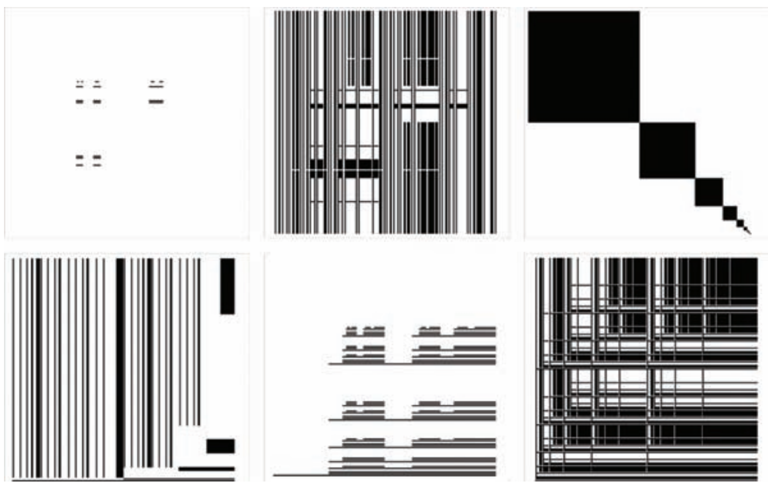

FIGURE 2
$M_f^{1,8}$ matrices with linear d^* among elementary NCCAs and DCAs: Top: rules 40, 56 and 132. Down: rules 152, 168 and 184.

the motion representation of this rule is $M_1 = \{\overset{\curvearrowright}{10}\}$: particles only look at the cell on their right, and move to it if it is empty.

Consider now initial configuration with center 0 and the message that Bob needs to send to Alice. It is easy to see that it suffices if Bob tells Alice the trajectory of his leftmost particle, say β (other particles on Bob's side will never be seen by Alice's particles). Moreover, only the part of this trajectory which happens within the area shaded in Figure 3 is relevant: above this zone, Alice's particles cannot see β; below this zone, whatever β does can no longer affect the result of the computation.

However, the information needed is even less. If two different paths of β differ in only one time step (at which β moves forward in one case, and stays in the other), then it can be checked that this perturbation can only propagate to the cells located one place to the left in the next time step. Thus, as long as the difference is not in the last position of β within the shaded area, it will have no effect. Moreover, any two trajectories with the same last position can be converted into each other through intermediate steps that differ at only one time-step. It follows that the only necessary information is the last position of β within the shaded area, and this must be either on the diagonal that marks its bottom border, or on the diagonal directly above it; the number of cases is hence n (including the case where β is not seen at all).

The case with an initial 1 at the center is similar; this 1 at the center will be then the leftmost particle, β, and the analysis follows as before. For Alice's protocol we use a nice property of rule 184: if we make the replacement $0 \leftrightarrow 1$, we obtain rule 226 which is the reflexion of 184; in other words, it can be alternatively view as 0's moving to the left against a background of 1's, instead of 1's moving to the right on a background of 0's. The same protocol as before can then be applied, and Alice will give the last "position" of its rightmost 0 within an area symmetric to the one shaded before.

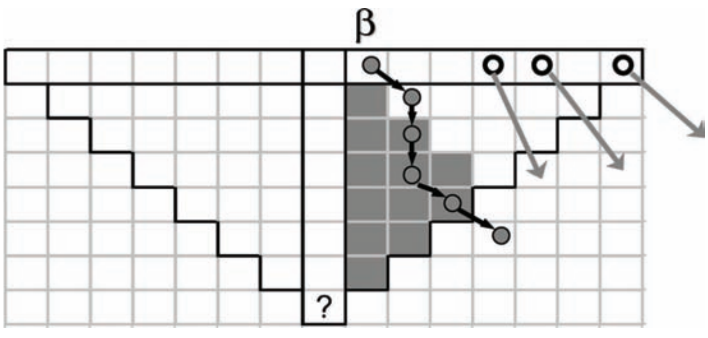

FIGURE 3
β is the leftmost particle within Bob's initial configuration, and he knows its path. The question mark identifies the cell whose state is being predicted.

Generalizations

Rule 184 is usually considered the simplest microscopic model for highway traffic, with cars moving ahead as long as there is space available. In turn, the simplest generalization corresponds to deterministic Fukui-Ishibashi traffic CA [10, 5], where the maximum speed is increased. This defines a family of rules M_k where a particle will move ahead as many positions as it can, with a maximum of k. Thus M_1 is as before (and its CA is 184), while

$$M_2 = \{ \overset{\frown}{100} , \overset{\frown}{10}1 \} \quad , \quad M_3 = \{ \overset{\frown}{1000} , \overset{\frown}{100}1 , \overset{\frown}{10}1 \}$$

The NCCA corresponding to rule M_2 has radius 2 to the left and 1 to the right; its Wolfram code is 43944. Similarly, M_3 corresponds to a CA with radius 3 to the left, 1 to the right, and code 2863377064.

The protocol described for rule 184 can be readily generalized to these rules (and for the general M_k): the number of possible final positions increases as $k \times n$, and is thus always linear in n. Where the generalization fails is in the case of Alice talking to Bob: the symmetry that reduced that case to its reflexion for rule 184 does no longer apply, since a $0 \leftrightarrow 1$ replacement of states fails to produce a mirror rule.

Figure 4, top, shows the growth numerically observed in the communication complexity of M_2 and M_3; in the detailed data (not shown) it can be seen that some symmetry in the protocol remains (the number of columns for $c = 0$ equals the number of rows for $c = 1$ and viceversa). The growth appears to be linear.

Another way (though less standard) to generalize rule 184 is by allowing more particles in a cell. Consider for instance the rule T with three states and radius 1 with code 6171534259461; its particle representation is

$$M_T \;=\; \{\overset{2}{\overset{\frown}{20}}, \overset{1}{\overset{\frown}{21}}, \overset{\frown}{10}, \overset{\frown}{11}\}$$

A sample iteration is shown at the bottom of Figure 4; the growth of ϕ (or rather, of the number of different rows/cells) is seen in the graph at the top, and is exactly linear. In fact, a protocol very similar to that of rule 184 can be written for this case; moreover, the left/right symmetry is recovered (with the transformation $0 \to 2, 1 \to 1, 2 \to 0$), allowing to solve both directions at once. For a sample iteration and a matrix of M_T see the bottom of Figure 6, rule 28.

5 NUMERICAL EXPLORATION OF NCCAS BEYOND ECAS

We just saw how generalizations of rule 184, to larger neighbourhoods and to additional states, yield rules with the same linear order of communication complexity. We searched for more complex behavior by performing

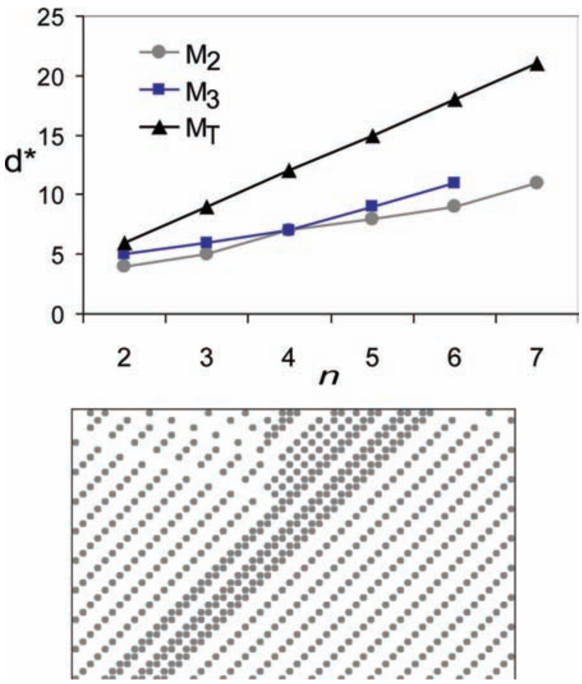

FIGURE 4
Top: growth of d^* for generalized versions of rule 184, increasing either maximum speed or the number of particles per cell. Center: sample iterations of M_3 (left) and of T, the generalization to 3 states (right). Big bullets denote two particles, small ones denote one, empty is zero; time goes top-down. Bottom: $M_T^{0,6}$; here grey represents 1, black is 2, white is 0.

an exhaustive computation, for small n, on all NCCAs with two states and radius 2, or with three states and radius 1[1].

NCCA with radius 1, 3 states 144 rules fall in this category, which in particular includes the rule T seen above. Due to rule equivalences (through left/right symmetry and $0 \leftrightarrow 1$ substitution) only 48 rules need to be considered. We computed d^* for $2 \leq n \leq 6$; the different curves of d^* are shown in Figure 5, and no evident classes appear. However, a more detailed examination identifies different orders of growth, which roughly correlate with $d^*(6)$. We computed the first, second and third differences of d^* for each rule, and selected several rules for further study: one linear, one quadratic and two possibly exponential cases. Tables 1 and 2 give

[1] The lists of rules can be retrieved at http://www.dim.uchile.cl/ãnmoreir/ncca/q3n3.txt and http://www.dim.uchile.cl/ãnmoreir/ncca/q2n5.txt, respectively.

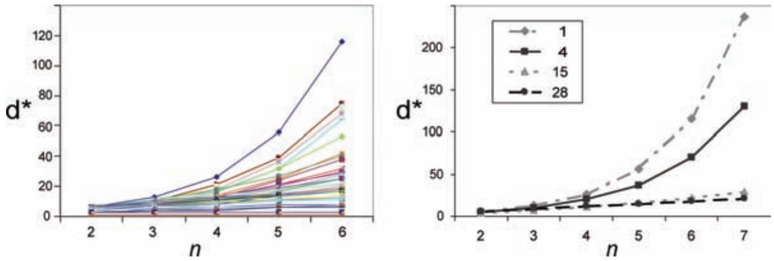

FIGURE 5
Left: d^* for different n, for all NCCA with 3 states and radius 1 (each line corresponds to a different rule). Right: d^* for selected rules.

the motion rules and conjectured growth for the whole set of rules; a version with including all the Wolfram codes and the detailed CA rule can be found at http://www.dim.uchile.cl/ãnmoreir/ncca/jac_q3r1.pdf. The selected rules are listed under the numbers 1, 4, 15 and 28 and have Wolfram codes 7163388257853, 6761211796341, 6893341954965 and 6171534259461 respectively (rule 28 corresponds to rule T from the previous section). Figure 6 shows sample behaviour and communication matrices for these rules.

Notice that rules 15 and 28 appear to have similar growth in Figure 5; however, this is a consequence of the scale. Moreover, rule 15 has clearly linear first differences in its values, which instead are constant for rule 28. The conjectured exponential behaviour for rules 1 and 4 is hard to test numerically, due to the rapid growth of the matrices; however, the fit is better than either quadratic or cubic polynomials. All the rules seem to have interesting behaviours; even the linear case, rule 28, exhibits non-trivial dynamics, usually converging for density 0.5 to the configuration 1^*, but sometimes going to $(20)^*$ instead. Rule 1 seems to have propagating signals of different speeds, and rule 4 has what appears to be an additive dynamics going on between fixed walls.

NCCA with radius 2, 2 states This class has 428 rules; in particular, it includes M_2, the generalization of rule 184 discussed above. As before, the list can be reduced due to equivalences, leaving only 129 different rules to be inspected. Again, we computed d^* for $2 \leq n \leq 5$, and then ranked the rules according to the maximum values obtained for $n = 5$. As before, the rules do not appear to fit into clear categories; moreover, the limited range of computationally feasible values of n made an estimation of the growth unreliable. We also computed the first and second differences, but clear examples of different classes of growth are scarce. Judging by simple inspection and educated guesses, about 9 rules have constant ϕ, about 32 are linear, and about 50 are somewhere between linear and quadratic. The rest are somewhere between

	Motion representation	Conj.Gr.
1	01 02 12 21 22	exponential
2	02 0 021 022 11 120 220	exponential
3	01 02 0 021 022 11 120 220	exponential
4	10 020 02 1 022 121 221	exponential
5	11 12 02• 12• 22•	exponential
6	10 11 020 02 1 022 121 221	exponential
7	10 20 21	
8	02 11	
9	10 11 20	
10	10 11 02	quadratic
11	10 11 20	quadratic
12	01 •2 21	quadratic
13	10 20	quadratic
14	10 12	quadratic
15	11 20	quadratic
16	01 •2 0 •21 •22 21	
17	10 21	
18	10 02• 12• 22•	
19	10 020 12 0 121 122 220	
20	10 20 21 22	
21	10 02 0 02 1 022 120 121 220 221	
22	11 12 20 21 22	
23	10 02 0 021 022 12 0 121 122 220	
24	02 0 02 1 022 120 121 220 221	

TABLE 1
Motion rules and conjectured growth for NCCA with 3 states and radius 1, first part.

quadratic and exponential, with some cases strongly suggesting exponential growth.

A list with all the rules considered, along with their Wolfram codes and motion representations, can be found at http://www.dim.uchile.cl/ãnmoreir/ncca/jac_q2r2.pdf. As a sample, we picked two rules among those with a

	Motion representation	Conj.Gr.
25	02 12 22	
26	10 2•	
27	10 20 21	
28	10 11 20 21	linear
29	02• 12• 22•	linear
30	11 02	linear
31	•1 02 12 22	linear
32	01 2•	linear
33	01 02 11 12	linear
34	10 11 21	linear
35	11 12	linear
36	01 20	linear
37	•2 21	linear
38	20 11	linear
39	02 12	linear
40	10	linear
41	02	linear
42	020 021 022 120 220	linear
43	11	linear
44	01 020 021 022 11 120 121 122 220	linear
45	1• 2•	constant
46	•2	constant
47	20	constant
48		constant

TABLE 2
Motion rules and conjectured growth for NCCA with 3 states and radius 1, cont.

clearer growth rate, which we conjectured to be exponential in one case (V_1, Wolfram code 3136457410) and quadratic in the other (V_2, code 3102247912). Their particle representations are

$$V_1 = \{ \overset{\frown}{001}\,,\ 0\overset{\frown}{11}\,,\ 1\overset{\frown}{\bullet 1}1 \} \quad , \quad V_2 = \{ 01\overset{\frown}{0}0\,,\ \overset{\frown}{\bullet 1}1\,,\ 01\overset{\frown}{1}0 \}$$

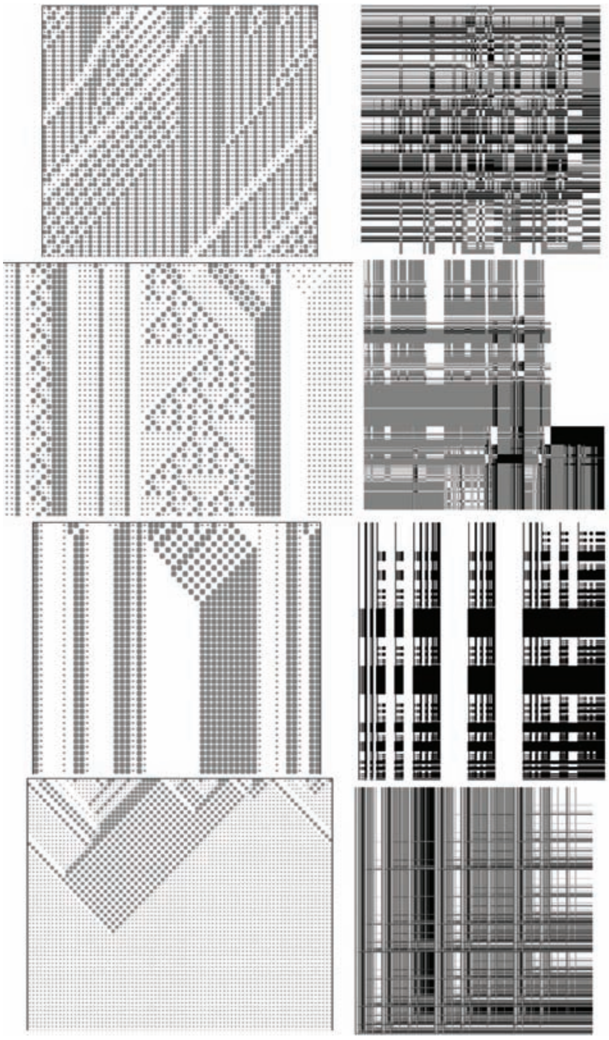

FIGURE 6
Sample behaviour (left) and $M^{2,6}$ (right) for the selected rules. From top to bottom: rules 1, 4, 15, 28.

and some sample behaviour can be seen in Figure 7; their matrices with $n = 5$ are shown in Figure 8.

6 CONCLUSIONS

The numerical exploration described here complements the previous work done on communication complexity of number-conserving and monotone

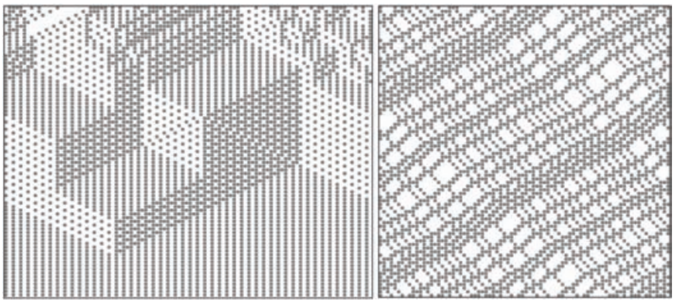

FIGURE 7
Sample behaviour of V_1 (left) and V_2 (right).

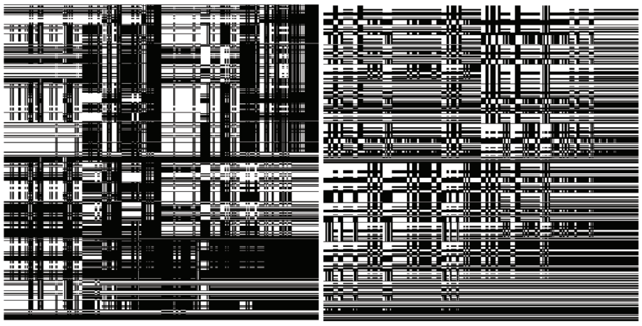

FIGURE 8
$M^{2,5}$ for V_1 (left) and V_2 (right).

ECAs. While protocols and lower bounds are usually hard to find, an experimental approach can give some insights about the presence of complexity in larger classes of rules. Since ECAs include only one non-trivial NCCA (rule 184), with linear communication complexity, and since the theorem that guarantees the existence of exponential NCCAs requires rather large neighbourhoods and set states, the question was: how far do we have to go from ECAs, in order to get high communication complexity in NCCA?

We addressed this question in two ways (both numerical). First, by considering the generalizations of rule 184; they turned out to maintain the linear behaviour of that rule, even when its symmetries seem to break. Second, by moving beyond ECAs to the two closest classes: NCCAs with radius 2 (but 2 states) and 3 states (but radius 1). Exhaustive evaluation in these classes showed that high communication complexity appears readily in both cases, giving a picture of high heterogeneity and rich behaviour which starts as soon as non-elementary NCCAs are considered. As an aside, we think that this study supports communication complexity as a useful tool for combing through CA classes in search of rules with complex behaviours.

ACKNOWLEDGMENTS

This work was partially supported by Fondecyt project 1080592 (EG and AM), Conicyt Anillo ACT-88 (EG and AM), as well as DGIP-UTFSM grant 241016 (AM).

REFERENCES

[1] N. Boccara and H. Fukś. (1998). Cellular automaton rules conserving the number of active sites. *J. Phys. A: Math. Gen.*, 31:6007–6018.

[2] N. Boccara and H. Fukś. (2001). Number-conserving cellular automaton rules. *Fund. Inform.*, 52:1–13.

[3] C. Durr, I. Rapaport, and G. Theyssier. (2004). Cellular automata and communication complexity. *Theoretical Computer Science*, 322:355–368.

[4] E.Goles, P.-E. Meunier, I. Rapaport, and G. Theyssier. (2009). Communications in cellular automata. In T. Neary, D. Woods, T. Seda, and N. Murphy, editors, *International Workshop on the Complexity of Simple Programs (CSP08)*, volume 1 of *Electronic Proceedings of Theoretical Computer Science*, pages 81–92, Cork, Ireland. Elsevier.

[5] M. Fukui and Y. Ishibashi. (1996). Traffic flow in 1d cellular automaton model including cars moving with high speed. *Journal of the Physical Society of Japan*, 65:1868–1870.

[6] E. Goles, P. Guillon, and I. Rapaport. (2011). Traced communication complexity of cellular automata. *Theoretical Computer Science*, 412:3906–3916.

[7] E. Goles, C. Little, and I. Rapaport. (2008). Understanding a non-trivial cellular automaton by finding its simplest underlying communication protocol. In S.-H. Hong, H. Nagamochi, and T. Fukunaga, editors, *Proceedings of the 19th International Symposium on Algorithms and Complexity (ISAAC 2008)*, volume 5369 of *Lecture Notes in Computer Science*, pages 593–605, Gold Coast, Australia. Springer.

[8] E. Goles, P.-E. Meunier, I. Rapaport, and G. Theyssier. (2011). Communication complexity and intrinsic universality in cellular automata. *Theoretical Computer Science*, 412:2–21.

[9] E. Goles, A. Moreira, and I. Rapaport. (2011). Communication complexity in number-conserving and monotone cellular automata. *Theor. Comput. Sci.*, 412:3616–3628.

[10] Sven Maerivoet and Bart De Moor. (2005). Cellular automata models of road traffic. *Physics Reports*, 419:1 – 64.

[11] A. Moreira. (2003). Universality and decidability of number-conserving cellular automata. *Theor. Comput. Sci.*, 292:711–721.

[12] M. Pivato. (2002). Conservation laws in cellular automata. *Nonlinearity*, 15:1781–1794.