# Exact hairy black holes and their modification to the universal law of gravitation

Andrés Anabalón[1] and Julio Oliva[2]

[1]*Departamento de Ciencias, Facultad de Artes Liberales y Facultad de Ingeniería y Ciencias, Universidad Adolfo Ibáñez, Viña del Mar, Chile*

[2]*Instituto de Ciencias Físicas y Matemáticas, Universidad Austral de Chile, Valdivia, Chile*

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In this paper two things are done. First, it is pointed out the existence of exact asymptotically flat, spherically symmetric black holes when a self-interacting, minimally coupled scalar field is the source of the Einstein equations in four dimensions. The scalar field potential is recently found to be compatible with the hairy generalization of the Plebanski-Demianski solution of general relativity. This paper describes the spherically symmetric solutions that smoothly connect the Schwarzschild black hole with its hairy counterpart. The geometry and scalar field are everywhere regular except at the usual Schwarzschild-like singularity inside the black hole. The scalar field energy momentum tensor satisfies the null-energy condition in the static region of spacetime. The first law holds when the parameters of the scalar field potential are fixed under thermodynamical variation. Second, it is shown that an extra, dimensionless parameter, present in the hairy solution, allows to modify the gravitational field of a spherically symmetric black hole in a remarkable way. When the dimensionless parameter is increased, the scalar field generates a flat gravitational potential that, however, asymptotically matches the Schwarzschild gravitational field. Finally, it is shown that a positive cosmological constant can render the scalar field potential convex if the parameters are within a specific rank.

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## I. INTRODUCTION

It is already more than forty years since Wheeler's original conjecture that black holes have no hair. Although there was an intensive analysis on this claim (for a review and references see Ref. [1]), the situation, for the minimally coupled scalar field, was clarified in the 1990s. When the spacetime is asymptotically flat there are two theorems that strongly constrain the existence of black holes when the energy momentum satisfies the dominant-energy [2] and the weak-energy conditions [3]. When just the null-energy condition is required there is numerical evidence on the existence of asymptotically flat four-dimensional black holes [4]. There is also a very general no-hair theorem in Brans-Dicke theories which satisfy the weak energy condition in the Einstein frame [5].

It is particularly timely to consider the astrophysical relevance of this problem. Actually, it has been pointed out that the angular and quadrupolar momentum, $J$ and $Q$, respectively, of the black hole located at the center of our galaxy, SgrA*, can be determined by the orbital precession of stars very near to it, therefore allowing to check the relation $Q = -\frac{J^2}{Mc}$ that follows from the Kerr solution [6,7]. Due to the uniqueness and no-hair theorems of asymptotically flat, four-dimensional, general relativity this would test the experimental validity of the hypothesis that they involve.

In an attempt to construct a hairy rotating black hole, an exact family of stationary and axisymmetric Petrov type D spacetimes was found when either a (non)minimally coupled scalar field or a generic nonlinear sigma model is the source of the Einstein equations [8]. The work [8] considered a cohomogeneity two Weyl rescaling of the Carter-Debever ansatz, which when replaced in the vacuum Einstein equations contains all the Ricci flat—Petrov type D—spacetimes [9]. Within this ansatz it was found, on-shell, the most general scalar field potential compatible with the Einstein equations for an arbitrary nonlinear sigma model. While the form of the spacetime metric is independent of the scalar manifold metric, the form of the off-shell potential is relative to the sigma model, and for the single scalar field is presented here for the first time. This paper provides a discussion on some physical implications that follow from the existence of these hairy black holes.

The simplest case of a single, minimally coupled, scalar field is very interesting. Since the Lagrangian has no continuous symmetry, is not possible to associate it with a conserved current, and therefore is a natural candidate to be a dark matter component. The only source for the scalar field can be its self-interaction. The analysis of Ref. [8] allows to find, in this case, a generic potential compatible with the Einstein equations:

$$\begin{aligned}V(\phi) = \frac{\alpha}{\nu^2\kappa}\bigg(&\frac{\nu-1}{\nu+2}\sinh((1+\nu)\phi l_\nu)\\&+\frac{\nu+1}{\nu-2}\sinh((1-\nu)\phi l_\nu)+4\frac{\nu^2-1}{\nu^2-4}\sinh(\phi l_\nu)\bigg)\\&+\frac{\Lambda(\nu^2-4)}{6\kappa\nu^2}\bigg(\frac{\nu-1}{\nu+2}e^{-(\nu+1)\phi l_\nu}+\frac{\nu+1}{\nu-2}e^{(\nu-1)\phi l_\nu}\\&+4\frac{\nu^2-1}{\nu^2-4}e^{-\phi l_\nu}\bigg).\end{aligned} \tag{1}$$

This is the most general potential allowing for uncharged, stationary and axisymmetric, Petrov type D solutions of the

 

Plebanski-Demianski form. Here $\kappa = 8\pi G$, where $G$ is the Newton constant, $l_\nu = (\frac{2\kappa}{\nu^2-1})^{\frac{1}{2}}$, $\Lambda$ is the cosmological constant, and $\alpha$, $\nu$ are parameters of the potential. This potential still makes sense when $\Lambda = 0$ and, as is shown in this paper, the spherically symmetric solutions of the Einstein equation are continuous deformations of the Schwarzschild black hole. Therefore, an interesting result of this paper is to single out the first exact, uncharged, asymptotically flat, black hole with everywhere regular geometry and a single scalar field except at the usual Schwarzschild singularity.

When $\Lambda = 0$, the existence of these modifications to the physically relevant Schwarzschild spacetime are pertinent to the motion of test particles following geodesics in this geometry. It is important to stress that no difference is expected to arise in the gravitational field outside of a star since, in these cases, a discontinuity appears in the derivative of the gravitational field at the surface of the star, and the lack of a conserved current for the scalar field make this discontinuity incompatible with its existence in the first place. The modified gravitational field introduced and discussed in this paper is relevant as a model for black holes. In this case, the dimensionless parameter $\nu$, plays a very important role in setting the strength of the gravitational field. Actually, for large enough values of the parameter $\nu$, the strength of the gravitational field of a hairy black hole can be made essentially flat all the way from its surface up to regions as far from the location of the event horizon as the model would require. Moreover, the fact that the parameter $\nu$ does not enter in the Komar mass allows to introduce an extra parameter in the gravitational field of a black hole, providing a new astrophysical tool to fit the measured gravitational field of any black hole configuration to this exact, analytical model, derived from general relativity. The existence of scalar fields has been already considered to be relevant to stellar kinematics [10], and it has been noted that these models will be experimentally tested in the future gravitational wave measurements [11].

To close this introduction, we would like to acknowledge that the fact that it is possible to relax the boundary conditions for gravitating scalar fields in anti-de Sitter spacetime [12] is what fueled the expectation that exact solutions of this system should exist [13]. However, it turns out that the black holes of Ref. [8] still make sense when the cosmological constant vanishes.

The outline of the paper is as follows: in the Sec. II the solutions and the potential are described, special limits are described, and the Eddington-Finkelstein coordinates are introduced. Section III describes the existence of two kinds of solutions in this hairy black hole family and the location of the horizon. Section IV is devoted to the computation of the Komar mass and how it satisfies the first law of black hole thermodynamics. Section V describes the behavior of the gravitational potential that a geodesic test particle feels in this background, the dimensionless parameter $\nu$ sets the strength of it. Finally, some remarks are made in Sec. VI.

The notation follows Ref. [14]. The conventions of curvature tensors are such that a sphere in an orthonormal frame has positive Riemann tensor and scalar curvature. The metric signature is taken to be $(-, +, +, +)$. Greek letters are in the coordinate tangent space. Since we set $8\pi G = \kappa$ and $c = 1 = \hbar$, the gravitational constant has units of length squared $[\kappa] = L^2$.

## II. THE EXACT HAIRY BLACK HOLES

As discussed in the introduction, this paper studies the first exact, asymptotically flat, solutions of the classical model

$$S(g,\phi) = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \qquad (2)$$

with field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}, \qquad (3)$$

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\alpha\phi\partial_\beta\phi g^{\alpha\beta} - g_{\mu\nu}V(\phi), \qquad (4)$$

$$V(\phi) = \frac{\alpha}{\nu^2\kappa}\left(\frac{\nu-1}{\nu+2}\sinh((1+\nu)\phi l_\nu) + \frac{\nu+1}{\nu-2}\sinh((1-\nu)\phi l_\nu) + 4\frac{\nu^2-1}{\nu^2-4}\sinh(\phi l_\nu)\right). \qquad (5)$$

The following configurations are exact solutions of this system [8]:

$$ds^2 = \Omega(r)\left(-F(r)dt^2 + \frac{dr^2}{F(r)} + d\Sigma^2\right), \qquad \frac{\nu^2\eta^{\nu-1}r^{\nu-1}}{(r^\nu - \eta^\nu)^2}, \qquad (6)$$

$$F(r) = \frac{r^{2-\nu}\eta^{-\nu}(r^\nu - \eta^\nu)^2}{\nu^2} + \left(\frac{1}{(\nu^2-4)} - \left(1 + \frac{\eta^\nu r^{-\nu}}{\nu-2} - \frac{\eta^{-\nu}r^\nu}{\nu+2}\right)\frac{r^2}{\eta^2\nu^2}\right)\alpha,$$
$$\phi = l_\nu^{-1}\ln(r\eta^{-1}), \qquad (7)$$

where $l_\nu = (\frac{2\kappa}{\nu^2-1})^{\frac{1}{2}}$ and $d\Sigma^2$ is the line element of a unit two-sphere. $\eta$ is the only integration constant of the black hole. The solution and theory are invariant under the transformation $\nu \to -\nu$. The energy momentum of the scalar field, in a comoving tetrad, has the form $T^{ab} = \mathrm{diag}(\rho, p_1, p_2, p_2)$ and, in the static regions of spacetime, defined by $F(r) > 0$, satisfies the null-energy condition.

It is easy to see from the form of the metric, and without any reference to the details of the solution itself, that it is possible to introduce Eddington-Finkelstein coordinates $u_\mp = t \pm \int \frac{dr}{F(r)}$ that allows to cover either the black hole $(u_-)$ or the white hole $(u_+)$. The asymptotically flat solution has a single horizon from which it follows that the

Penrose diagram is the same as for the Schwarzschild black hole.

In the hairless limit, $\nu = 1$, the change of coordinates $r = \eta - \frac{1}{y}$ brings the hairy solution (6) and (7) to the familiar Schwarzschild black hole

$$ds^2 = -\left(1 - \frac{2M}{y}\right)dt^2 + \frac{dy^2}{1 - \frac{2M}{y}} + y^2 d\Sigma, \qquad (8)$$

where $M = \frac{3\eta^2+\alpha}{6\eta^3}$.

From the form of the metric functions, it follows from these expressions that the leading behavior of the metric is the same than the Schwarzschild solution with the radial coordinate given by $\rho = \eta - \frac{1}{r}$. As is discussed in Sec. III, the solution (6) and (7) are actually two different solutions, depending whether $0 > r > \eta$ or $\eta > r > \infty$, where $r = 0$ or $r = \infty$ are the locations of the singularities, where the rank of the coordinate $r$ is defined in a Eddington–Finkelstein-like patch such that the horizon is smoothly covered.

The cases with $\nu = 2$ and $\nu = \infty$ are special, however, these values can be treated by a simple limiting procedure. The solution with $\nu = 2$ plus the corresponding limit of the part which is proportional to the cosmological constant of (1) was considered in the context of the existence of topological AdS black holes in Ref. [15]. Along the same lines, the limit when $\nu = \infty$ recovers the de Sitter black hairy black holes compatible with inflation was considered in Ref. [16].

## III. THE TWO BRANCHES AND THE LOCATION OF THE HORIZON

One can notice that the potential (5) has a different behavior at $\phi = \pm\infty$. From this observation it follows that there are two solutions, depending on the branch that one is considering. For further analysis is better to parameterize the solution with the dimensionless coordinate $x = \frac{r}{\eta}$, such that now the asymptotic region is at $x = 1$. The convention of this paper is such that the solution with $x > 1$ is defined as the positive branch while the negative branch is the one defined for $x < 1$.

The form of the lapse function makes very hard the task of finding the exact location of the horizon. However, there is a simple argument that gives a necessary condition for its existence, which is the change of sign of the lapse function $\Omega(x)F(x)$; since $\Omega(x)$ is positive definite is enough to focus on $F(x)$. It can be noticed that $\Omega(x = 1)F(x = 1) = 1$, and it should be remembered that $x = 1$ is the location of infinity. Now, it follows that, when $\nu < 2$, $F(x)$ is regular at the origin and takes the value

$$F_{\nu<2}(x = 0) = \frac{\alpha}{\nu^2 - 4}. \qquad (9)$$

When $\nu > 2$, the expansion of $F_\nu(x)$ around $x = 0$ diverges as

$$F_{\nu>2}(x = 0) = \frac{(\eta^2(\nu - 2) - \alpha)x^{2-\nu}}{\nu^2(\nu - 2)}. \qquad (10)$$

Indeed, the region $x \in (0, 1)$ corresponds to the negative branch. The positive branch is in the region $x \in (1, \infty)$ and $F(x)$ has the following asymptotic expansion:

$$F_\nu(x = \infty) = \frac{(\eta^2(\nu + 2) + \alpha)x^{2+\nu}}{\nu^2(\nu + 2)}. \qquad (11)$$

When $\nu = 2$, then there is a logaritmic divergence at $x = 0$

$$F_2(x=0) = \alpha \ln(x), \quad F_2(x=\infty) = \left(\frac{\eta^2}{4} + \frac{\alpha}{16}\right)x^4. \qquad (12)$$

When $\nu = \infty$, then

$$F_\infty(\rho=0) = \frac{\eta^2 - \alpha}{\rho}, \quad F_\infty(\rho=0) = (\eta^2 + \alpha)\rho. \qquad (13)$$

The conditions (9)–(11) are enough to analyze if the lapse function vanishes a single time in a given interval. In the case that there is more than one horizon more information is necessary. A necessary condition for the existence of more than one horizon is the vanishing of the derivative of $F_\nu(x)$ in the interval of interest. This condition can be easily solved and the existence of black holes can be classified as follows:

(i) When $\nu \le 2$, the negative branch has black holes if $\alpha > 0$ and the positive branch if $\alpha < -\eta^2(\nu + 2)$.

(ii) When $\infty > \nu > 2$, the negative branch has black holes if $\eta^2(\nu - 2) < \alpha$ and the positive branch if $\alpha < -\eta^2(\nu + 2)$.

(iii) When $\nu = \infty$, the negative branch has black holes if $\eta^2 < \alpha$ and the positive branch if $\alpha < -\eta^2$.

Two main conclusions can be extracted from this analysis:

(1) There are black holes for an open set in the parameter space.

(2) If a black hole solution is realized in one of the branches, then, automatically, the other branch contains a naked singularity.

## IV. THE MASS AND THE FIRST LAW

The computation of the Komar mass is straightforward. The result is given by

$$M_- = \frac{3\eta^2 + \alpha}{6\eta^3 G}. \qquad (14)$$

The subscript $(-)$ indicates that this is the mass for the region $x < 1$. The change in the orientation of the outward normal implies that the positive branch has mass $M_+ = -M_-$.

The entropy is one quarter of the area

$$S = \frac{A}{4G} = \frac{\pi\Omega(r_+)}{G} = \frac{\pi}{G}\frac{\nu^2 x_+^{\nu-1}}{\eta^2(x_+^\nu - 1)^2}, \qquad (15)$$

where $x_+ = \frac{r_+}{\eta}$ is the solution to $F(x_+) = 0$. The temperature is defined to make the Euclidean continuation smooth

$$T_\pm = \pm \frac{\eta}{4\pi} \frac{x_+(1 - x_+^\nu)((\nu - 2)(1 + x_+^{\nu+2}) + (\nu + 2)(x_+^\nu - x_+^2))}{(x_+^{\nu+2}(\nu^2 - 4) - \nu^2 x_+^\nu + (\nu + 2)x_+^2 + (2 - \nu)x_+^{2\nu+2})}, \tag{16}$$

where the subscript $\pm$ indicates the temperatures for the black holes defined by the positive and negative branch, respectively. With these results at hand it is straightforward to check that

$$\delta M_K = T\delta S, \tag{17}$$

where, the variation let the parameters of the Lagrangian, $\alpha$ and $\nu$, fixed. All these quantities have smooth limits when $\nu = 2$.

## V. THE GRAVITATIONAL FIELD

A test particle moving in this gravitational field satisfies the geodesic equation $(\Omega(r)\dot{r})^2 + F(r)L^2 + \Omega(r)F(r) = E^2$, where $L = \Omega(r)\dot{\phi}$ and $E = \Omega(r)F(r)\dot{t}$ are conserved quantities and the dot stands for the derivative respect to the geodesic affine parameter. If the coordinate $\dot{\rho} = \Omega(r)\dot{r}$ is introduced it is clear that the $\frac{\Omega(r)F(r)-1}{2}$ term is the actual gravitational potential that a test particle feels on this background.

Therefore, to understand the departure of the geodesic motion from the expected from the Schwarzschild black hole ($\nu = 1$), it is enough to compare the minus of the lapse function, $-g_{tt} = \Omega(x)F(x)$, for different values of $\nu$. It is important to remark that all the corrections of the geodesic motion in the Schwarzschild geometry to the Newtonian behavior are proportional to the angular momentum of the particle. The lapse function in the $\nu = 1$ case is just the Keplerian potential.

### A. Different gravitational fields for the same black hole area

To model the same black hole in the different theories, defined by the different values of $\alpha$ and $\nu$, it is necessary to fix its area. Thus, let us fix it as follows:

$$\Omega(x_+, \eta, \nu) = \rho_+^2 \Rightarrow \eta = \frac{|\nu| x_+^{\frac{\nu-1}{2}}}{\rho_+ |x_+^\nu - 1|}, \tag{18}$$

where $\rho_+$ defines the area of the horizon to be $4\pi\rho_+^2$ and $x_+$ is the location of the horizon defined by $F(x_+) = 0$. This definition of $\eta$ allows to write $\Omega$ as

$$\Omega(x, x_+, \rho, \nu) = \rho_+^2 \frac{x^{\nu-1}(x_+^\nu - 1)^2}{(x^\nu - 1)^2 x_+^{\nu-1}}. \tag{19}$$

The definition of the location of the horizon can be used to find the value of the parameter $\alpha$ in terms of $x_+$ and $\eta$ and $\nu$. Finally, $\eta$ and $\alpha$ can be replaced in the negative of the lapse function to obtain $F(x)\Omega(x)$ as a function of $x$,

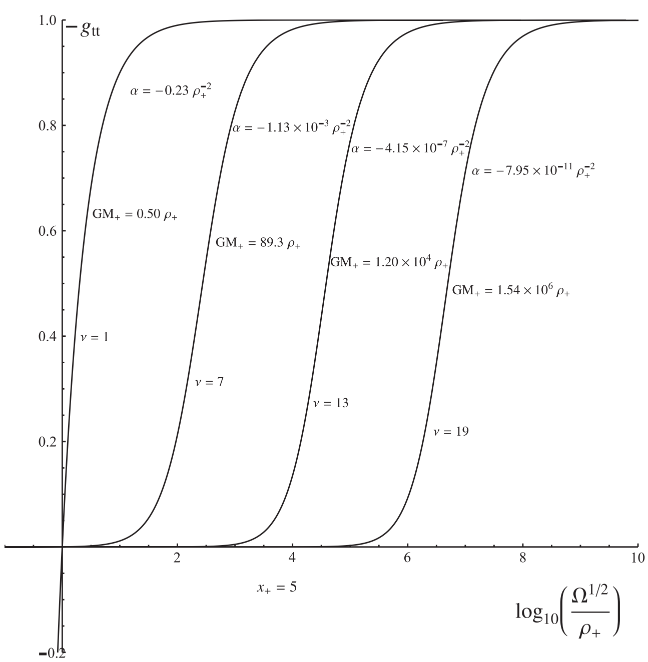


FIG. 1. The positive branch. $\Omega(x)F(x)$ for a fixed horizon area. The horizontal axis is $\log_{10}(\sqrt{\Omega(x)}/\rho_+)$. This function is the analogue of the logarithm of the radial coordinate in the Schwarzschild geometry, normalized to be zero at the horizon.

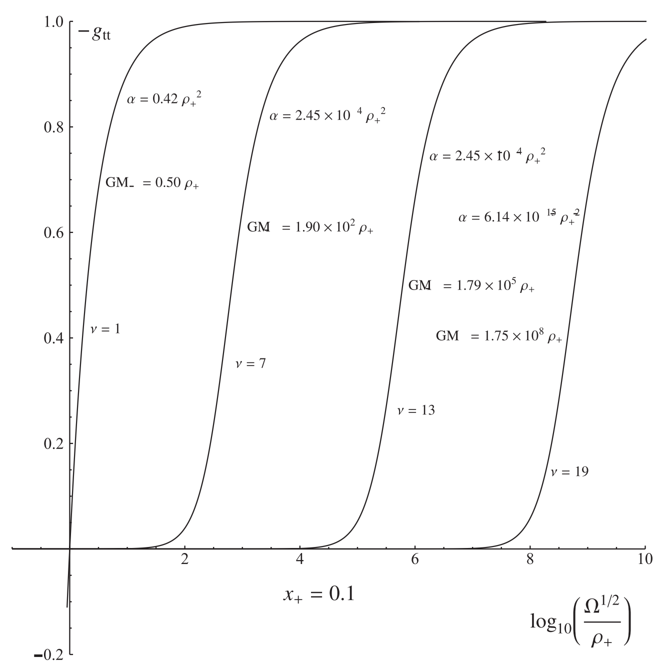


FIG. 2. The negative branch. $\Omega(x)F(x)$ for a fixed horizon. The horizontal axis is $\log_{10}(\sqrt{\Omega(x)}/\rho_+)$. This function is the analogue of the logarithm of the radial coordinate in the Schwarzschild geometry, normalized to be zero at the horizon.

$x_+$, and $\nu$. These graphs are independent of $\rho_+$. They depend on $x_+$ through the relation of the Komar mass and $\rho_+$ and the relation of $\alpha$ and $\rho_+^{-2}$.

The fact that the horizontal axis is on a logarithmic scale shows how flat the gravitational field can be made in the presence of the scalar field (Figs. 1 and 2). This effect is a purely general relativistic effect. It is very interesting to note that the second derivative of the scalar field potential is zero at $\phi = 0$, and therefore the scalar field is massless at infinity. However this massless scalar, due to the nonlinear structure of the Einstein equations, has a strong effect in the gravitational field; it smoothly enhances the gravitational field from the Schwarzschild solution to large deviations from it.

## VI. FINAL REMARKS

Before this paper there was only one, exact, uncharged, spherically symmetric black hole in four-dimensional general relativity; now there is an infinite family of them. Indeed, we make reference to configurations that are everywhere regular except at the usual Schwarzschild singularity.

As follows from the analysis of the lapse function, the scalar field allows to have much larger than Schwarzschild amounts of mass in a given region of spacetime. If this scalar field would actually exist, relatively small black holes can be much more massive than one would expect based on the Schwarzschild solution. These smaller but massive black holes strongly modify the universal law of gravity not only in its near surroundings but also in arbitrarily far regions from their location. It is tempting to conjecture that these geometries can be useful to describe the geodesic motion of actual test particles in an astrophysical situation, like the well-known flat galactic rotation curves. However, this can be considered as a serious possibility only after a study of the stability of these solutions is made. This, indeed, will be addressed in a future publication.

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