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An integer linear programming approach for bilinear integer programming

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ABSTRACT

We introduce a new Integer Linear Programming (ILP) approach for solving Integer Programming (IP) problems with bilinear objectives and linear constraints. The approach relies on a series of ILP approximations of the bilinear IP. We compare this approach with standard linearization techniques on random instances and a set of real-world product bundling problems.

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1. Introduction

We study the bilinear optimization problem

$$\max \left(\sum_i \alpha_i x_i\right)\left(\sum_j \beta_j y_j\right) \tag{1a}$$

$$\sum_{i=1}^{N} a_{k,i} x_i + \sum_{j=1}^{M} b_{k,j} y_j \le d_k \quad \forall k = 1, \ldots, K \tag{1b}$$

$$x \in \mathbb{N}^N, \quad y \in \mathbb{N}^M \tag{1c}$$

where coefficients α_i and β_j are nonnegative integers (through scaling we can also consider rational coefficients).

This Integer Linear Programming (ILP) problem with bilinear objective and linear constraints is a special case of non-convex quadratic IP problems and more generally of non-convex non-linear IP problems, both of which have received significant attention recently [2,5,9,10]. If all variables are bounded this bilinear IP can be transformed to a Integer Linear Programming (ILP) Problem using well known linearization techniques [1,3,4,7,8,11]. One such technique replaces x_i by its binary decomposition $\sum_{r=0}^{\lceil \log k_i \rceil} 2^r w_{i,r}$, where $w_{i,r}$ is a binary variable and k_i is an upper bound of x_i. It then adds auxiliary integer variables $z_{i,j,r}$ such that $z_{i,j,r} = w_{i,r} \cdot y_j$ by enforcing a standard linearization of this requirement to obtain the ILP formulation of (1) given by

$$\max \sum_{i=1}^{N} \sum_{j=1}^{M} \sum_{r=0}^{\lceil \log k_i \rceil} \alpha_i \beta_j 2^r z_{i,j,r} \tag{2a}$$

$$\sum_{i=1}^{N} \sum_{r=0}^{\lceil \log k_i \rceil} 2^r a_{k,i} w_{i,r} + \sum_{j=1}^{M} b_{k,j} y_j \le d_k \quad \forall k = 1, \ldots, K \tag{2b}$$

$$z_{i,j,r} \le y_j \quad \forall j = 1, \ldots, M,\ i = 1, \ldots, N,\ r = 1, \ldots, \lceil \log k_i \rceil \tag{2c}$$

$$y_j \le z_{i,j,r} + (1 - w_{i,r}) \cdot k_j \quad \forall j = 1, \ldots, M,\ i = 1, \ldots, N,\ r = 1, \ldots, \lceil \log k_i \rceil \tag{2d}$$

$$0 \le z_{i,j,r} \le w_{i,r} \cdot k_j \quad \forall j = 1, \ldots, M,\ i = 1, \ldots, N,\ r = 1, \ldots, \lceil \log k_i \rceil \tag{2e}$$

$$w_{i,r} \in \{0, 1\} \quad \forall i = 1, \ldots, N,\ r = 1, \ldots, \lceil \log k_i \rceil \tag{2f}$$

$$y \in \mathbb{N}^M \tag{2g}$$

where k_j is an upper bound on y_j. We can obtain an alternative formulation by also replacing y_j by its binary decomposition, but we would obtain a formulation that is even larger than the $\Theta(\log(k_i)NM)$ variables and constraints of formulation (2). Because the size of formulation (2) can be prohibitively large, we propose a new linearization technique that leads to the solution of a series of ILP problems of the same size as the original problem.

Our proposed approach is detailed in Section 2. In Section 3 we present two families of problems that we use to test our methodology. One of them comes from a product bundling problem of a major food company. Finally, in Section 4 we show computational results of these models to compare our approaches with standard linearization techniques.

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2. An ILP solution approach

In this section we present an ILP solution approach that is based on a linearization of the objective function. This approach is based on the following simple lemma.

Lemma 1. *Let a_1, a_2, b_1, b_2 be positive numbers. If $a_1+a_2 \geq b_1+b_2$ and $a_1 \cdot a_2 < b_1 \cdot b_2$ then* $\min\{a_1, a_2\} < \min\{b_1, b_2\}$.

Proof. Assume that $a_1 \leq a_2$ and $b_1 \leq b_2$. Let $K_a = a_1 + a_2$, $K_b = b_1 + b_2$ and $f_K(x) = Kx - x^2$. Note that if $K_a \geq K_b$ then $f_{K_a}(x) \geq f_{K_b}(x) \forall x \in [0, K_b/2]$. Since $a_1 a_2 = f_{K_a}(a_1) < f_{K_b}(b_1) = b_1 b_2$ and due to the fact that $f_{K_a}(x)$ is strictly increasing in $[0, K_b/2]$ then $a_1 < b_1$. □

This lemma suggests the following idea to solve the problem: Instead of maximizing $(\sum_i \alpha_i x_i)(\sum_j \beta_j y_j)$ we can maximize $(\sum_i \alpha_i x_i) + (\sum_j \beta_j y_j)$. An optimal solution for this new objective function may be sub-optimal for the original objective function, but if this is the case, we can use the lemma to obtain strict lower bounds for terms $\sum_i \alpha_i x_i$ and $\sum_j \beta_j y_j$ in an optimal solution. We can then use this idea to obtain an optimal solution to our original problem by solving a sequence of ILP models with a linear objective as follows.

For a given lower bound LB for terms $\sum_i \alpha_i x_i$ and $\sum_j \beta_j y_j$ we construct the ILP model with a linear objective given by

$$\max \sum_i \alpha_i x_i + \sum_j \beta_j y_j \tag{3a}$$

$$\sum_i \alpha_i x_i \geq LB \tag{3b}$$

$$\sum_j \beta_j y_j \geq LB \tag{3c}$$

$$\sum_{i=1}^{N} a_{k,i} x_i + \sum_{j=1}^{M} b_{k,j} y_j \leq d_k \quad \forall k = 1, \ldots, K \tag{3d}$$

$$x_i, y_j \in \mathbb{N}. \tag{3e}$$

We refer to this model as (LIN). In order to obtain the solution of our original problem, we start solving this ILP with $LB = 1$. Next, we update this value by letting $LB = \min\{\sum_i \alpha_i x_i^*, \sum_j \beta_j y_j^*\} + 1$, where (x^*, y^*) is an optimal solution of the previous run, and we re-solve the ILP with the new parameter. We repeat this until the ILP becomes infeasible. The best solution found (in terms of the original objective function $(\sum_i \alpha_i x_i)(\sum_j \beta_j y_j)$) is an optimal solution of our original problem.

It is important to note that we assume that optimal solution of the bilinear problem does not have objective value equal to zero. In this case, the original bilinear problem is infeasible or it has an unbounded solution if and only if (LIN) is infeasible or unbounded, respectively. Thus, we can safely use $LB = 1$. If the optimal value of the bilinear problem is equal to zero, then (LIN) will be infeasible. Hence, a final step is required to check the existence of a solution with either $x_i = 0$ for all i such that $\alpha_i > 0$, or $y_j = 0$ for all j such that $\beta_j > 0$.

Note that in each iteration, the original objective function might not improve. Hence, one way to enhance this approach is to modify the ILP model to ensure that the new solution will always increase the original objective function. This is achieved as follows.

Let BEST and T be two parameters. The following formulation allows us to find a solution such that either (i) each term $\sum_i \alpha_i x_i$ and $\sum_j \beta_j y_j$ are lower bounded by $LB + T$ or (ii) each term $\sum_i \alpha_i x_i$ and $\sum_j \beta_j y_j$ is lower bounded by $LB + t$ for some $t < T$ and $(\sum_i \alpha_i x_i)(\sum_j \beta_j y_j) >$ BEST. If T is large enough, the original objective function increases at each step:

$$\max \sum_i \alpha_i x_i + \sum_j \beta_j y_j \tag{4a}$$

$$\sum_i \alpha_i x_i \geq \sum_{t=0}^{T} s_t (LB + t) \tag{4b}$$

$$\sum_j \beta_j y_j \geq \sum_{t=0}^{T} s_t (LB + t) \tag{4c}$$

$$\sum_i \alpha_i x_i + \sum_j \beta_j y_j \geq \sum_{t=0}^{T-1} s_t \left((LB + t) + \left\lceil \frac{\text{BEST} + 1}{LB + t} \right\rceil \right) \tag{4d}$$

$$\sum_{t=0}^{T} s_t = 1 \tag{4e}$$

$$\sum_{i=1}^{N} a_{k,i} x_i + \sum_{j=1}^{M} b_{k,j} y_j \leq d_k \quad \forall k = 1, \ldots, K \tag{4f}$$

$$x_i, y_j \in \mathbb{N} \tag{4g}$$

$$s_t \in \{0, 1\}. \tag{4h}$$

The aim of introducing the s variables is to find an incumbent solution or to increase LB in T units. Constraints (4b) and (4c) ensure that if $s_t = 1$ for some t then terms $\sum_i \alpha_i x_i$ and $\sum_j \beta_j y_j$ are greater than or equal to $LB + t$. Finally, constraint (4d) assures that the original objective function will increase for $t < T$. This is because if terms $\sum_i \alpha_i x_i$ and $\sum_j \beta_j y_j$ are greater than or equal to $LB + t$, and the sum of both terms is greater than or equal to $(LB+t)+\lceil \frac{\text{BEST}+1}{LB+t} \rceil$, then by Lemma 1 the product $(\sum_i \alpha_i x_i)(\sum_j \beta_j y_j)$ should be greater than or equal to $(LB + t) \cdot \lceil \frac{\text{BEST}+1}{LB+t} \rceil >$ BEST. If no increment in the objective function is obtained for $t < T$, then for all feasible solutions of this problem we have that $s_T = 1$, so the terms $\sum_i \alpha_i x_i$ and $\sum_j \beta_j y_j$ are lower bounded by $LB + T$. We refer to this model as (LIN+).

As before, in order to obtain the solution of our original problem, we start solving this formulation with BEST $= 0$ and $LB = 1$. Next, we update these values with BEST $= (\sum_i \alpha_i x_i)(\sum_j \beta_j y_j)$ and $LB = \min\{\sum_i \alpha_i x_i, \sum_j \beta_j y_j\} + 1$, if a better solution has been found, or we update $LB = LB + T$, if not. Then, we re-solve the formulation with the new parameters until no feasible solution exists.

3. Testing instances

To compare our ILP approaches with the linearization (2), we test them over two different problem: a nonlinear bipartite matching problem, and a real product bundling problem.

Bipartite graph problem

The first set of problems has been created to compare these different approaches. Given a complete bipartite graph $G = (L \cup R, L \times R)$, for each vertex v we set a price p_v and a weight w_v, and for each edge (u, v) we set a maximum weight M_{uv}. The problem is to select quantities $x_u, u \in L$ and $y_v, v \in R$ such that the objective function $(\sum_{u \in L} p_u x_u)(\sum_{v \in R} p_v y_v)$ is maximized, subject to $w_u x_u + w_v y_v \leq M_{uv}$ for each edge $uv \in L \times R$.

Product bundling problem

Our second set of problems comes from a major food company that is evaluating the possibility of delivering its products directly to small grocers, avoiding its current wholesalers and distributors. Because of distribution and storage logistics, the producer would like to select product bundles [6,12] that maximize the total sale

of products through bundles, subject to constraints on client's demands.

Because bundles are more convenient for the producer, we can expect the price of products obtained through the bundle to be equal or smaller than their individual prices. Hence, we can assume that clients will cover their demands by first buying as many units of the bundle as possible without exceeding their demand for an individual product and cover the remaining demand with individual purchases. Under this assumption, a mathematical programming model for obtaining the bundle design that maximizes the number of bundles sold can be constructed as follows.

Let P be a set of products. A bundle of products is an integer vector $x = (x_p)_{p\in P} \in \mathbb{N}^{|P|}$ where x_p indicates the number of units of product p in the bundle. Let C be a set of clients, $D_{c,p}$ be the demand of client $c \in C$ for product $p \in P$ and y_c be the number of bundles bought by client $c \in C$. A mathematical programming model for an optimal bundle design is given by

$$\max \sum_{c\in C} \sum_{p\in P} x_p \cdot y_c \tag{5a}$$

$$x_p \cdot y_c \le D_{c,p} \quad \forall c \in C, p \in P \tag{5b}$$

$$x_p \in \mathbb{N} \quad \forall p \in P \tag{5c}$$

$$y_c \in \mathbb{N} \quad \forall c \in C. \tag{5d}$$

Note that Eq. (5b) is not linear and hence our ILP approaches cannot be applied directly. However, using standard linearization techniques (e.g. [4]) it is possible to linearize such constraints in a compact way that is not available for the objective function:

$$y_c \le \left\lfloor \frac{D_{cp}}{i} \right\rfloor + \left(k_p - \left\lfloor \frac{D_{cp}}{i} \right\rfloor\right) \left(|b(i)| - \sum_{j:b(i)_j=1} w_{p,j}\right) \quad \forall c \in C,\ \forall p \in P,\ i = 1, \ldots, k_p \tag{6a}$$

$$x_p = \sum_{i=0}^{\lceil \log k_p \rceil} 2^i w_{p,i} \quad \forall p \in P \tag{6b}$$

$$w_{p,i} \in \{0, 1\} \quad \forall p \in P,\ i = 1, \ldots, \lceil \log k_p \rceil \tag{6c}$$

where $b(i)$ is a vector containing the binary representation of the integer i (i.e. $i = \sum_j 2^{b(i)_j}$) and $|b(i)|$ is the number of non-zero bits in $b(i)$. In these constraints, note that $x_p = i$ if and only if the term $(|b(i)| - \sum_{j:b(i)_j=1} w_{p,j})$ is equal to 0. Hence, these constraints assure that if $x_p = i$ then $y_c \le \lfloor \frac{D_{cp}}{i} \rfloor$, so $x_p \cdot y_c \le D_{cp}$. Note that this technique cannot be applied to the original objective function or to constraints of the form $y_c \cdot x_p \le z$ where z is a variable. However, because formulation (2) has already linearized terms $x_p \cdot y_p$ it can enforce constraints (5b) by adding linear constraints of the form $\sum_{r=0}^{\lceil \log k_p \rceil} 2^r z_{p,q,r} \le D_{c,p}$ and hence does not require adding (6).

4. Computational results

We implement these formulations using IBM ILOG CPLEX 12.2 and we test them in a Intel Xeon server with 8 Gb of RAM. In both problems, we compare our approach to the model obtained by replacing a set of variables by its binary decomposition, as explained in Section 1. We denote the models obtained by this decomposition by (BIN).

Bipartite graph problem

For the first problem, we use a complete bipartite graph of 50 vertices on each part, and we select random weights w_v and prices p_v following a Poisson probabilistic distribution of mean 4. The maximum weight on the arcs $M_{u,v}$ is also selected randomly following a Poisson distribution of mean λ, where λ is equal to 8, 16 and 32. Note that for higher values of λ, the space of feasible solutions is increased, obtaining a harder problem to solve. We compare solution times of ten random instances for each value of λ, and we report average times and number of iterations in Table 1. In the case of formulation (LIN+), a large value of parameter T leads us to fewer (but slower) iterations. In these experiments we use $T = 50$, which shows a good performance for these datasets.

Table 1
Time results for our first problem using randomly generated data with $|L| = |R| = 50$.

λ	BIN time	LIN time	(it.)	LIN+ time	(it.)
8	6.0	0.56	(4.3)	2.99	(2.3)
16	4584.9	15.1	(31.2)	11.1	(19.3)
32	OOM	1637.6	(90.6)	1269.2	(63.2)

Table 2
Time results for product bundling problem on randomly generated data with $|C| = 10$ and $|P| = 30$.

ρ	λ	BIN time	LIN time	(it.)	LIN+ time	(it.)
0.8	10	1	3	(9)	1	(2)
0.8	50	8	40	(19)	6	(2)
0.8	200	25	262	(35)	55	(4)
0.5	10	19	17	(11)	5	(3)
0.5	50	997	277	(28)	70	(4)
0.5	200	5511	27,298	(77)	10,874	(7)
0.2	10	287	71	(18)	20	(4)
0.2	50	6160	2156	(52)	2982	(9)
0.2	200	OOM	352,889	(108)	114,219	(12)

It can be seen that formulation (BIN) is significantly slower in each case. Furthermore, in the larger problems, formulations (BIN) requires a large number of variables and constraints, resulting in an *Out of Memory (OOM)* error in the solver. Additionally, we see that in some instance formulation (LIN) requires a large number of iterations to finish, while formulation (LIN+) requires fewer iterations, but each iteration is harder to solve.

Product bundling problem

For the product bundling problem, we test the performance of our approaches on two sets of instances. The first set of instances are randomly generated with $|C| = 10$ and $|P| = 30$. In order to generate these instances, we first fix $D_{c,p} = 0$ with uniform probability ρ. Secondly, among the values not fixed to zero, we fix $D_{c,p}$ according to a Poisson distribution of parameter λ. The results on these instances are shown in Table 2.

It can be seen that the problem becomes easier to solve for instances with a large number of zeros on the demand matrix. Also, with highly non-homogeneous matrices (e.g. $\lambda = 200$) problem becomes easier to solve for formulation (BIN), mainly because the LP relaxation of the problem is close to the optimal value. As before, formulation (BIN) is unable to solve large instances of this problem. In contrast, formulations (LIN) and (LIN+) could deal with problems of these sizes, but they require a long time to solve them.

Our second set of instances for this problem is constructed using real data from a major food company, and we select subsets of clients and products of different sizes to compare the performance of our three formulations. Results are shown in Table 3. Results are similar to that obtained with randomly generated data. However, for instance *w10x60* (BIN) is significantly faster than (LIN) and (LIN+). One possible reason for this is that linearization (6) hurts the effectiveness of our ILP approaches. Another reason could be that, because demand values $D_{c,p}$ are very large for this instance, the solution of this problem includes 1 client that buys 29,283 units. Both ILP approaches found this solution in the first iteration. However, most of the time is expended proving the optimality of this solution. Larger values for T provide fewer but longer iterations, obtaining a similar time. Nevertheless, instance *w10x60d* has the same demand matrix as *w10x60* but divided by 12 (which rep-

Table 3
Time results for product bundling problem for instances on real data.

Name	$\|C\| \times \|P\|$	Max $D_{c,p}$	BIN time	LIN time	(it.)	LIN+ time	(it.)
w5x41m	5×41	99	27	44	(31)	8	(5)
w5x41	5×41	433	92	127	(47)	33	(12)
w10x60	10×60	2402	2393	199,907	(129)	29,213	(9)
w10x60d	10×60	201	357	422	(35)	107	(3)
wP32x311	32×311	26	OOM	1175	(14)	193	(6)

resents packs of a dozen of products), and now (LIN+) is faster than (BIN). As before, (BIN) is unable to solve our largest instance.

5. Impact on a real case

These models were used to construct bundles of food products for a major food company in Chile. We use the data of weekly demand on 497 products of 760 small retailers with a total weekly demand of approximately 2 million SKUs. In addition to the original constraints, constraints were added on the size and final price of the bundle. To construct a bundle, a subset of 10–12 products was selected using hierarchical clustering with Euclidean distance, and solved using these formulations. After the selection of the first bundle, we update the demand and construct a new bundle using the same procedure. The ILP model took approximately 5–10 h per problem. Five bundles were constructed using this procedure, each one of them with more than 150 potential clients. In the end, 29.75% of the total demand could be potentially delivered using these five bundles.

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