Fundamenta Informaticae XXI (2001) 1001–1019
IOS Press

Computational Complexity of Avalanches in the Kadanoff Sandpile Model

Enrico Formenti[*]
Université Nice–Sophia Antipolis,
I3S, UMR 6070 CNRS,
2000 route des Lucioles, BP 121,
F-06903 Sophia Antipolis Cedex.

Eric Goles[†]
Universidad Adolfo Ibañez,
Av. Diagonal Las Torres 2640,
Peñalolen,
Santiago, Chile.

Bruno Martin[‡]
Université Nice–Sophia Antipolis,
I3S, UMR 6070 CNRS,
2000 route des Lucioles, BP 121,
F-06903 Sophia Antipolis Cedex.

Abstract. This paper investigates the *avalanche problem* $\mathbf{AP}$ for the Kadanoff sandpile model (KSPM). We prove that (a slight restriction of) $\mathbf{AP}$ is in $\mathbf{NC}^1$ in dimension one, leaving the general case open. Moreover, we prove that $\mathbf{AP}$ is $\mathbf{P}$-complete in dimension two. The proof of this latter result is based on a reduction from the monotone circuit value problem by building logic gates and wires which work with an initial sand distribution in KSPM. These results are also related to the known *prediction problem* for sandpiles which is in $\mathbf{NC}^1$ for one-dimensional sandpiles and $\mathbf{P}$-complete for dimension 3 or higher. The computational complexity of the prediction problem remains open for the Bak's model of two-dimensional sandpiles.

Keywords: Kadanoff Sandpile Model, Bak Sandpile Model, Sandpiles Models, Complexity, Discrete Dynamical Systems, Self-Organized Criticality (SOC).

[*]Work supported by The French National Research Agency project EMC (ANR-09-BLAN-0164).
[†]FONDECYT 1100003, BASAL-CMM-U. de Chile, Anillo ACT-88, CNRS.
[‡]Work supported by The French National Research Agency project EMC (ANR-09-BLAN-0164).

1. Introduction

Discrete dynamical systems (DDS) are often used for modeling real phenomena. Therefore, predicting their overall behavior is both useful and most wanted in applications. Unfortunately, this task is in general very difficult if not impossible. Taking finite phase spaces, of course, gives precise answers but full numerical simulation is the only way to obtain those answers.

In many cases, parallel processing methods can help to speed up numerical simulations of DDS at the price of increasing enough the number of processors. In this paper we are going to single out a family of DDS and a simulation problem about them such that the complexity of the problem is both sensitive with respect to instance data and dimension (the meaning of the word "dimension" will be made clearer later) of the DDS phase space.

In this paper we consider the well-known discrete dynamical system of sandpiles proposed in [1] (SPM). Roughly speaking, its dynamics is as follows. Consider the toppling of grains of sand on a (clean) flat surface, one by one. After a while, a sandpile has formed. At some point, the simple addition of even a single grain may cause avalanches of grains to fall down along the sides of the sandpile. Then, the growth process of the sandpile starts again. Remark that this process can be naturally extended to arbitrary dimensions although for $d > 3$, the physical meaning is not clear. Moreover, this process can also be seen as a special case of sand automata [2] or cellular automata [3].

The first complexity results about SPM appeared in [8, 9] where the authors proved the computation universality of SPM. For that, they modelled wires and logic gates with sandpiles configurations. C. Moore and M. Nilsson considered the *prediction problem* ($\mathbf{PRED}$) for SPM *i.e.* the problem of computing the stable configuration (fixed point) starting from a given initial configuration of the sandpile. Later, C. Moore and M. Nilsson proved that $\mathbf{PRED}$ is in $\mathbf{NC}^3$ for dimension $d = 1$ and, by using similar constructions as in [8, 9], they proved that it is $\mathbf{P}$-complete for $d \geq 3$, leaving $d = 2$ as an open problem [15][1]. Later, P. B. Miltersen improved the bound for $d = 1$ showing that $\mathbf{PRED}$ is in $\mathbf{LOGDCFL}(\subseteq \mathbf{AC}^1)$ and that it is not in $\mathbf{AC}^{1-\varepsilon}$ for any $\varepsilon > 0$ [14]. Therefore, in any case, one-dimensional sandpiles are capable of (very) elementary computations such as computing the maximum of n bits.

In this paper, we address a slightly different problem: the avalanche problem ($\mathbf{AP}$). The system is started with a monotone configuration of the sandpile. When a grain of sand is added to the initial pile, this eventually causes an avalanche (a sequence of topples) and we address the question of the complexity of deciding whether a certain given position –initially with no grain of sand– will receive some grains in the future. Like the $\mathbf{PRED}$ problem, $\mathbf{AP}$ can be formulated in higher dimensions but in order to get acquainted with $\mathbf{AP}$, we introduce its one-dimensional version first.

One-dimensional sandpiles can be conveniently represented by a finite sequence of *piles* and each x_i represents the number of grains contained in pile i. In the classical SPM, a grain falls from pile i to $i+1$ if and only if the height difference $x_i - x_{i+1} \geq 2$. Kadanoff sandpile model (KSPM) generalises SPM [13, 7] by adding a parameter p. The setting is the same except for the local rule: $p-1$ grain falls to the $p-1$ adjacent piles if the difference between pile i and $i+1$ is at least p.

Assume $x_k = 0$, for a value of k "far away" from the first pile. The avalanche problem asks whether adding a grain at pile 1 will cause an avalanche such that at some point in the future $x_k \geq 1$, that is to

[1]Recall that $\mathbf{P}$-completeness plays for parallel computation a role comparable to $\mathbf{NP}$-completeness for non-deterministic computation. It corresponds to problems which cannot be solved efficiently in parallel or, equivalently, which are *inherently sequential*, unless $\mathbf{P}$=$\mathbf{NC}$ (for more on this subject see [12], for instance).

say that an avalanche is triggered and reaches the "flat" surface at the bottom.

This problem can be generalized to two-dimensional sandpiles. In this case, it requires to be started on an initial distribution of sand and it is not triggered by the addition of one grain to the first pile. This statement is related to the question addressed by C. Moore and M. Nilsson: indeed, if one has a solution for $\mathbf{PRED}$ then a solution for $\mathbf{AP}$ can be produced in constant time. On the other hand, answers to $\mathbf{AP}$ are a kind of rough lower bound for solutions to $\mathbf{PRED}$.

In this paper we prove that in the one-dimensional case, $\mathbf{AP}$ is in the class $\mathbf{NC}^1$ for a suitable class of sandpiles and that in the two-dimensional case, $\mathbf{AP}$ is $\mathbf{P}$-complete. The proof is obtained by reduction from the Monotone Circuit Value Problem where the circuit only contains gates without negations — that is, AND's and OR's (see Section 3 for details). We stress that our proof for the two-dimensional case needs some further hypothesis/constraints for monotonicity and determinism (see Section 3). If both properties are technical requirements for the proof's sake, monotonicity also has a physical justification. Indeed, if KSPM is used for modelling real physical sandpiles, then the image of a monotone non-increasing configuration has to be monotone non-increasing since gravity is the only force considered here. We have chosen to design the Kadanoff dynamics for $d = 2$ by considering a certain definition of the three-dimensional sandpile that does not correspond to the one of Bak's *et al.* in [1]. This hypothesis is not restrictive. It is just used for constructing the transition rules. Bak's construction was done similarly. Nevertheless, our result (as well as the results related to Bak's model) depends on the way the three dimensional sandpile is modelled. In our case, we have decided to formalise the sandpile as a monotone decreasing pile in three dimensions where $x_{i,j} \geq \max\{x_{i+1,j}, x_{i,j+1}\}$ (here $x_{i,j}$ denotes the sand grains initial distribution) together with Kadanoff's avalanche dynamics ruled by parameter p. The pile (i, j) can give a grain either to every pile $(i+1, j), \ldots, (i+p-1, j)$ or to every pile $(i, j+1), \ldots, (i, j+p-1)$ if the monotonicity is not violated. With such a rule and if we use the height difference for defining the monotonicity, we can define the transition rules of the dynamics for every value of the parameter p. We also give a deterministic way for applying the rules.

The paper is organized as follows. Section 2 formally defines the Kadanoff sandpile model in dimension one and presents the avalanche problem as well as its computational complexity. Section 3 generalizes the model and the problem to dimension two. Section 4 concludes the paper and proposes further research directions. In the conclusion, we also provide the Kadanoff model related to Bak's physical sandpile interpretation. In this case, our proof is similar for $p > 2$ and for the case $p = 2$ we exactly regain Bak's two dimensional automata. In the latter case the complexity of $\mathbf{PRED}$ remains open.

2. Sandpiles and Kadanoff model in dimension one

A sandpile *configuration* is a distribution of sand grains over a lattice (here $\mathbb{Z}$). Each pile of the lattice is associated with an integer which represents its sand content. A finite configuration can be identified with an ordered sequence of integers ${}^{\omega}x_{-p+1}, \ldots, x_1, \ldots, x_n^{\omega}$ where x_{-p+1} (resp. x_n) is the first (resp. the last) pile such that all the piles on the left of x_{-p+1} equal x_{-p+1} (resp. all the piles on the right of x_n equal x_n). Given a configuration x, $a \in \mathbb{N}$ and $j \in \mathbb{Z}$, we use the notation ${}^{\omega}ax_j$ (resp. $x_j a^{\omega}$) to say that $\forall i \in \mathbb{Z}, i < j \Rightarrow x_i = a$ (resp. $\forall i \in \mathbb{Z}, i > j \Rightarrow x_i = a$).

Remark that any configuration

$$x \equiv {}^{\omega}x_1, x_2, \ldots, x_{n-1}, x_n^{\omega}$$

can be identified with its *heights differences* sequence

$$h(x) \equiv {}^{\omega}0, h_1 = (x_1 - x_2), \ldots, h_{n-1} = (x_{n-1} - x_n), 0^{\omega} \ ,$$

where the integer $n-1$ will be referred to as the *length* of the configuration and it is denoted $|x|$ in the sequel. In other words, we associate the initial (infinite) configuration with a finite sequence of integers $h_1, h_2, \ldots, h_{|x|-1}$. This latter representation is more convenient and is widely used in the sequel. The particular choice of the indexes will be clear later on. A configuration is *finite* if only a finite number of its height differences has non-zero sand content.

Notation. For any $a, b \in \mathbb{Z}$ with $a \leq b$, let $[\![a, b]\!] = [a, b] \cap \mathbb{N}$. Similarly, $[\![a, b) = [a, b) \cap \mathbb{N}$, *etc.* In the sequel $\mathrm{KSPM}(d, p)$ is a shorthand for the KSPM model in dimension d and parameter p.

A configuration x is *monotone* if the sequence of its pile heights is monotone (*i.e.*, expressed in terms of height differences), $\forall i \in [\![1, |x|)$, $h_i \geq 0$. A monotone configuration x is *stable* if the sequence of its heights differences is stable, *i.e.* $\forall i \in [\![1, |x|)$, $h_i < p$, *i.e.* if the difference between any two adjacent piles is less than Kadanoff's parameter p.

Let $\mathbf{SM}(n)$ denote the set of stable monotone configurations of the form

$${}^{\omega}x_1, x_2, \ldots, x_{n-1}, x_n^{\omega}$$

and of length n, for $x_i \in \mathbb{N}$ and such that $h_i > 0$ for $i \in [\![1, n]\!]$. Finally, let $\mathbf{SM} = \cup_{n \in \mathbb{N}} \mathbf{SM}(n)$.

Consider a stable monotone configuration x. Adding one more sand grain, say at pile i ($i \in [\![-p+1, |x|]\!]$), may cause the pile i to topple some grains to its adjacent piles. In their turn the adjacent piles receive a new grain and may also topple, and so on. This phenomenon is called an *avalanche*. The avalanche ends when no more piles can topple *i.e.* the system is at a fixed point. More formally, an avalanche is a sequence $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$ such that $x^{(n)}$ is a fixed point and $x^{(i+1)}$ is obtained from $x^{(i)}$ via the application of the Kadanoff's rule after adding one grain to the first pile of $x^{(1)}$ (according to the chosen update policy).

In this paper, topplings are controlled by the *Kadanoff's parameter* $p \in \mathbb{N}, p \geq 2$ which completely determines the model and its dynamics. In $\mathrm{KSPM}(1, p)$, $p-1$ grains will fall from pile i if $h_i = (x_i - x_{i+1}) \geq p$ and the new configuration becomes

$${}^{\omega}(x_1) \cdots (x_{i-1}), (x_i - p + 1), (x_{i+1} + 1), \cdots, (x_{i+p-2} + 1), (x_{i+p-1} + 1), (x_{i+p}), (x_{i+p+1}), \cdots, (x_n), (0)^{\omega}$$

(in the above formula, parenthesis are added to better illustrate the content of piles). In other words, the pile i distributes one grain to each of its $(p-1)$ right adjacent piles. Equivalently, if we measure the heights differences after applying the dynamics, we get

$$(h_{i-1} + p - 1), (h_i - p), (h_{i+1}), (h_{i+2}), \cdots, (h_{i+p-2}), (h_{i+p-1} + 1) \ ,$$

and all remaining heights do not change. In other words, the effect of toppling pile i is that of adding $(p-1)$ to h_{i-1} while substracting p to h_i and incrementing h_{i+p-1} by 1.

Consider the problem of deciding whether some pile on the right of pile x_n (more precisely for x_k with $n < k \leq n+p-1$) will receive some grains according to the Kadanoff's dynamics. Since the initial configuration is stable, it is not difficult to prove that avalanches will reach at most the pile $n + p - 1$.

Remark that given a configuration, several piles could topple at the same time. Therefore, at each time step, one might have to decide which pile or piles are allowed to topple. According to the update policy chosen, there might be different images of the same configuration. However, it is known from [11] that for any given initial number of sand grains n, the orbit graph is a lattice and hence, for our purpose, we may only consider one decision problem to formalize $\mathbf{AP}$:

Problem AP (dimension 1)

INSTANCE: $p \in \mathbb{N}$ $(p \geq 2)$, a configuration $x \in \mathbf{SM}$ to which a grain is added to pile 1 and $k \in (|x|, |x| + p - 1]\!]$.

QUESTION: Does there exist an avalanche such that $x_k \geq 1$?

Notation. Given a problem $\Pi \equiv (I, Q)$, I is the set of *instances* (*i.e.* the problem data) and Q is a predicate over I (*i.e.* a question about the instance). For $i \in I$, $\Pi(i)$ denotes the correspondent instantiation of Π. Moreover, denote $\Pi|_A$ the problem $\Pi' \equiv (A, Q)$ for $A \subseteq I$.

Let us consider some instances of $\mathbf{AP}$. Fix $p = 3$ and consider a stable configuration whose height differences are as follows ${}^{\omega}00\underline{2}2022120000^{\omega}$, $k \in (8, 10]\!]$. We add a single grain to pile 1 (underlined in the configuration). Then, the next step is ${}^{\omega}02\underline{0}21222120000^{\omega}$. In one step, one can see that no avalanches can be triggered, hence the answer to $\mathbf{AP}$ is negative. As a second example, consider the following sequence of height differences (always with $p = 3$): ${}^{\omega}0\underline{2}122122221201200^{\omega}$, $k \in (14, 16]\!]$. We add a single grain to pile 1 (underlined in the configuration). There are several possibilities for avalanches from the left to the right but none of them arrives to the rightmost 0's region. So the answer to the decision problem is still negative. To get an idea of what happens for a positive instance of the problem, consider the initial configuration: ${}^{\omega}0\underline{2}12222100^{\omega}$ with $p = 3$ and $k \in (8, 10]\!]$; a grain has still to be added to pile 1, underlined in the configuration.

Consider the height difference representation. A site i is said to *fire* if the KSPM rule is applied to it. Lemma 2.1 bounds the number of firings that a site can perform before arriving to the fixed point, when at most one grain is added to the first pile. This result was also proved independently in [16].

Lemma 2.1. Given a configuration $x \in \mathbf{SM}$, a site can fire at most once in $\mathrm{KSPM}(1, p)$ for $p > 1$, when at most one grain is added to the first column x_1.

Proof:
Consider a stable monotone initial configuration. By contradiction, assume that there exists a site i firing twice and it is the first (*w.r.t.* time) with this property. Remark that the initial value h_i is possibly incremented by at most p before site i fires for the second time. Indeed, there are only two sites which can cause h_i to rise: if site $i + 1$ fires then h_i increases by $p - 1$, if site $i - p + 1$ fires then h_i increases by 1. Then, since initially it is $h_i \leq p - 1$, when site i is supposed to fire for the second time, one finds $h_i \leq p - 1 + p - p < p$ which is a contradiction with the fact that site i should fire.

□

Given a configuration $x \in \mathbf{SM}(n)$, an ordered set of index of piles $S = \{j_1, j_2, \ldots, j_q\} \subseteq [\![1, n]\!]$ of x is *critical* if and only if for any $k \in [\![1, q]\!]$, $h_{j_k} = p - 1$ and $j_{k+1} - j_k < p$. Given $k \in [\![1, p - 1]\!]$, S is

k-*maximal* for x if $j_1 = 1$, $j_q \geq |x| + k - p + 1$ and for any i, $h_i = p - 1$ implies $i \in S$. The following lemma will be used in the proof of the Theorem 2.1.

Lemma 2.2. Consider a configuration $x \in \mathbf{SM}(n)$ and let S be a maximal critical set for x. If a grain is added at x_1, then all sites $i \in [\![1, \max S]\!]$ will fire.

Proof:
Order the elements of S increasingly $j_1 = 1 < j_2 < \ldots < j_q$. Let us prove the lemma by induction on $i \in [\![1, q]\!]$. If $j_1 = 1$, then, $h_1 = p - 1$ and hence the site 1 will fire. Since $h_i > 0$ for $i \in [\![2, p-2]\!]$, all sites in $[\![2, p-2]\!]$ will also fire.

Assume now that the statement is true for all sites $i \leq j_s$. We are going to prove it for all $j \in [\![j_s + 1, j_{s+1}]\!]$. Indeed, by construction $j_{s+1} - j_s \leq p - 1$, hence the site $j_{s+1} - p + 1$ has been fired since $j_{s+1} - p + 1 \leq j_s$. This implies the site j_{s+1} will fire and as a consequence all sites with index in $j \in [\![j_s + 1, j_{s+1})$ will also fire. □

Theorem 2.1. $\mathbf{AP}|_{\mathbf{SM}} \in \mathbf{NC}^1$ for $\mathrm{KSPM}(1, p)$ and $p > 1$.

Proof:
Consider a configuration $x \in \mathbf{SM}(n)$ for some $n \in \mathbb{N}$ and fix $k \in (n, n + p - 1]\!]$. We claim that $\mathbf{AP}|_{\mathbf{SM}}$ has a positive solution if there exists a k-maximal critical set $S = \{j_1, j_2, \ldots, j_q\}$ for x (for $j_1 < j_2 < \ldots < j_q$).

Indeed, assume that no k-maximal set S exists. Several cases may occur. First, $j_1 \neq 1$ then $h_1 + 1 < p$ and there is no avalanche. Second, there exists s such that $j_{s+1} - j_s > p - 1$. This implies that $h_{j_{s+1}} < p - 1, h_{j_{s+2}} < p - 1, \ldots, h_{j_{s+p-1}} < p - 1$. Hence, $h_{j_{s+p-1}} + 1 < p$ and the avalanche does not continue to the right of j_{s+p-1}. Finally, assume that $j_q < n - p + 1$. By Lemma 2.1, any site can fire only once, hence no avalanche can go further than site n and hence $\mathbf{AP}|_{\mathbf{SM}}$ has no positive solution.

Assume now that $S = \{j_1, j_2, \ldots, j_q\}$ (with $j_1 < j_2 < \ldots < j_q$) is a k-maximal critical set for x. By Lemma 2.2, j_q will fire and hence $h_{j_q+p-1} > 0$ but by maximality of S, $j_q + p - 1 \geq n + k - p + 1 + p - 1 = n + k$ which means that $\mathbf{AP}|_{\mathbf{SM}}$ has a positive answer.

Computing the vector of height differences can be performed in time $\mathcal{O}(\log n)$ by a parallel iterative algorithm on a PRAM with a polynomial number of processors. The verification of the existence of a maximally critical set for x can also be performed in $\mathcal{O}(\log n)$ by a PRAM with a polynomial number of processors according to the following steps. When computing the height differences in previous step, one can assign an identifier to each processor corresponding to the index of the pile being computed. Consider also a global vector of size n of boolean variables `v[]`. The element `v[i]=true` if the following tests are verified. Processor 1 (resp. $p - 1$) sets `v[1]=true` if $h_1 = p - 1$. For $i \in [\![2, n + k - p + 1)$, processor i sets `v[i]=true` if there exists $j \in [\![i - p + 1, i]\!]$ such that $h_j = p - 1$. Finally, processor n sets `v[n]=true` if there exists $j \in [\![n + k - p + 1, n]\!]$ such that $h_j = p - 1$. Remark that all those tests can be performed in constant time and can be launched in $\mathcal{O}(\log n)$ time by a parallel iterative algorithm. Another easy parallel algorithm can compute in time $\mathcal{O}(\log n)$, `V=`$\wedge_1^n$ `v[i]`, where $\wedge$ denotes the logical AND operation. By the first part of the proof, $\mathbf{AP}|_{\mathbf{SM}}$ has a positive answer if and only if `V` is true. □

For $p = 2$, the following easy result shows that the unrestricted version of $\mathbf{AP}$ is still feasible.

Proposition 2.1. $\mathbf{AP} \in \mathbf{NC}^1$ for $\mathrm{KSPM}(1, 2)$.

Proof:
For $p = 2$, Kadanoff's model coincides with the well-known SPM model. According to the characterization of fixed points given, for instance in [10], $\mathbf{AP}$ has a positive answer if and only $h_i = 1$ for all $i \in [\![1, n]\!]$. This can be easily tested in $\mathcal{O}(\log n)$ time by a parallel iterative algorithm with the help of a boolean vector of variables for collecting results similarly to what is done in the proof of Theorem 2.1.

□

It would be interesting to study if the same is true for other values of p, for $p = 3$ for example. Indeed, we suspect that for larger values of p, $\mathbf{AP}$ becomes $\mathbf{P}$-complete.

3. Sandpiles and Kadanoff model in two dimensions

Before dealing with the Kadanoff dynamics for two-dimensional sandpiles, the definitions introduced in Section 2 have to be generalised.

A two-dimensional sandpile *configuration* is a distribution of sand grains over the $\mathbb{N} \times \mathbb{N}$ lattice. Therefore, a configuration on $\mathbb{N} \times \mathbb{N}$ will be identified by a mapping from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{N}$, giving a number of grains of sand to every position in the lattice. Thus, a configuration will be denoted by $x_{i,j}$ as $(i, j) \mapsto \mathbb{N}$. A configuration x is *monotone* if $\forall i, j \in \mathbb{N} \times \mathbb{N}$, $x_{i,j}$ is such that $x_{i,j} \geq 0$ and $x_{i,j} \geq \max\{x_{i+1,j}, x_{i,j+1}\}$. So we get a monotone sandpile, in the same sense as in [4]. Example 3.1 illustrates a case which violates the condition of monotonicity of the Kadanoff dynamics.

Example 3.1. Consider the initial configuration given in the bottom left matrix

$$\begin{array}{cccc} 0 & 1 & 0 & 0 \\ \boxed{2} & 3 & 0 & 0 \\ 8 & 4 & 2 & 2 \\ 8 & 4 & 3 & 2 \end{array}$$

$$\uparrow v$$

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 8 & \boxed{6} & 2 & 2 \\ 8 & 4 & 3 & 2 \end{array} \quad \xrightarrow{h} \quad \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 8 & 4 & 3 & \boxed{3} \\ 8 & 4 & 3 & 2 \end{array}$$

Values count for the number of grains of a pile. For $p = 3$, Kadanoff dynamics cannot be applied from the square boxed pile. Indeed, the resulting configurations do not remain monotone neither by applying the dynamics horizontally nor vertically (resp. $\uparrow v$ and $\xrightarrow{h}$). A pile which violates the condition has been highlighted by a box with rounded corners in the resulting configurations (it might be not unique).

Any configuration can be identified by the mapping of its *horizontal height differences* (resp. vertical): $h_{\rightarrow} : (i, j) \mapsto x_{i,j} - x_{i+1,j}$ (resp. $h_{\uparrow} : (i, j) \mapsto x_{i,j} - x_{i,j+1}$). A configuration is *finite* if its height differences matrices have a finite number of non-zero sand content. A monotone configuration x is *horizontally stable* (resp. *vertically*) if $\forall i, j \in \mathbb{N} \times \mathbb{N}$, $h_{\rightarrow}(i, j) = x_{i,j} - x_{i+1,j} < p$ (resp. $h_{\uparrow}(i, j) = x_{i,j} - x_{i,j+1} < p$) and is *stable* if both horizontally and vertically stable.

In other words, it is a generalisation of the Kadanoff model in one dimension which requires the configuration to be stable if the difference between any two adjacent piles is less than the Kadanoff

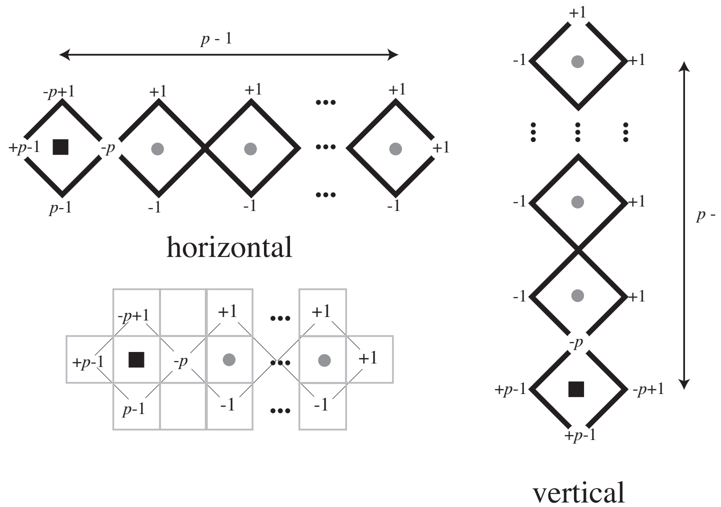

Figure 1. Horizontal (pictured on the left) and vertical (on the right) chenilles in the $\mathbb{N} \times \mathbb{N}$ lattice for an arbitrary parameter $p \geq 2$. The pile (i, j) –denoted by a black square– gives one grain of sand to the pile $(i + p - 1, j)$ horizontally and one grain of sand to the pile $(i, j+p-1)$ vertically. Some cells correspond to locations and others to height differences. The figure gives the height differences of this dynamics and the change of the lattice structure between the dynamics on the grains and the corresponding dynamics on the height differences.The representation on the bottom left will be used for defining the templates. Observe that the squares without values –containing yet the black square and the gray bullets– correspond to the piles. We will be working on the dual graph which contains the height differences at the end of the edges drawing a diamond shape. Later on, we will only keep the square grid for our explanations.

parameter p. To this configuration, we apply the Kadanoff dynamics for a given integer $p \geq 1$. This can be done if and only if the new configuration remains monotone. Said differently, prior its application, the dynamics requires to test if the local application produces height difference matrices with non-negative entries.

The Kadanoff dynamics applied to pile (i, j) for a given p consists in giving a grain of sand to any pile in the horizontal or vertical line having a distance at most $p-1$ from (i, j), *i.e.* $\{(i, j+1), ...(i, j+p-1)\}$ or $\{(i+1, j), ...(i+p-1, j)\}$. Notice that when considering the dynamics defined over height differences, we work with a different lattice though isomorphic to the initial one. Figure 1 explains how the dark pile with coordinates (i, j) with a height difference of p gives grains either horizontally (Figure 1 left) or vertically (Figure 1 right). Figure 1 also depicts the relationship between the sandpiles lattice and the height differences lattice. The local dynamics depicted by Figure 1 will be called *Chenilles* (horizontal and vertical, respectively).

For a better understanding of the dynamics, recall that in one dimension an avalanche at pile i changes the height differences of sites $i - 1, i$ and $i + p - 1$. In two dimensions, there are height changes on the line but also to both sides of it. The dynamics is simpler to depict than to write down formally. An example of the Kadanoff's dynamics applied horizontally (resp. vertically) is given in Figure 3. More precisely, the Kadanoff's dynamics for a value of parameter $p = 4$ is depicted in Figure 4. Observe that we do not need to take into account the number of grains of sand in the piles. It suffices to take the graph

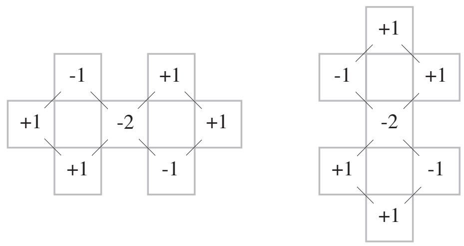

Figure 2. Templates for Bak's dynamics with $p = 2$. We have used the drawing conventions explained on Figure 1.

of the edges adjacent to each pile (depicted by thick lines) and to store the height differences. So, from now on, we will restrict ourselves to the lattice and to the dynamics defined over the height differences. In Figure 4, we only keep the information required for applying the dynamics in the simplified view.

Example 3.2. (Obtaining Bak's)
In the case $p = 2$ and if we assume the real sandpile defined as in [4] (*i.e.* $x_{i,j} \geq \max\{x_{i+1,j}, x_{i,j+1}\}$), we get the templates from Figure 2.

3.1. P-completeness

When changing from dimension 1 to 2 (or greater), the statement of $\mathbf{AP}$ has to be adapted. Like in [15], for technical reasons one has to start from an initial configuration which is not stable and which will be updated according to the deterministic policy chosen (we apply the chenille's templates in any way such that it is spatially periodic, for instance starting at position $(0, 0)$ from the left to the right and from the top to the bottom). Indeed, consider an initial finite configuration x which is non-zero for piles (i, j) with $i, j \geq 0$, monotone and let Q be the total sum of the height differences (both horizontal and vertical). Denote by n the maximum index of non-zero height differences along both axes. Then, $\mathbf{M}(n)$ denotes the set of monotone configurations of the form given by a lower-triangular matrix of size $n \times n$ (a matrix where the entries above the main diagonal are zero). To generalise the avalanche problem in two dimensions, we have to find a generic position which is far enough from the initial sandpile but close enough to be attained. To get very rough bounds, the following approach was followed. For the upper bound, the worst case occurs when all the grains are arranged on a single pile (with Q as a height difference) which is at an end of one of the axes -at distance n from the origin- and they fall down (even if it violates the monotonicity of the configuration). For the lower bound, the pile containing the grains is at the origin and the grains fall along the main diagonal (refer to Figure 5), even if it is not allowed by

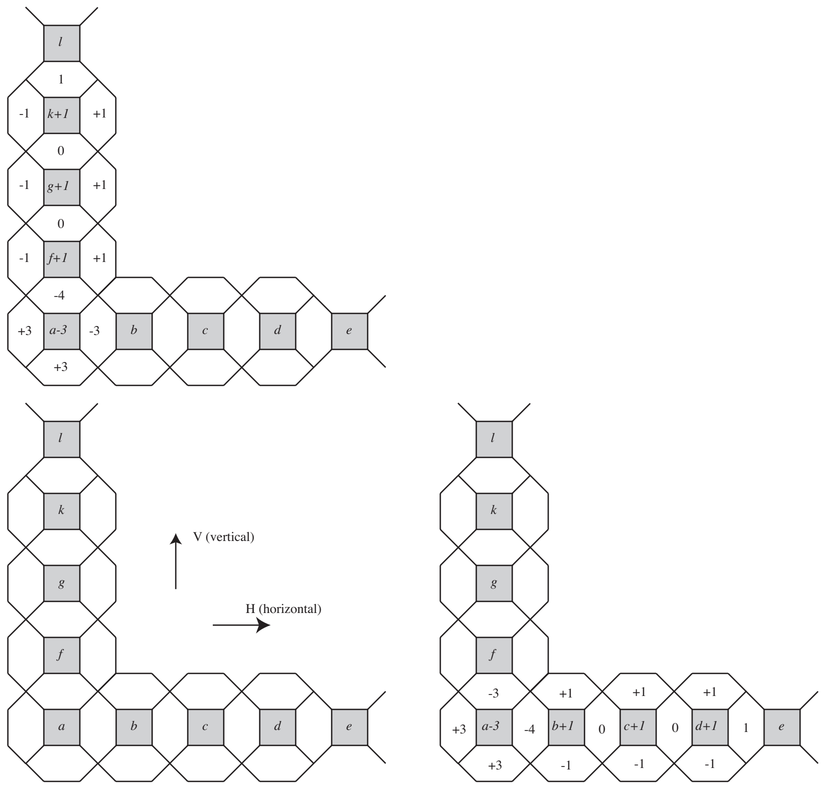

Figure 3. Horizontal and vertical chenilles for $p = 4$. Shaded squares count the number of grains on each pile and the hexagons between the squares the height difference between the corresponding two adjacent piles. The initial configuration is on the bottom-left. The Kadanoff's dynamics is applied from the shaded pile labelled a horizontally or vertically (resp. $\uparrow v$ and $\xrightarrow{h}$) to get the resulting configurations.

the dynamics. Thus, our decision problem can be restated as follows.

Problem $\mathbf{AP}$ (dimension 2)

INSTANCE: An initial monotone configuration $x \in \mathbf{M}(n)$, $(k, \ell) \in \mathbb{N} \times \mathbb{N}$ such that $x_{k,\ell} = 0$ and $\frac{\sqrt{2}}{2}n \leq \|(k, \ell)\| \leq n + Q$.

QUESTION: Does there exist an avalanche (obtained by using the vertical and horizontal chenilles) such that $x_{k,\ell} \geq 1$?

In the above statement, Q is the sum of the height differences and $\|.\|$ is the standard Euclidean norm.

To prove the $\mathbf{P}$-completeness of $\mathbf{AP}$ we proceed by reduction from the monotone circuit value problem ($\mathbf{MCVP}$), proved $\mathbf{P}$-complete under many-one $\mathbf{NC}^1$ reduction [12, Theorem 6.2.2]. Remark that the original proof [6] uses a logspace reduction but it should be noted that any logspace reduction is also a $\mathbf{NC}$ many-one reduction [12, page 54]. $\mathbf{MCVP}$ statement is: given the standard encoding

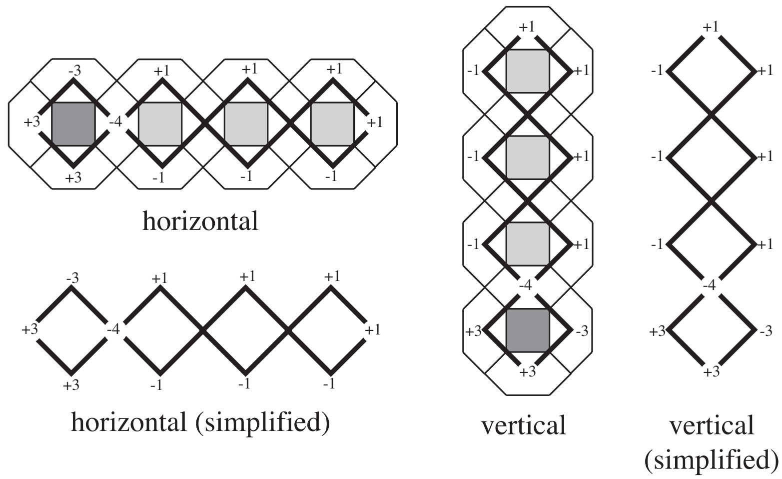

Figure 4. Horizontal and vertical chenilles for $p = 4$. The dynamics is applied to the dark-shaded pile (the leftmost one on the left part of the figure and the lowest one on the right part of the figure). The numbers express the height differences after the application of the dynamics. The two simplified views remove the number of sand grains information and only keeps the height difference information. It corresponds to a change in the lattice structure when height differences are considered instead of pile heights.

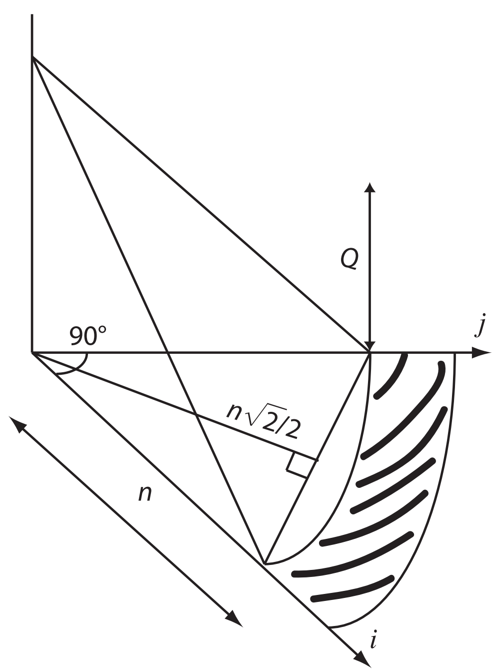

Figure 5. Getting the two dimensional bounds.

of a Boolean circuit (which ensures a topological numbering of the gates) with n inputs $\{\alpha_1, ..., \alpha_n\}$, a designated output β and logic gates AND, OR we want to decide the truth of the output value β on binary input $\{\alpha_1, ..., \alpha_n\}$ [12, page 122]. Without loss of generality, we also assume that each gate of the circuit has a fan in of two and fan out of at most two and that the gates are laid out in levels with connections only going to adjacent levels. The problem remains $\mathbf{P}$-complete with these restrictions [6]. For the reduction, we have to construct, by using sandpile configurations, wires and turning the signal on the grid (Figure 6), logic AND gates (Figure 7 (Right)), logic OR gates (Figure 8 (Left)) and cross-overs (Figure 7 (Left)). We also need to define a way to deterministically update the network; to do this, we can apply the chenille's templates in any way such that it is spatially periodic, for instance starting at position $(0, 0)$ from the left to the right and from the top to the bottom. Our main result is thus:

Theorem 3.1. $\mathbf{AP}$ is $\mathbf{P}$-complete for any $p \geq 2$.

Proof:
The fact that $\mathbf{AP}$ is in $\mathbf{P}$ can be seen with a sequential simulation of the dynamics. Recall that we deterministically update the network by applying the chenille's templates in any way such that it is spatially periodic, for instance starting at position $(0, 0)$ from the left to the right and from the top to the bottom. A double nested loop suffices for the simulation if the number of topplings of the system is polynomially bounded. To get this polynomial bound, we use the following energy function $E(t) = \sum_{i,j}(i + j)x^t_{i,j}$ where t denotes the time and $x^t_{i,j}$ the number of grains contained in the pile at time t. $E(t)$ is strictly decreasing as t grows. $E(t + 1) - E(t) = \sum_{i,j}(i + j)(x^{t+1}_{i,j} - x^t_{i,j})$. We express the horizontal chenille on the toppling of pile (i, j) (The same property holds if we had considered the vertical chenille).

$$\begin{cases} x^{t+1}_{i,j} = x^t_{i,j} - p + 1 & \\ x^{t+1}_{i+k,j} = x^t_{i+k,j} + 1 & \text{for } k \in [\![1, p-1]\!] \end{cases}$$

Computing the sum of the differences leads to the fact that $E(t + 1) < E(t)$. Moreover, the energy function is bounded: $|E(t)| \leq Q$ where $Q = \sum_{i,j} x^0_{i,j}$. Hence, the sequence of topplings must converge to a stable state, where no more application of the dynamics can occur within $O(Q)$ updates and is thus polynomial in the system's size.

For the reduction, one has to take an arbitrary instance of the $\mathbf{MCVP}$ variant previously defined and to build an initial configuration of a sandpile for the Kadanoff dynamics for $p \geq 2$. Thus, we have to design the following gadgets:

- straight wire and turning wire (Figure 6);
- (safe) wire crossing (Figure 7 (Left));
- AND gate (Figure 7 (Right));
- OR gate (Figure 8 (Left)).

Simulated gates can be made up like classical gates (up to an additive constant depending upon their size) with a fan in of two parallel wires and a fan out upper-bounded by two. The sandpile circuit is built directly from the standard encoding of the instance of the $\mathbf{MCVP}$ variant. This construction can be done by a $\mathbf{NC}$ algorithm [6, 12].

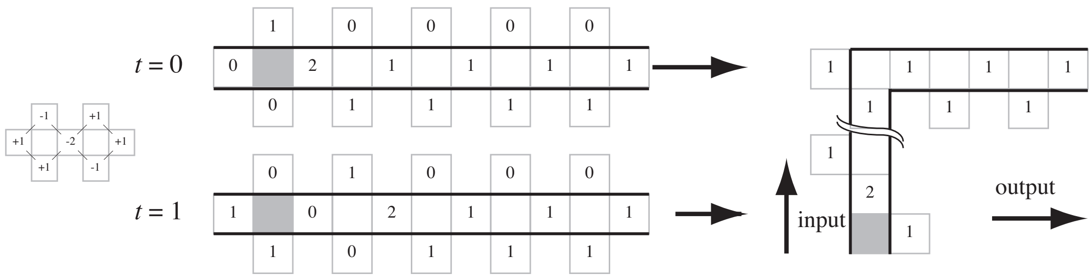

Figure 6. Information propagation in a wire for $p = 2$ at times $t = 0$ and $t = 1$ using the templates for Bak dynamics (recalled on the left of the figure) and wire for turning the information to the right. We have used the drawing conventions explained on Figure 1 ruled by the dynamics given in Figure 2.

The construction of each gadget is shown graphically for $p = 2$, which corresponds to the simplest case, but it can be done for greater values of the parameter. As an example of such a construction, we give the design of an AND gate for $p = 3$ in Figure 8 (Right). Generalising for greater values of p is not hard though tedious. For $p = 2$, the horizontal and vertical chenilles are given in Figure 2. To construct the entry vector to an arbitrary circuit we have to construct from the starting pile wires to simulate any variable $\alpha_i = 1$. (If $\alpha_i = 0$ nothing is done: we do not construct a wire from the initial pile. Else, there will be a wire to simulate the value 1). Also remark that the signal propagation does not require a global synchronising clock. Actually, this helps for designing the circuit since one signal arriving at a logical gate can "wait" for the second signal to arrive. Notice also that we do not require a diode (for which truth signals can flow only in one direction) since all the signals used by our gadgets are obviously directed by the dynamics and the monotonicity of the sandpile. Designing such a gadget is though possible by arranging the action of an AND gate, if necessary.

Then, by construction, positive instances of the $\mathbf{MCVP}$ variant are in 1:1 correspondence with positive instances of $\mathbf{AP}$. □

4. Conclusion and future work

We have proved that the avalanche problem for the class of stable monotone sand piles in $\mathbf{SM}(n)$ with one grain added in the KSPM model is in $\mathbf{NC}$ fo one dimension and that the class of monotone sand piles is $\mathbf{P}$-complete for two dimensions. Our results work with a sandpile defined as in [4] and for any value of p. Let us also point out that in the case where $p = 2$, this model corresponds to the two-dimensional Bak's model with a pile such that $x_{i,j} \geq 0$ and $x_{i,j} \geq \max\{x_{i+1,j}, x_{i,j+1}\}$.

Let us briefly recall Bak's original model which is depicted in Figure 9 (left). For the nearest neighbors, also called the von Neumann neihborhood, a pile gives a grain to each of its four neighbors if and only if it has enough tokens. Denote by $x_{i,j}$ the number of grains piled up at position (i, j). Then, for $(i, j) \in \mathbb{Z}^2$, we have $x_{i,j} \geq x_{i+1,j}$ but not necessarily $x_{i,j} \geq x_{i,j+1}$. This is the main difference with the

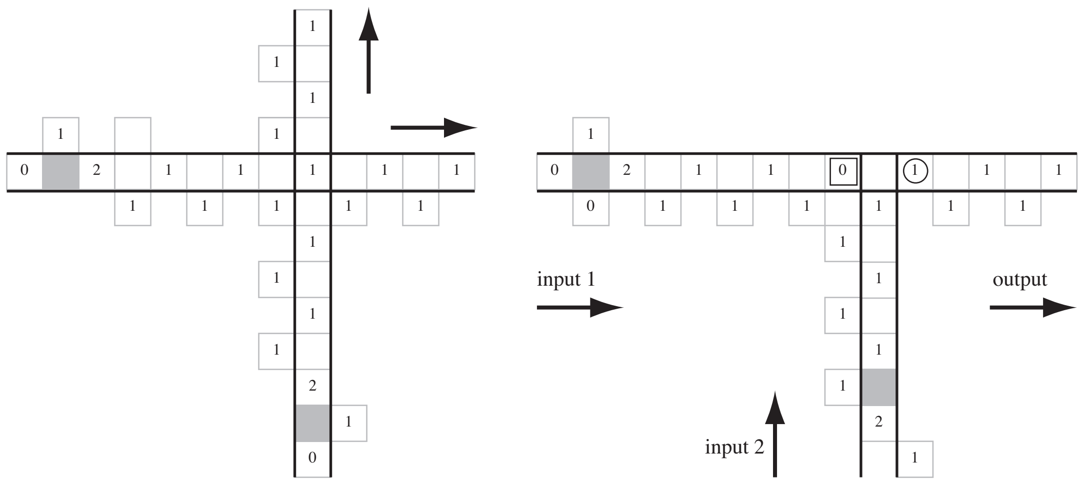

Figure 7. (Left) Crossing over two wires for $p = 2$; arrows show the directions of propagation. (Right) A logic AND gate with two inputs for $p = 2$. The upcoming “2” has to reach the horizontal “2” to change the value of the boxed “0” to “1”. Then, the upcoming “2” can apply the vertical chenille template and changes the circled “1” into “2”. In other words, the AND is computed by applying 3 horizontal chenilles and 4 vertical ones. We have used the drawing conventions explained on Figure 1 ruled by the dynamics given in Figure 2.

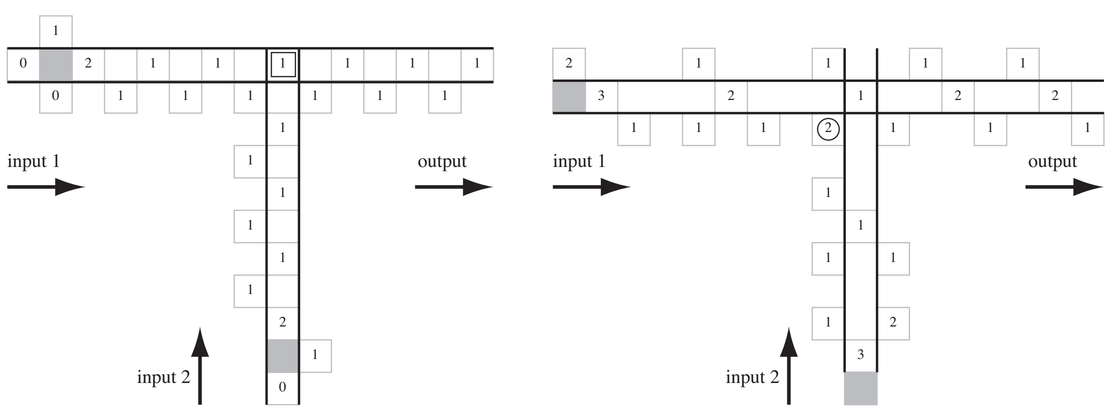

Figure 8. (Left) A logic OR gate with two inputs for $p = 2$. The boxed cell indicates the OR gate point of computation. (Right) The logic AND gate for two inputs for $p = 3$. The circled “2” is put in order to get enough tokens for the horizontal and vertical inputs. We have used the drawing conventions explained on Figure 1 ruled by the dynamics given in Figure 2.

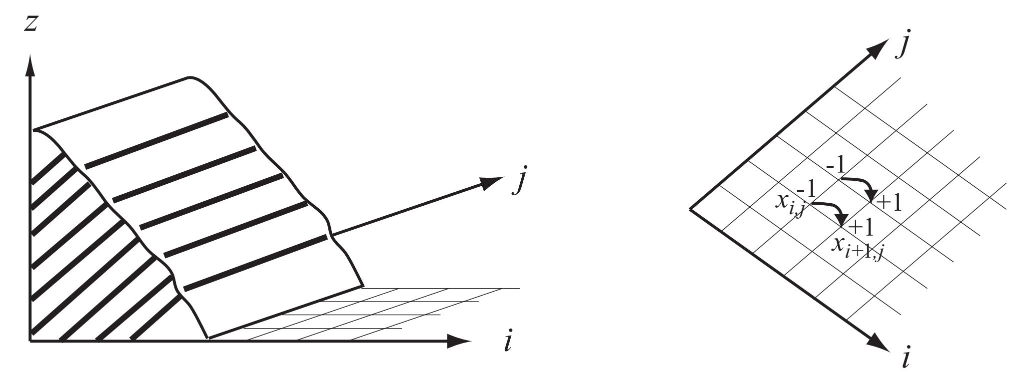

Figure 9. A Bak's sandpile (left) and the tumble of two adjacent grains from (i, j) to $(i+1, j)$ and from $(i, j+1)$ to $(i+1, j+1)$. We assume a legal tumbling: $x_{i,j} - x_{i+1,j} \geq 2$ and $x_{i,j+1} - x_{i+1,j+1} \geq 2$.

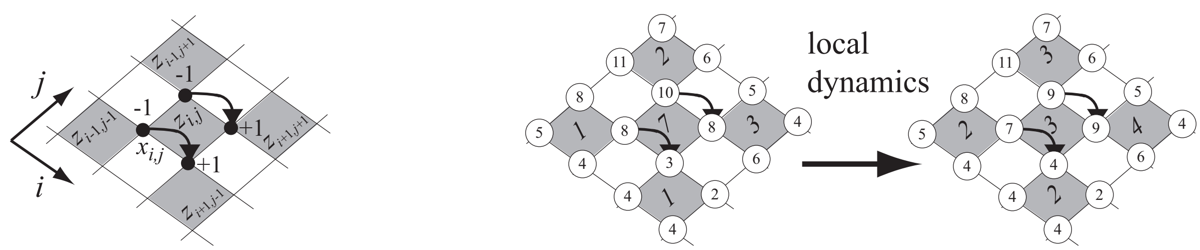

Figure 10. (Left) The tumble of two adjacent grains from (i, j) to $(i+1, j)$ and from $(i, j+1)$ to $(i+1, j+1)$ in terms of height differences. The shaded cells compute the "average" height difference and each vertex is labelled by the number of grains $x_{i,j}$. (Right) A numerical example of the local dynamics.

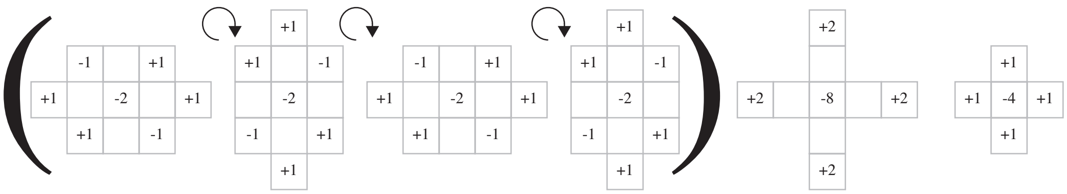

Figure 11. From Bak's to Kadanoff's operators. All the Kadanoff's operators between the brackets have been applied to get the pattern (a). The Bak's pattern (b) is obtained by eliminating the holes in (a) and by dividing the number of tokens by two.

model we are considering. In this context, Bak's dynamics is depicted on the right of Figure 9. In terms of height differences, Bak's dynamics (Figure 10) is interpreted as follows. In every shaded cell of the grid, one adds the two possible height differences (a kind of "average" height difference) ${z_{i,j}}^2$ according to $z_{i,j} = (x_{i,j} - x_{i+1,j}) + (x_{i,j+1} - x_{i+1,j+1})$. So, for a tumbling from site (i, j) and for another from $(i, j+1)$, then

$$\begin{cases} z'_{i,j} \leftarrow z_{i,j} - 4, \\ z'_{i,j\pm 1} \leftarrow z_{i,j\pm 1} + 1, \\ z'_{i\pm 1,j} \leftarrow z_{i\pm 1,j} + 1; \end{cases}$$

where $z'_{i,j}$ denotes the new height difference after the tumbling of a grain. Figure 10 (Right) illustrates the application of this local dynamics; we can observe that the two "firing" adjacent grains are critical since $8 - 3 \geq 2$ and $10 - 8 \geq 2$ (ordered increasingly *w.r.t.* j-coordinate).

It is important to notice that, by directly taking the two-dimensional Bak's tokens game, its computation universality was proved in [9] by designing logical gates in non-planar graphs. Furthermore, by using the previous construction, C. Moore *et al.* proved the $\mathbf{P}$-completeness of the prediction problem for lattices of dimensions d with $d \geq 3$ (the prediction problem aims to predict the final state from an initial distribution of sand). But the problem remained open for two-dimensional lattices. Furthermore, it was proved in [5] that, in the latter case, it is not possible to build complex circuits because wires cannot cross.

In our framework, the two-dimensional Bak's model corresponds to the application of the four rotations of the template for $p = 2$ (see Figure 11) by applying to a pile, if there are at least four tokens, the horizontal ($\rightarrow$) and the vertical ($\downarrow$) chenille. But this model is not anymore the representation of a two-dimensional sandpile as presented in [4], that is with $x_{i,j} \geq 0$ and $x_{i,j} \geq \max\{x_{i+1,j}, x_{i,j+1}\}$. To define a reasonable two-dimensional model, consider a monotone sandpile decreasing in $i \geq 0$ and $j \geq 0$. Over this pile we define the extended Kadanoff's model as a local avalanche in the growing direction of the $(i + j)$ axis such that monotonicity is conserved. Certainly, one may define other local applications of Kadanoff's rule which also match with the physical sense of monotonicity. For instance, by considering the set $(i + 1, j), (i + 1, j + 1), (i, j + 1)$ as the piles to be able to receive grains from pile (i, j).

[2] we have kept the original notation coming from [1].

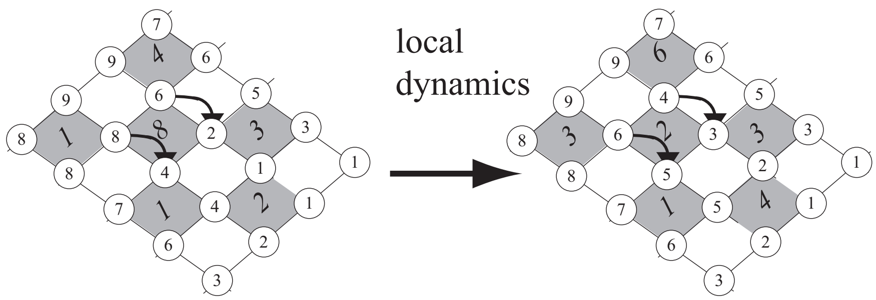

Figure 12. The tumble of adjacent grains in terms of height differences in the case $p = 3$ (p odd). The shaded cells compute the “average” height difference and each vertex is labelled by the number of grains $x_{i,j}$.

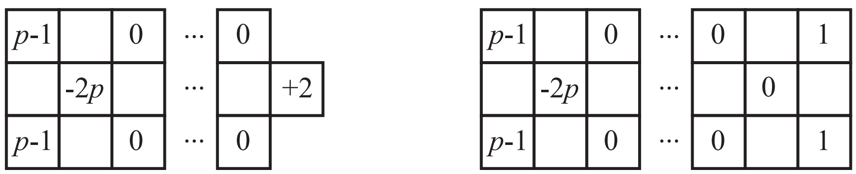

Figure 13. The chenilles associated to the Kadanoff model associated with Bak’s sandpile for odd value of p (left) and even value of p (right).

Like in Figures 9 and 10, we may construct the Kadanoff model associated with Bak’s sandpile (see Figures 12 and 13). The corresponding chenilles are depicted in Figure 13 for odd and even values of the parameter p. From that, it is easily observed that for two dimensions and in the case $p > 2$, we may construct gates and wires in a way similar with Section 3 so, the problem remains $\mathbf{P}$-complete. But, since for $p = 2$, we recuperate exactly Bak’s automaton, the computational complexity in this case remains open.

Similarly, for p arbitrary, one may simultaneously apply other combinations of chenilles which, in general, allows us to get $\mathbf{P}$-complete problems. For instance, when there are enough tokens, the applications of the four chenilles (*i.e.* $\leftarrow$,$\rightarrow$,$\uparrow$ and $\downarrow$) give rise to a new family of local templates called *butterflies* (because of their four wings). It is not so difficult to construct wires and circuits for butterflies. Hence, for this model, the decision problem will remain $\mathbf{P}$-complete. One thing to analyze from an algebraic and complexity point of view is to classify every local rule derived from the chenille application. Further, one may define a more general sandpile dynamics which contains both Bak’s and Kadanoff’s ones i.e given an integer $p \geq 2$, we allow the application of every Kadanoff’s update for $q \leq p$. That means that an active pile with more than p grains can distribute up to q grains to the adjacent piles. We are studying this dynamics and, as a first result, we observe yet that in one dimension there are several fixed points and also, given a monotone circuit with depth m and with n gates, we may simulate it on a line with this generalized rule for a given $p \geq m + n$. Many other generalization of either Bak’s or Kadanoff’s models can be conceived by changing the points of the lattice which are taken into account or the way to apply the dynamics.

For the one-dimensional case, we proved that for a restricted version of $\mathbf{AP}$ it belongs to $\mathbf{NC}^1$

for $p \geq 2$. we are currently trying to prove that $\mathbf{AP}$ is $\mathbf{P}$-complete when considering the unrestricted version.

Acknowledgements

The authors are grateful to K. Imai from the Hiroshima University Graduate School of Engineering for his valuable help implementing a two-dimensional Kadanoff sandpile model which fits with our settings and allowed to check our constructions.

References

[1] P. Bak, C. Tang, and K. Wiesenfeld. Self-organized criticality. *Phys. Rev. A*, 38(1):364–374, 1988.

[2] Julien Cervelle, Enrico Formenti, and Benoît Masson. From sandpiles to sand automata. *Theor. Comput. Sci.*, 381(1-3):1–28, 2007.

[3] Alberto Dennunzio, Pierre Guillon, and Benoît Masson. Sand automata as cellular automata. *Theor. Comput. Sci.*, 410(38-40):3962–3974, 2009.

[4] E. Duchi, R. Mantaci, H. D. Phan, and D. Rossin. Bidimensional sand pile and ice pile models. *Pure Math. Appl. (PU.M.A.)*, 17(1-2):71–96, 2006.

[5] A. Gajardo and E. Goles. Crossing information in two dimensional sand piles. *Theoretical Computer Science*, 369(1-3):463–469, 2006.

[6] L.M. Goldschlager. The monotone and planar circuit value problems are log space complete for P. *ACM sigact news*, 9(2):25–29, 1977.

[7] E. Goles and M. Kiwi. Sand pile dynamics in one dimensional bounded lattice. In N. Boccara et al, editor, *Cellular Automata and Cooperative Systems*, volume 396 of *NATO-ASI*, pages 203–210. Ecole d'Hiver, Les Houches, Kluwer, 1993.

[8] E. Goles and M. Margenstern. Sand piles as a universal computer. *Journal of Modern Physics-C*, 7(2):113–122, 1996.

[9] E. Goles and M. Margenstern. Universality of the chip firing game on graphs. *Theoretical Computer Science*, 172:121–134, 1997.

[10] E. Goles, M. Morvan, and H. D. Phan. Sand piles and order structure of integer partitions. *Discrete Applied Mathematics*, 117:51–64, 2002.

[11] E. Goles Ch., M. Morvan, and H. D. Phan. The structure of a linear chip firing game and related models. *Theor. Comput. Sci.*, 270(1-2):827–841, 2002.

[12] Raymond Greenlaw, H. James Hoover, and Walter L. Ruzzo. *Limits to parallel computation: P-completeness theory*. Oxford University Press, Inc., New York, NY, USA, 1995.

[13] L.P. Kadanoff, S.R. Nagel, L. Wu, and S. Zhou. Scaling and universality in avalanches. *Phys. Rev. A*, 39(12):6524–6537, 1989.

[14] P. B. Miltersen. The computational complexity of one-dimensional sandpiles. *Theory of Computing Systems*, 41:119–125, 2007.

[15] C. Moore and M. Nilsson. The computational complexity of sandpiles. *Journal of statistical physics*, 96(1-2):205–224, 1999.

[16] K. Perrot and E. Rémila. Avalanche structure in the Kadanoff sand pile model. In *LATA*, volume 6638 of *LNCS*, pages 427–439. Springer Verlag, 2011.