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Condensation of classical optical waves beyond the cubic nonlinear Schrödinger equation

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ABSTRACT

A completely classical nonlinear wave is known to exhibit a process of condensation whose thermodynamic properties are analogous to those of the genuine Bose–Einstein condensation. So far this phenomenon of wave condensation has been studied essentially in the framework of the nonlinear Schrödinger (NLS) equation with a pure cubic Kerr nonlinearity. We study wave condensation by considering two representative generalizations of the NLS equation that are relevant to the context of nonlinear optics, the nonlocal nonlinearity and the saturable nonlinearity. For both cases we derive analytical expressions of the condensate fraction in the weakly and the strongly nonlinear regime. The theory is found in quantitative agreement with the numerical simulations of the generalized NLS equations, without adjustable parameters.

1. Introduction

The propagation of incoherent nonlinear optical waves is a subject that is attracting a growing interest in various different fields of investigations, such as, *e.g.*, wave propagation in homogeneous [1–18] or periodic media [19], nonlinear imaging [20], cavity systems [21–30], or nonlinear interferometry [31]. In particular, the study of the long-term evolution of partially coherent nonlinear optical waves has been considered in various circumstances [3,4,13,19,32–34], as well as in various optical media characterized by different nonlinearities [2,14,15,17]. When the propagation of the optical wave is almost conservative (the dissipation can be neglected) and it can be accurately described by a Hamiltonian model equation (*e.g.*, NonLinear Schrödinger (NLS) equation), then the evolution of the incoherent wave is characterized by a nonequilibrium process of thermalization [8,12–14,16,17]. This phenomenon of optical wave thermalization can be interpreted in analogy with the kinetic gas theory: the incoherent wave exhibits an irreversible evolution toward the thermodynamic equilibrium state, *i.e.*, the Rayleigh–Jeans (RJ) equilibrium distribution that realizes the maximum of entropy (‘disorder’).

Nonlinear waves manifest themselves in a myriad of different forms, such as, *e.g.*, sound, elastic, vibrational, surface or electromagnetic waves. However, in most of these wave systems dissipation plays a non-negligible role in the wave evolution. But electromagnetic light waves in nonlinear optics constitutes a promising field of investigation of thermal wave relaxation because of the availability of low-loss nonlinear media in which light propagation is accurately ruled by NLS-like equations over long distances (see, *e.g.*, Ref. [35]).

A remarkable property of the phenomenon of thermalization is that it can be characterized by a ‘self-organization process,’ in the sense that it is thermodynamically advantageous for the optical wave to generate a large-scale coherent structure in order to reach the most disordered equilibrium state [36]. An important example of this type of self-organization is provided by the condensation of classical nonlinear waves in a defocusing Kerr-like nonlinear medium [1,37–43]. Optical wave condensation is characterized by the spontaneous formation of an almost plane-wave starting from an initial incoherent field: the plane-wave solution (‘condensate’) remains immersed in a sea of small scale fluctuations (‘uncondensed quasiparticles’), which store the information necessary for the reversible evolution of the wave. The wave turbulence (WT) theory [38,44–51] is known to provide a detailed description of wave condensation, in both the weakly and the highly nonlinear regimes of condensation [41]. WT theory reveals that the thermodynamic properties of this condensation process are analogous to those of the genuine Bose–Einstein condensation [40], despite the fact that the considered optical wave is completely classical. Contrary to the recent observation of Bose–Einstein condensation of photons [52], wave condensation refers here to a pure classical regime. This is because the

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occupation number of photons is large compared to unity (see Section 2.3), and because the wave number range associated to the quantum behavior is much larger than the wave number range involved in an experiment of classical wave condensation (see Ref. [43]). We also note that, because this condensation process results from the natural thermalization of the optical wave to thermal equilibrium, it is of a different nature than the condensation processes recently discussed in dissipative optical cavities [21,22,27].

This phenomenon of classical wave condensation has been essentially studied in the framework of the NLS equation in the presence of a pure cubic Kerr nonlinearity [40,41,50,53–58]. In many cases, however, realistic optical experiments are not modelled by a cubic Kerr nonlinearity. An example is provided by the recent work [43], which reported experimental evidence of the phenomenon of wave condensation. Our aim in this Letter is to show that optical wave condensation can take place with more complex nonlinearities. We consider the examples of the nonlocal nonlinearity and of the saturable nonlinearity, which refer to natural extensions of the cubic nonlinearity that have been the subject of many studies in the nonlinear optics literature. A spatial nonlocal wave interaction means that the response of the nonlinearity at a particular point is not determined solely by the wave intensity at that point, but also depends on the wave intensity in the neighborhood of this point. Nonlocality, thus, constitutes a generic property of a large number of nonlinear wave systems [59–63], and the dynamics of nonlocal nonlinear waves has been widely investigated in this last decade [64–68]. On the other hand, a saturable nonlinearity is encountered whenever the optical field in a material is strong enough so that the higher-order nonlinearities become non-negligible [35].

We show that the generalized NLS equation accounting for a nonlocal or a saturable nonlinearity describes a process of wave condensation completely analogous to that described in the framework of the cubic Kerr nonlinearity. We extend the previous works [40,41] in which the cubic nonlinearity was considered and derive analytical expressions of the condensate fraction, in both the weakly and the strongly nonlinear regimes of propagation. For both the saturable and the nonlocal nonlinearity, we obtain a quantitative agreement with the numerical simulations, without any adjustable parameter. We also show that the condensate amplitude exhibits strong fluctuations near by the transition to condensation, while the fluctuations are suppressed in the highly condensed regime. We believe that this work will stimulate new optical experiments aimed at studying the condensation of light in the framework of different accessible optical settings.

2. Nonlocal nonlinearity

2.1. NLS model equation

In this section we study the transverse spatial evolution of a partially coherent wave that propagates in a nonlinear medium characterized by a nonlocal nonlinear response. A nonlocal nonlinear response is found in several systems such as,*e.g.*, atomic vapors [59], nematic liquid crystals [60], thermal susceptibilities [61,62] and plasmas physics [63]. For this reason the dynamics of nonlocal nonlinear waves has been widely investigated [64,66–68]. We consider here the standard nonlocal NLS model equation, $i\partial_z A(z,\mathbf{r}) = -\alpha\nabla^2 A + \gamma A \int V(\mathbf{r}-\mathbf{r}')\,|A(z,\mathbf{r}')|^2\,d\mathbf{r}'$, where $A(z,\mathbf{r})$ is the amplitude of the propagating electric field, $\mathbf{r}=(x,y)$ denotes the position in the transverse surface section of the beam and ∇^2 denotes the corresponding transverse Laplacian, $\nabla^2 = \partial_x^2 + \partial_y^2$. The nonlinear coefficient is γ, and the diffraction coefficient is $\alpha = 1/(2k_0)$, where $k_0 = 2\pi n_0/\lambda$ is the wave vector modulus of the laser source, λ is its wavelength and n_0 is the linear contribution to the refractive index of the nonlinear material. The nonlocal response function (or kernel function), $V(\mathbf{r})$, is real and even, and its spatial extension, say σ, characterizes the amount of nonlocality in the system. We assume that the incoherent wave exhibits a (quasi-)homogeneous statistics and normalize the wave amplitude with respect to its average amplitude, $\sqrt{\bar{I}}$, where $\bar{I} = 1/S\int |A|^2\,d\mathbf{r}$ is the average intensity of the wave, S being a surface of integration. In this way, we normalize the propagation length, z, with respect to the nonlinear length, $L_0 = 1/(\gamma\bar{I})$, while the transverse variables are normalized with respect to the healing length, $\Lambda = \sqrt{\alpha L_0} = 1/\sqrt{2k_0\gamma\bar{I}}$. In dimensionless units, the NLS equation then takes the following form

$$i\partial_z A(z,\mathbf{r}) = -\nabla^2 A + A\int V(\mathbf{r}-\mathbf{r}')\;|A(z,\mathbf{r}')|^2\;d\mathbf{r}'. \quad (1)$$

The variables can be recovered in real units through the transformations $z\to zL_0$, $\mathbf{r}\to\mathbf{r}\Lambda$, $A\to A\sqrt{\bar{I}}$, $V\to V/\Lambda^2$, $\sigma\to\sigma\Lambda$. With this normalization, σ is in units of the healing length Λ, and the dimensionless quantity $\sigma/\Lambda = \sigma\sqrt{2k_0\gamma\bar{I}}$ is the only independent parameter of Eq. (1). Note that the nonlocal kernel in (1) is normalized in such a way that $\int V(\mathbf{r})\,d\mathbf{r} = 1$, so that in the limit of a local response [$V(\mathbf{r}) = \delta(\mathbf{r})$, $\delta(\mathbf{r})$ being the Dirac function], Eq. (1) recovers the standard local NLS equation. We considered in Eq. (1) the defocusing regime of interaction, so as to guarantee the modulational stability of the plane-wave solution (condensate).

Besides the momentum, Eq. (1) conserves two important quantities, the total power of the wave ('number of particles')

$$N = \int |A|^2\;d\mathbf{r}, \quad (2)$$

and the total energy (Hamiltonian), $H = E + U$, which has a linear kinetic contribution:

$$E = \int |\nabla A|^2\;d\mathbf{r}, \quad (3)$$

and a nonlinear contribution:

$$U = \frac{1}{2}\iint V(\mathbf{r}-\mathbf{r}')|\psi(\mathbf{r})|^2\;|\psi(\mathbf{r}')|^2\;d\mathbf{r}\,d\mathbf{r}'. \quad (4)$$

Note that with the adopted normalization we have, $n = N/S = 1$.

2.2. Wave condensation

The phenomenon of wave condensation has been the subject of a large number of studies in the past in the framework of the NLS equation with a cubic Kerr nonlinearity. The numerical simulations reported here for the nonlocal NLS Eq. (1) have been realized by following the same procedure as those reported in Refs. [16,40,41,53–58] for the cubic nonlinearity, and we refer the interested reader to these previous works for details. We consider an initial condition of the random optical wave, $A(\mathbf{r}, z=0)$, that we assume to be of zero mean ($\langle A\rangle = 0$), and characterized by a homogeneous statistics. This latter assumption refers to the physical limit in which the correlation length, λ_c, of the optical wave is much smaller than the size of the optical beam (the so-called 'quasi-homogenous statistics'), a condition which can easily be realized in an experiment (see, *e.g.*, [43]).

Wave condensation manifests itself by a dramatic narrowing of the spectrum of the optical wave as it propagates through the nonlinear medium. Specifically, the condensation process is characterized by the spontaneous formation of a plane-wave, so that the power of the optical wave progressively accumulates into the fundamental plane-wave mode, $\mathbf{k} = 0$. This is illustrated in Fig. 1a, which reports the fraction of power condensed into the

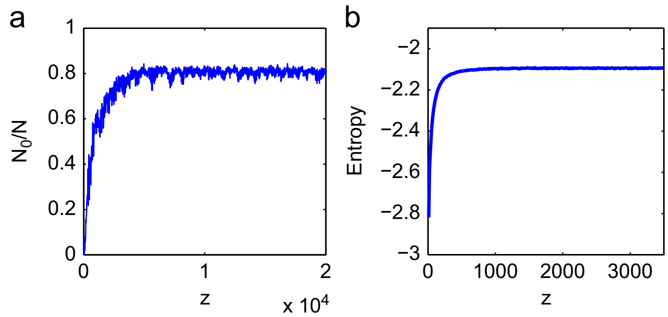

Fig. 1. Numerical simulation of the nonlocal NLS Eq. (1), which illustrates the evolution of the condensate fraction, N_0/N (a), and the nonequilibrium entropy, $\mathcal{S}=\int \ln[n_{\boldsymbol{k}}(z)]\, d\boldsymbol{k}$ (b), during the propagation. The condensation process results from the natural thermalization of the wave toward the equilibrium state: it is thermodynamically advantageous for the wave to generate a coherent plane-wave (condensate), in order to reach the most 'disordered' state. The spatial discretization of the NLS Eq. (1) is $dx=1$ ($k_c=\pi$), the number of modes $N_*=128^2$. The nonlocal response function is given in Eq. (5), with $\sigma=1$ (in units of Λ). The equilibrium state of this simulation is marked by an orange diamond ($\diamond$) in Fig. 2a, for $N_0/N\simeq 0.8$.

fundamental mode, *i.e.*, the condensate fraction N_0/N, during the propagation in z. In this example we considered a Gaussian-shaped response (kernel) function,

$$V(r)=\frac{\exp(-r^2/(2\sigma^2))}{2\pi\sigma^2}. \tag{5}$$

However, we underline that the particular choice of the kernel function $V(\boldsymbol{r})$ does not affect the generality of our results. In particular, similar evolutions of the system are obtained for an exponential-shaped kernel function.

The phenomenon of condensation results from the natural thermalization of the wave, as illustrated by the saturation of the process of entropy growth reported in Fig. 1b. These important properties of wave condensation have been previously discussed, *e.g.*, in Refs. [16,41], and we refer the reader to these works for details. We underline that all the power of the initial incoherent wave cannot be completely transferred to the fundamental mode because such evolution would imply a loss of information for the optical wave, which would violate the *formal* reversibility of the system. Actually, the plane-wave remains immersed in a sea of small-scale fluctuations, whose energy is equipartitioned among all the modes. These small-scale fluctuations refer to the 'uncondensed particles', and thus carry the power difference $N-N_0$. Such thermalized small-scale fluctuations store the information necessary for a reversible evolution of the wave, a feature which is consistent with the formal reversibility of the NLS Eq. (1).

2.3. Rayleigh-Jeans distribution

As commented here above, the phenomenon of wave condensation finds its origin in the natural thermalization of the optical wave toward the thermodynamic equilibrium spectrum. The thermalization of a nonlinear wave can be considered as an irreversible process of diffusion in phase-space, which is due to the nonintegrable character of the NLS Eq. (1) [69]. Despite the formal reversibility of the NLS Eq. (1), the optical wave exhibits an irreversible evolution to thermal equilibrium—the RJ spectrum. This irreversible evolution of the system can be proven rigorously in the weakly nonlinear regime ($U/E \ll 1$) by the WT theory [38,48–51]. The kinetic approach relies on a natural asymptotic closure induced by the dispersive properties of the waves, which leads to a kinetic description of the wave interaction that is formally based on an irreversible Boltzmann-like kinetic equation. Starting from the NLS Eq. (1), one can derive an irreversible kinetic equation describing the evolution of the averaged spectrum of the wave, $n(\boldsymbol{k},z)$, which is defined by $\langle a(\boldsymbol{k}_1,z)\, a^*(\boldsymbol{k}_2,z)\rangle = n(\boldsymbol{k}_1,z)\delta(\boldsymbol{k}_1-\boldsymbol{k}_2)$, where $a(\boldsymbol{k},z)$ is the Fourier transform of $A(\boldsymbol{r},z)$. The irreversible behavior of the kinetic equation is expressed by a H-theorem of entropy growth [45,48,38], $\partial_z\mathcal{S}\ge 0$, where $\mathcal{S}(z)=\int \ln[n_{\boldsymbol{k}}(z)]\, d\boldsymbol{k}$ is the nonequilibrium entropy. Note that this H-theorem is completely analogous to the Boltzmann's H-theorem relevant for gas kinetics [70]. Accordingly, the kinetic equation describes an irreversible relaxation toward the RJ distribution, *i.e.*, the equilibrium state that realizes the maximum of entropy,

$$n^{eq}(k)=\frac{T}{\omega(k)-\mu}, \tag{6}$$

where $\omega(k)=k^2$ is the linear dispersion relation of the NLS Eq. (1), while T and μ respectively denote the 'temperature' and the 'chemical potential' by analogy with thermodynamics. Actually, T and μ correspond to the Lagrangian multipliers respectively associated to the conservation of the energy $E/S=\int \omega_k n_{\boldsymbol{k}}\, d\boldsymbol{k}$ and of the power $N/S=\int n_{\boldsymbol{k}}\, d\boldsymbol{k}$ in the kinetic equation [48].

Note that the RJ distribution (6) is a formal solution only, because it does not lead to converging expressions for the energy E, nor the total power N. In order to regularize the divergences of the integrals, one usually introduces an ultraviolet frequency cut-off, k_c. Such a cut-off is known to appear in real physical systems through viscosity or diffusion effects at the microscopic scale [48]. The frequency cut-off also arises naturally in the numerical simulations, due to the discretization of the NLS equation, $k_c=\pi/dx$ where dx is the spatial discretization [40]. In this case the frequency cut-off mimics the evolution of the random wave in the presence of a box-shaped waveguide potential, $V(r)$, which is characterized by a finite potential depth, $V_0\sim k_c^2$. Then in the context of optics, k_c corresponds to the spatial frequency cut-off inherent to a multimode waveguide configuration of optical wave propagation [42].

The equilibrium distribution (6) is a Lorentzian spectrum in which the correlation length at equilibrium is determined by the chemical potential, $\lambda_c^{eq}\sim 1/\sqrt{-\mu}$. Note that in the tails of the spectrum (6) the optical wave exhibits an energy equipartition among the modes, $\varepsilon(k)=\omega(k)n_k\simeq T$ for $k\gg\sqrt{-\mu}$. This power-law behavior of the spectrum is clearly visible in the numerical simulations (see *e.g.*, [32]).

Wave condensation arises when $-\mu\to 0$, *i.e.*, the coherence length λ_c^{eq} tends to infinity [40]. This entails a divergence of the equilibrium distribution (6) for $k=0$, which means that the fundamental mode becomes macroscopically populated. In this respect, classical wave condensation is reminiscent of the Bose–Einstein condensation inherent to quantum systems. Indeed, the RJ distribution for a wave system can be deduced from the quantum Bose distribution in the limit where the modes are highly occupied, $n_k\gg 1$ [48].[1] The idea that highly occupied modes of a quantum Bose field are well described by a classical field is well known, *e.g.*, in laser physics, where highly occupied modes are described by classical equations. In this way, it has been shown that a modified ("projected") NLS equation can be used to model the highly populated modes ($n_k\gg 1$) of a finite-temperature quantum Bose gas [71,72].

2.4. Condensate fraction

The fraction of power which eventually condenses into the fundamental plane-wave mode depends on the initial degree of

[1] The cross-over between the classical and the quantum regimes may be given by the so-called Wien law, $\hbar\omega(k)/k_B\Theta\sim 1$ where $2\pi\hbar$ is the Planck constant, k_B is the Boltzmann constant, Θ is the room temperature.

coherence of the optical wave – the higher the coherence, the higher the condensate fraction, N_0/N. A natural possible measure of the amount of incoherence is provided by the kinetic energy of the wave, since it is related to the second order moment of the wave spectrum, $E \sim \int k^2 n_{\boldsymbol{k}}\, d\boldsymbol{k}$. However, the kinetic energy is not an appropriate variable, because it is not a conserved quantity of the NLS Eq. (1) – it is only conserved in the weakly nonlinear regime of propagation, $U/E \ll 1$. Moreover, as we shall see in Section 2.4.2, as soon as the condensate fraction becomes large, the nonlinear term in the NLS Eq. (1) becomes of the same order as the linear diffraction term, so that there is not a clear separation between the kinetic and the nonlinear energy contributions. Therefore, the appropriate quantity to measure the amount of incoherence is provided by the conserved total energy, H. Note that the whole Hamiltonian can easily be measured in an optical experiment, since E and U can be retrieved respectively from the far and near field of the optical wave. In the following we derive a closed relation between the condensate fraction and the Hamiltonian, N_0/N vs H – the so-called 'condensation curve'.

The condensation curve has been derived in detail in Refs. [40,41] for the NLS equation with a cubic Kerr nonlinearity. More precisely, it was shown in Ref. [41] that wave condensation originates in the spontaneous regeneration of a non-vanishing first-order cumulant in the hierarchy of the cumulants equations. Although mathematically rigorous, the presentation of wave condensation given in [41] is quite technical. Here we derive the analytic expressions of the condensate fraction by following a more intuitive approach which is based on the analysis outlined in Ref. [40]. In particular, we focus the presentation on the differences introduced by the nonlocal character of the nonlinearity, and refer the reader to [40] for details.

2.4.1. Small condensate fraction: the WT regime

As in the usual interpretation of Bose–Einstein condensation [73], we isolate the ground state from the contributions of the 'excited states'. The total power N of the wave at equilibrium can thus be written

$$N = N_0 + \sum_{\boldsymbol{k}}{}' \frac{T}{k^2-\mu}, \tag{7}$$

where $\sum'_{\boldsymbol{k}}$ denotes a sum over all modes $\boldsymbol{k}$, except the fundamental one $\boldsymbol{k}=0$. In the same way, we calculate the energy of the wave H by following the procedure of Ref. [40]. The difference between the cubic Kerr nonlinearity and the nonlocal nonlinearity considered here relies in the computation of the averaged nonlinear energy $\langle U \rangle$ in Eq. (4). This calculation is performed under the assumption that the random wave exhibits a Gaussian statistics. This assumption is justified in the weakly nonlinear regime ($U/E \ll 1$) which characterizes the small condensate fraction considered in this Section. Note that the statistics does not require to be Gaussian initially: in the weakly nonlinear regime, linear diffraction effects dominate the interaction and bring the system to a state of Gaussian statistics (a feature that is clearly visible in the simulations). Under this assumption one obtains

$$\langle H \rangle = \sum_{\boldsymbol{k}} k^2 n_{\boldsymbol{k}} + n_0 \sum_{\boldsymbol{k}}{}' \hat{V}_{\boldsymbol{k}} n_{\boldsymbol{k}} + \frac{1}{2}N + \frac{1}{2S}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}{}' \hat{V}_{\boldsymbol{k}_1-\boldsymbol{k}_2} n_{\boldsymbol{k}_1} n_{\boldsymbol{k}_2} \tag{8}$$

where $\hat{V}_{\boldsymbol{k}}$ denotes the Fourier transform of the nonlocal potential $V(\boldsymbol{r})$, and $n_0 = N_0/S$, $n = N/S = 1$, where we recall that S represents the system size. Note that '$n_{\boldsymbol{k}}$' stands for '$n_{\boldsymbol{k}}(z)$' in Eq. (8) and denotes the averaged spectrum of the wave at the propagation length z. At equilibrium, this averaged spectrum can be substituted by the corresponding RJ distribution (6). Remarking furthermore that the 'temperature' in the RJ distribution can be substituted by $T=(N-N_0)/\sum'_{\boldsymbol{k}}(k^2-\mu)^{-1}$, one obtains the following coupled equations:

$$\frac{\langle H(\mu)\rangle}{S} = (1-n_0)\frac{\sum'_{\boldsymbol{k}} \dfrac{k^2+n_0\hat{V}_{\boldsymbol{k}}}{k^2-\mu}}{\sum'_{\boldsymbol{k}} \dfrac{1}{k^2-\mu}} + \frac{1}{2} + \frac{(1-n_0)^2}{2}\frac{\sum'_{\boldsymbol{k}_1,\boldsymbol{k}_2} \dfrac{\hat{V}_{\boldsymbol{k}_1-\boldsymbol{k}_2}}{(k_1^2-\mu)(k_2^2-\mu)}}{\left(\sum'_{\boldsymbol{k}} \dfrac{1}{k^2-\mu}\right)^2}, \tag{9}$$

$$\frac{N_0(\mu)}{N} = \frac{1}{-\mu}\frac{1}{\sum_{\boldsymbol{k}} \dfrac{1}{k^2-\mu}}. \tag{10}$$

The coupled Eqs. (9) and (10) provide the condensation curve in the weakly nonlinear regime. It is obtained by means of a parametric plot of Eq. (9) and (10) with respect to the chemical potential, μ. The condensation curve is reported in Fig. 2a (green line), and its validity is restricted to the small condensation regime, $n_0 \ll 1$. In this expression of the condensation curve, finite size effects of the system are taken into account through the

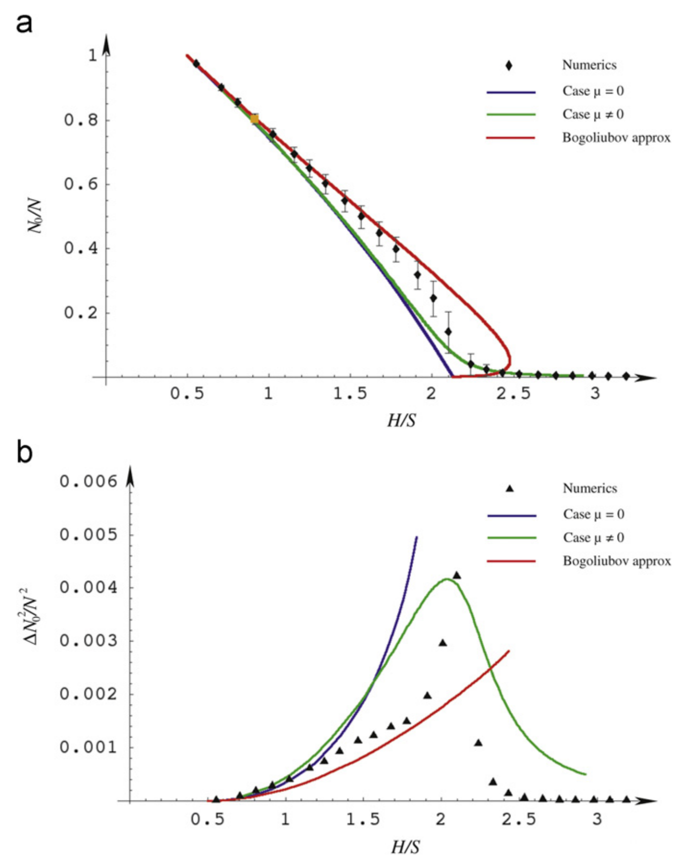

Fig. 2. (a) Wave condensation for the nonlocal NLS Eq. (1): the red line refers to the condensation curve in the presence of a strong condensate (Bogoliubov regime) [from Eq. (15)]. The condensation curve in the presence of a weak condensate is reported in green [from Eqs. (9), (10)], and the corresponding limit $\mu \to 0$ ('thermodynamic limit' [40,41]) in blue [from Eq. (11)]. The symbols refer to the numerical simulation of the NLS Eq. (1). Each point corresponds to an averaging of N_0/N over 60 000 unit lengths once the equilibrium state is reached. The 'error-bars' denote the standard deviation of the fluctuations of N_0/N once equilibrium is established. The spatial discretization of the NLS Eq. (1) is $dx=1$ ($k_c=\pi$), and the number of modes $N_* = 128^2$. The nonlocal response function is given in Eq. (5), with $\sigma=1$. Periodic boundary conditions have been used to integrate numerically the NLS Eq. (1). (b) The variance, $\frac{1}{N^2}(\langle N_0^2\rangle - \langle N_0\rangle^2)$, of the condensate fraction as a function of the total energy of the wave system. As in (a) the lines correspond to the theoretical computations: The green line corresponds to Eq. (16), and the corresponding 'thermodynamic limit' ($-\mu \to 0$) is reported in blue. The red line corresponds to the fluctuations in the Bogoliubov approximation (see the text). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)

nonvanishing chemical potential, $-\mu>0$. Indeed, in analogy with quantum Bose–Einstein condensation, classical wave condensation occurs beyond the thermodynamic limit in two spatial dimensions, *i.e.*, condensation is reestablished thanks to finite size effects. These aspects have been discussed in detail in Refs. [41]. In particular, in the thermodynamic limit ($-\mu\to 0$ in Eq. (9)), one obtains

$$\frac{\langle H(\mu)\rangle}{S}=(1-n_0)\frac{\sum_{\boldsymbol{k}}'\frac{k^2+n_0\hat{V}_{\boldsymbol{k}}}{k^2}}{\sum_{\boldsymbol{k}}'\frac{1}{k^2}}+\frac{1}{2}+\frac{(1-n_0)^2}{2}\frac{\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}'\frac{\hat{V}_{\boldsymbol{k}_1-\boldsymbol{k}_2}}{k_1^2\,k_2^2}}{\left(\sum_{\boldsymbol{k}}'\frac{1}{k^2}\right)^2}. \tag{11}$$

This condensation curve is reported in Fig. 2a (blue line). The energy threshold, $H_{\rm th}$, for wave condensation can be obtained by setting $n_0=0$ in Eq. (11), which gives

$$\frac{\langle H\rangle_{\rm th}}{S}=\frac{N_*-1}{\sum_{\boldsymbol{k}}'\frac{1}{k^2}}+\frac{1}{2}+\frac{1}{2}\frac{\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}'\frac{\hat{V}_{\boldsymbol{k}_1-\boldsymbol{k}_2}}{k_1^2k_2^2}}{\left(\sum_{\boldsymbol{k}}'\frac{1}{k^2}\right)^2}, \tag{12}$$

where N_* denotes the number of modes (degrees of freedom) in the simulation. The value of the energy threshold, $\langle H\rangle_{\rm th}/S$ depends essentially on the ultraviolet wavenumber cut-off k_c, and the number of modes N_*. It is important to note that, in the thermodynamic limit ($N\to\infty$, $S\to\infty$, keeping constant N/S), the condensation curve for $\mu\neq 0$ given by Eqs. (9) and (10) tends to be the condensation curve given in Eq. (11) for $\mu=0$. For this reason, one can say that Eqs. (9) and (10) describe wave condensation 'beyond the thermodynamic limit'.

2.4.2. High condensate fraction: Bogoliubov regime

As the coherence of the initial random wave is increased (*i.e.*, its energy H decreases), the condensate fraction becomes stronger, so that the dynamics enters the nonlinear regime of interaction that invalidates the WT theory (the condition $U/E\ll 1$ is no longer verified). Actually, the existence of the condensate plane-wave changes the nature of the interaction and the Bogoliubov's regime is established: the dynamics turns out to be dominated by the three-wave interaction instead of the four-wave interaction [1]. In this regime, one can still derive a closed relation between the condensate amplitude and the energy into the Bogoliubov basis [40,41].

We briefly sketch here the derivation of the condensation curve in the strong condensation regime following the analysis of Ref. [40]. The equilibrium distribution in the Bogoliubov regime reads

$$n_k^{eq}=\frac{T}{\omega_B(k)}, \tag{13}$$

with the Bogoliubov dispersion relation [74]

$$\omega_B(k)=\sqrt{k^4+2n_0k^2\hat{V}_k}. \tag{14}$$

Note that this dispersion relation can readily be obtained by a linearization of the NLS Eq. (1) around the plane wave solution. The formation of the condensate breaks the conservation of the power in the incoherent wave component [41], which merely explains the absence of the chemical potential in the equilibrium distribution (13). Note that the power of the incoherent component is not conserved because it interacts with the plane-wave condensate, which plays the role of a reservoir of particles. Then the corresponding expressions of the power N and of the averaged energy are obtained by expanding the wave amplitude of the optical wave into the Bogoliubov basis [40,41]. This yields the following expression of the condensation curve into the strong condensation regime

$$\frac{\langle H\rangle}{S}=(1-n_0)\frac{N_*-1}{\sum_{\boldsymbol{k}}'\frac{k^2+n_0\hat{V}_k}{\omega_B^2(k)}}+\frac{1}{2}+\frac{(1-n_0)^2}{2}\frac{\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}'\hat{V}_{\boldsymbol{k}_1-\boldsymbol{k}_2}\left(\frac{k_1^2+n_0\hat{V}_{k_1}}{\omega_B^2(k_1)}\right)\left(\frac{k_2^2+n_0\hat{V}_{k_2}}{\omega_B^2(k_2)}\right)}{\left(\sum_{\boldsymbol{k}}'\frac{k^2+n_0\hat{V}_k}{\omega_B^2(k)}\right)^2}. \tag{15}$$

This condensation curve is reported in Fig. 2 (red line). Its validity is in principle restricted to the strong condensation regime, and, in particular, the hysteresis observed at the transition to condensation ($H\sim H_{\rm th}$) is not physical [41]. Indeed, for a small condensate amplitude, the transition to wave condensation is continuous, as described by the WT theory discussed above through Eqs. (9)–(11).

The existence of the condensation process for the nonlocal NLS Eq. (1) has been confirmed by the numerical simulations. The numerical results have been reported in Fig. 2a. Note that a relatively small number of modes was considered ($N_*=128^2$) in order to compute the condensation curve with a large number of points and, more importantly, in order to perform accurate averages of the condensate fraction and of its fluctuations over a large propagation length z. Indeed, each point was obtained by averaging $n_0(z)$ over 60 000 nonlinear lengths of interaction (L_0), once the equilibrium state has been reached (*i.e.*, once the process of entropy growth has saturated to a constant value, see Fig. 1b). We underline that there is a quantitative agreement between the numerical simulations and the theory, in both regimes of weak and strong condensation, without any adjustable parameters.

2.4.3. Fluctuations

We report in Fig. 2b the variance of the fluctuations of the condensate fraction once the equilibrium state was reached, for different values of the energy H. First of all, we remark that the condensate fraction exhibits stronger fluctuations near by the transition to condensation, *i.e.*, for $H\sim H_{\rm tr}$, while fluctuations are suppressed in the highly condensed Bogoliubov regime, as well as in the uncondensed regime for larger energies (that is for $H/S>2.5$ in Fig. 2a). To assess the fluctuations theoretically, we first remark that the amount of fluctuations of the condensate fraction are identical to the fluctuations of the uncondensed waves. Indeed, by expanding $\Delta N'^2=\langle (N-N_0)^2\rangle-\langle N-N_0\rangle^2$, one readily obtains $\Delta N'^2=\Delta N_0^2$. An estimation of the amount of fluctuations of the 'uncondensed' waves, $\Delta N'^2$, can readily be obtained by following standard equilibrium statistical mechanics [70], which gives

$$\frac{\Delta N_0^2}{N^2}=\frac{\Delta N'^2}{N^2}=\frac{\sum_{\boldsymbol{k}}'\frac{1}{(k^2-\mu)^2}}{\left(\sum_{\boldsymbol{k}}\frac{1}{k^2-\mu}\right)^2}. \tag{16}$$

The corresponding fluctuations in the strong condensation regime can be derived from Eq. (16) with the substitution $\omega(k)\to\omega_B(k)$. We remark in Fig. 2b that there is only a qualitative agreement between the theoretical expressions and the numerical simulations. This preliminary study of the fluctuations of the condensate fraction will be the subject of future investigations. In particular, we plan to derive rigorous mathematical expressions of the fluctuations by using the WT theory in both the weak and strong nonlinear regimes of interactions.

3. Saturable nonlinearity

3.1. NLS model equation

In this Section we study the case of a saturable nonlinearity by following the same presentation as that outlined above for the nonlocal nonlinearity. We consider the standard NLS model equation describing the propagation of on optical wave in a saturable nonlinear medium [35]

$$i\partial_z A = -\nabla^2 A + g(|A|^2)\, A, \qquad (17)$$

where $g(|A|^2)$ is the nonlinear law of the intensity of the optical wave, $I=|A|^2$. Although $g(|A|^2)$ may have different forms, we consider here the concrete example of a rational nonlinear saturation which is encountered in many physical situations

$$g(|A|^2) = \frac{\rho|A|^2}{1+\rho|A|^2}. \qquad (18)$$

The normalization adopted here is the same as that considered in the nonlocal NLS Eq. (1), except that because of the saturation of the nonlinearity, the nonlinear coefficient γ has the units of the inverse of a length, so that the propagation length z is normalized with respect to $L_0 = 1/\gamma$. With these normalized units, the dimensionless parameter $\rho = \overline{I}/I_s$ denotes the ratio between the average intensity and the saturation intensity of the nonlinear material. For $\rho \gg 1$ the system becomes essentially linear, while for $\rho \ll 1$ one recovers a pure Kerr nonlinearity, so that the interesting new physics due to the saturable nonlinearity arises for $\rho \sim 1$.

As for the nonlocal NLS equation, Eqs. (17) and (18) conserve the total power of the wave

$$N = \int |A|^2\, d\boldsymbol{r} \qquad (19)$$

and the total energy (Hamiltonian), $H = E + U$, whose linear kinetic contribution is

$$E = \int |\nabla A|^2\, d\boldsymbol{r}, \qquad (20)$$

and the nonlinear contribution is

$$U = \int \left(|A|^2 - \frac{1}{\rho}\ln(1+\rho|A|^2) \right) d\boldsymbol{r}. \qquad (21)$$

Note that the NLS Eq. (17) for a generic nonlinear law $g(I)$ also exhibits a Hamiltonian structure (see footnote 2).

3.2. Condensate fraction

As discussed above in the case of the nonlocal nonlinearity in Section 2.4, we consider here separately the regimes of small and strong condensation.

3.2.1. Small condensate fraction: WT regime

The theory of wave condensation developed for the cubic Kerr nonlinearity can be extended to the case of a saturable nonlinearity. One obtains the following coupled equations between the condensate fraction and the averaged energy

$$\frac{\langle H(\mu)\rangle}{S} = (1-n_0)\frac{\sum_{\boldsymbol{k}}' \frac{k^2}{k^2-\mu}}{\sum_{\boldsymbol{k}}' \frac{1}{k^2-\mu}} + \frac{\langle U(n_0)\rangle}{S}, \qquad (22)$$

$$\frac{N_0(\mu)}{N} = \frac{1}{-\mu}\frac{1}{\sum_{\boldsymbol{k}} \frac{1}{k^2-\mu}}. \qquad (23)$$

As for the nonlocal nonlinearity, the difference with the cubic Kerr nonlinearity relies on the computation of the averaged nonlinear energy $\langle U(n_0)\rangle$ in Eq. (22). This calculation can be performed under the assumption that the incoherent contribution to the wave amplitude, $\delta A = A - \sqrt{n_0}$, exhibits a Gaussian statistics. Then we use the standard transformation rules [75] to calculate the probability density function (PDF) of the wave intensity $I = |\sqrt{n_0} + \delta A|^2$, which gives

$$f_I(I) = \theta(I)\frac{\exp\left(-\frac{I+n_0}{1-n_0}\right)}{1-n_0} I_0\left(\frac{2\sqrt{n_0 I}}{1-n_0}\right), \qquad (24)$$

where $\theta(x)$ is the Heaviside function and $I_0(x)$ is the modified Bessel function of zero-th order. Note that in the limit $n_0 \to 0$, Eq. (24) reduces to the exponential PDF which characterizes a Gaussian field, $f_I(I) = \theta(I)\exp(-I)$. In the opposite limit, $n_0 \to 1$, one obtains $f_I(I) = \delta(I-1)$, as expected for a homogeneous plane-wave solution. According to (24), the averaged nonlinear energy reads

$$\frac{\langle U(n_0)\rangle}{S} = 1 - \frac{1}{\rho(1-n_0)}\int_0^\infty \ln(1+\rho I)\exp\left(-\frac{I+n_0}{1-n_0}\right) I_0\left(\frac{2\sqrt{n_0 I}}{1-n_0}\right) dI. \qquad (25)$$

Eqs. (22), (23) and (25) form a closed set of equations, which provides the condensation curve in the weakly nonlinear regime.[2] It is reported in Fig. 3a (green line) through a parametric plot of Eqs. (22) and (23) with respect to the variable μ. The corresponding thermodynamic limit of the condensation curve is obtained by taking the limit $-\mu \to 0$ in Eq. (22)

$$\frac{\langle H(n_0)\rangle}{S} = (1-n_0)\frac{N_* - 1}{\sum_{\boldsymbol{k}}' \frac{1}{k^2}} + \frac{\langle U(n_0)\rangle}{S}, \qquad (26)$$

where $\langle U(n_0)\rangle$ is given in Eq. (25). This expression of the condensation curve is reported in Fig. 3a (blue line). The energy threshold for wave condensation is obtained by imposing $n_0 = 0$ in Eq. (26), which gives

$$\frac{\langle H\rangle_{\rm th}}{S} = \frac{N_* - 1}{\sum_{\boldsymbol{k}}' \frac{1}{k^2}} + 1 - \frac{1}{\rho}\int_0^\infty \ln(1+\rho I)\exp(-I)\, dI. \qquad (27)$$

Note that, as for the nonlocal nonlinearity, the condensation curve for $\mu \neq 0$ given by Eqs. (22) and (23) tends to be the condensation curve given in Eq. (26) for $\mu = 0$ in the thermodynamic limit. Also, as in the case of a nonlocal nonlinearity [Eq. (12)], Eq. (27) depends on the frequency cut-off (k_c) and the total numbers of modes (N_*).

3.2.2. High condensate fraction: the Bogoliubov regime

The WT theory in the Bogoliubov regime reported in [40,41] for the pure Kerr nonlinearity can be extended to take into account the saturable nonlinearity considered here. Note however that caution should be exercised in the procedure of diagonalization of the quadratic Hamiltonian in [40].[3] Following a procedure similar to that reported in Ref. [40], one can derive the

[2] Note that, for a generic nonlinear law $g(I)$, the nonlinear energy reads, $U = \int G(I)\, d\boldsymbol{r}$, with $G(I) = \int_0^I g(x)\, dx$. The corresponding averaged nonlinear energy thus takes the form, $\langle U(n_0)\rangle/S = \int_0^I G(I) f_I(I)\, dI$.

[3] With the notations of [40], one needs to remove to $\langle U(n_0)\rangle$ the term which is necessary to diagonalize the quadratic Hamiltonian H_2, namely the term $\rho n_0(1-n_0)/(1+\rho n_0)^2$.

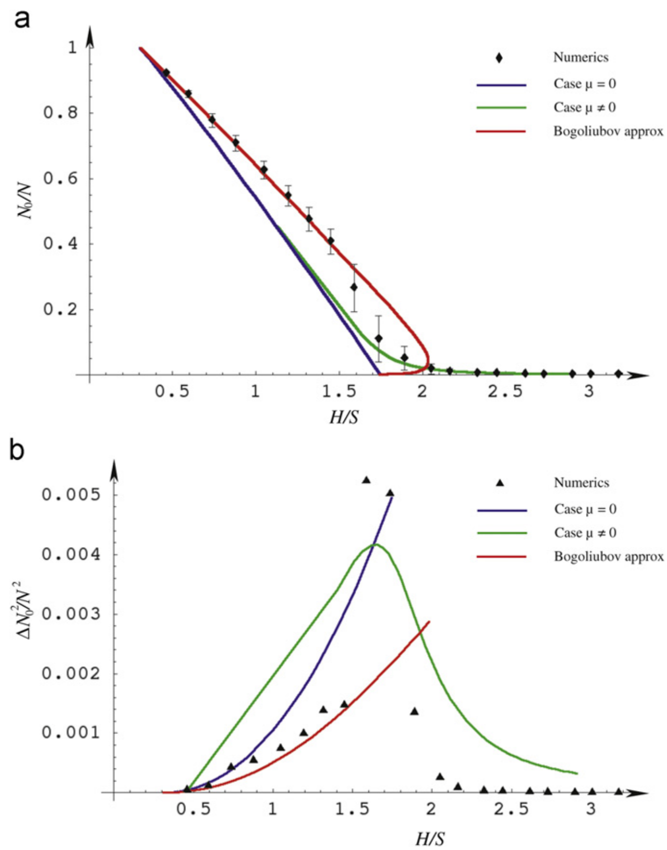

Fig. 3. (a) Wave condensation for the saturable NLS Eqs. (17), (18): the red line refers to the condensation curve in the presence of a strong condensate (Bogoliubov regime) [from Eq. (28)]. The condensation curve in the presence of a weak condensate is reported in green [from Eqs. (22)–(25)], and the corresponding 'thermodynamic limit' $\mu \to 0$ in blue [from Eq. (26)]. The points refer to the numerical simulation of the NLS Eqs. (17), (18). Each point corresponds to an averaging of N_0/N over 40 000 unit lengths once the equilibrium state is reached. The error-bars denote the standard deviation of the fluctuations of N_0/N once equilibrium is established. The spatial discretization of the NLS Eqs. (17), (18) is $dx=1$ ($k_c=\pi$), $\rho=1$, the number of modes is $N_*=128^2$. (b) The variance, $\frac{1}{N^2}(\langle N_0^2\rangle - \langle N_0\rangle^2)$, of the condensate fraction as a function of the total energy of the wave system. As in (a) the lines correspond to the theoretical computations: The green line corresponds to Eq. (16), and the corresponding 'thermodynamic limit' $(-\mu \to 0)$ is reported in blue. The red line corresponds to the fluctuations in the Bogoliubov approximation. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)

condensation curve in the strong condensation regime

$$\frac{\langle H \rangle}{S} = \frac{\langle U(n_0)\rangle}{S} - \frac{\rho n_0(1-n_0)}{(1+\rho n_0)^2} + \frac{(1-n_0)(N_*-1)}{\sum_k' \dfrac{k^2+\rho n_0/(1+\rho n_0)^2}{\omega_B^2(k)}}, \tag{28}$$

where the averaged nonlinear energy $\langle U(n_0)\rangle$ is given by Eq. (25), and $\omega_B(k)$ denotes the Bogoliubov dispersion relation

$$\omega_B(k) = \sqrt{k^4 + \frac{2\rho n_0}{(1+\rho n_0)^2}k^2}. \tag{29}$$

The condensation process for the saturable NLS Eqs. (17), (18) has been confirmed by numerical simulations. The numerical results concerning the condensation curve have been reported in Fig. 3a. Each point refers to an averaging of the condensate fraction, $n_0(z)$, over 40 000 unit lengths, once the equilibrium state has been reached. As for the nonlocal nonlinearity discussed above, we underline that a quantitative agreement between the numerical simulations and the theory has been obtained in both regimes of weak and strong condensation, and without any adjustable parameter.

Note that for completeness, we have also reported in Fig. 3b the amount of fluctuations in the condensate fraction once equilibrium was reached. As for the nonlocal wave system discussed above in Section 2.4.3, we remark that fluctuations are pronounced near by the transition to condensation $H \sim H_{\rm tr}$, while they are suppressed far away from the transition. We leave for future works the rigorous theoretical treatment of the fluctuations in the framework of the WT theory.

4. Discussion and conclusion

In summary, we have shown theoretically and numerically that the NLS equation accounting for a nonlocal or a saturable nonlinearity describes a process of wave condensation. The condensation curves derived analytically in both the small and strong condensation regimes have been found in quantitative agreement with the numerical simulations, without adjustable parameters. It is interesting to briefly comment the influence of the range of a nonlocal interaction on the process of thermalization. The simulations reported here have been performed in the 'short-range' regime, *i.e.*, the regime in which the nonlocality of the interaction is of the same order as the healing length ($\sigma = 1$ in Figs. 1 and 2).

However, when the system enters a regime of long-range interaction ($\sigma \gg 1$), the situation may change in two interesting respects.

(i) As the parameter σ increases, the Bogoliubov dispersion relation is no longer a monotonic growing function, but exhibits a minimum at $k_0 \sim 1/\sigma$. The shape of the dispersion relation reminds the well-known Landau 'roton' spectrum of liquid helium [76]. In this case, the dynamics of the NLS Eq. (1) may display in the low energy limit a different behavior, because the lowest energy state is no longer a homogeneous plane-wave, but a spatially modulated intensity pattern [77]. We shall discuss this phenomenon of 'crystallization' in a separate article. Note however that for a Gaussian- (exponential-)shaped kernel (5) considered here, the system does not seem to exhibit a crystallization process. Actually, in 1D a necessary and sufficient condition for the formation of a crystal is that the Fourier transform of the response function ($\hat{V}_{\boldsymbol{k}}$) changes sign in a frequency interval [77–79], while the problem is more complicate in 2D and 3D [80].

(ii) The highly nonlocal regime ($\sigma \gg 1$) has been recently considered in the focusing regime of interaction [34]. To comment this aspect, it proves to be convenient to first discuss the meaning of wave condensation with a focusing nonlinearity. In the focusing regime, the fundamental condensate state refers to the soliton state, which is the state that realizes the minimum of the Hamiltonian, in the same way as the homogeneous plane-wave realizes the minimum of the Hamiltonian in the defocusing regime. Wave condensation thus manifests itself by the spontaneous emergence of a coherent soliton that remains immersed in a sea of small-scale fluctuations – a process known as 'soliton turbulence' [18,36,81,82]. Note that this process can survives in 2D thanks to a nonlocal nonlinearity, which is known to stabilize soliton collapse [65,66]. It is important to note that this soliton turbulence process breaks down as the range of the nonlocal response increases ($\sigma \gg 1$). Indeed, a highly nonlocal response is responsible for a process of 'incoherent soliton turbulence', which is characterized by the spontaneous formation of an *incoherent* soliton structure, instead of a coherent soliton state. This process of incoherent soliton turbulence can be described in detail in the framework of a long-range Vlasov-like kinetic equation [34]. This indicates that a highly nonlocal nonlinearity significantly slows down the process of thermalization, in analogy with recent studies developed in the framework of long-range interacting systems (in the mean field approximation) [83]. Also note that in the limit of a highly nonlocal nonlinear interaction, the range of the response function can be much larger than the width of the optical beam, so that the dynamics of the system turns out to be ruled by an effectively linear Schrödinger equation, which cannot describe wave thermalization [68]. The slowing down of the thermalization process due to a highly nonlocal response is an important phenomenon which will be the subject of future investigations [84], in relation with the condensation process discussed here and the 'exact' statistical description of the random nonlinear wave provided by the long-range Vlasov equation.

Note that we have not considered here the propagation of optical waves in materials characterized by a noninstantaneous response. Indeed, it has been shown that the causality property inherent to the noninstantaneous response function prevents the thermalization process to take place. More precisely, the WT theory reveals in this case that the relevant kinetic equation describing wave propagation in noninstantaneous nonlinear media has a form analogous to the weak Langmuir turbulence equation considered in plasma physics [85]. This kinetic equation supports the existence of spectral incoherent solitons: Contrary to conventional incoherent solitons [86,87], spectral incoherent solitons do not exhibit a localization in the spatio-temporal domain, but solely in the spectral domain [3,88]. It turns out that, instead of the expected processes of thermalization and condensation, an incoherent optical wave propagating in noninstantaneous nonlinear media self-organizes into spectral incoherent solitons, which can thus be viewed as nonequilibrium stable states of the incoherent field.

An interesting aspect which deserves to be studied in future works concerns the role of vortices in the process of wave condensation in the presence of a nonlocal nonlinearity. Vortices can exhibit intricate dynamics, *e.g.*, vortices of opposite circulation can annihilate by pairs, but they may also exhibit complex motions analogous to those known in 2D-Euler systems (for a review see [89]). In particular, two close like-signed vortices can rotate one around each other, while vortices of opposite signs (vortex dipole) can move along a straight line perpendicular to the line connecting the vortex pair. In the strong condensation regime ($H \ll H_{tr}$), the general idea is that vortices affect the nonequilibrium transient process leading to the formation of the condensate, while in the final stage, when all vortices have annihilated, the system comprises a coherent uniform component immersed in a sea of turbulent fluctuations [53–58]. It would be interesting to study the role of vortices in wave condensation, in relation with the new physics introduced by a nonlocal nonlinearity which is known, *e.g.*, to significantly alter the dynamics of dark–soliton interactions [67].

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