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Decompositions for the edge colouring of reduced indifference graphs

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Abstract

The chromatic index problem—finding the minimum number of colours required for colouring the edges of a graph—is still unsolved for indifference graphs, whose vertices can be linearly ordered so that the vertices contained in the same maximal clique are consecutive in this order. We present new positive evidence for the conjecture: every non neighbourhood-overfull indifference graph can be edge coloured with maximum degree colours. Two adjacent vertices are twins if they belong to the same maximal cliques. A graph is reduced if it contains no pair of twin vertices. A graph is overfull if the total number of edges is greater than the product of the maximum degree by $\lfloor n/2 \rfloor$, where n is the number of vertices. We give a structural characterization for neighbourhood-overfull indifference graphs proving that a reduced indifference graph cannot be neighbourhood-overfull. We show that the chromatic index for all reduced indifference graphs is the maximum degree. We present two decomposition methods for edge colouring reduced indifference graphs with maximum degree colours.

Keywords: Algorithms; Graph algorithms; Indifference graphs; Edge colouring

1. Introduction

In this paper, G denotes a simple, undirected, finite, connected graph. The sets $V(G)$ and $E(G)$ are the vertex and edge sets of G. Denote $|V(G)|$ by n and $|E(G)|$ by m. A graph with just one vertex is called *trivial*. A *clique* is a set of vertices pairwise adjacent

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in G. A *maximal clique* of G is a clique not properly contained in any other clique. A *subgraph* of G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $X \subseteq V(G)$, denote by $G[X]$ the *subgraph induced by* X, that is $V(G[X]) = X$ and $E(G[X])$ consists of those edges of $E(G)$ having both ends in X. For $Y \subseteq E(G)$, the *subgraph induced by* Y is the subgraph of G whose vertex set is the set of endpoints of edges in Y and whose edge set is Y; this subgraph is denoted by $G[Y]$. The notation $G \backslash Y$ denotes the subgraph of G with $V(G \backslash Y) = V(G)$ and $E(G \backslash Y) = E(G) \backslash Y$. A graph G is H-*free* if G does not contain an isomorphic copy of H as an induced subgraph. Denote by C_n the chordless cycle on n vertices and by $2K_2$ the complement of the chordless cycle C_4. A *matching* M of G is a set of pairwise nonadjacent edges of G. A matching M of G *covers* a set of vertices X of G when each vertex of X is incident to some edge of M. The graph $G[M]$ is also called a *matching*.

For each vertex v of a graph G, the *adjacency* $\mathrm{Adj}_G(v)$ of v is the set of vertices that are adjacent to v. The *degree* of a vertex v is $\deg(v) = |\mathrm{Adj}_G(v)|$. The *maximum degree* of a graph G is then $\Delta(G) = \max_{v \in V(G)} \deg(v)$. We use the simplified notation Δ when there is no ambiguity. We call Δ-*vertex* a vertex with maximum degree. The set $N[v]$ denotes the *neighbourhood* of v, that is, $N[v] = \mathrm{Adj}_G(v) \cup \{v\}$. A subgraph induced by the neighbourhood of a vertex is simply called a *neighbourhood*. We call Δ-*neighbourhood* the neighbourhood of a Δ-vertex. Two vertices v and w are *twins* when $N[v] = N[w]$. Equivalently, two vertices are twins when they belong to the same set of maximal cliques. A graph is *reduced* if it contains no pair of twin vertices. The *reduced graph* G' of a graph G is the graph obtained from G by collapsing each set of twins into a single vertex and removing possible resulting parallel edges and loops.

The *chromatic index* $\chi'(G)$ of a graph G is the minimum number of colours needed to colour the edges of G such that no adjacent edges get the same colour. A celebrated theorem by Vizing [11,13] states that $\chi'(G)$ is always Δ or $\Delta+1$. Graphs with $\chi'(G) = \Delta$ are said to be in *Class* 1; graphs with $\chi'(G) = \Delta + 1$ are said to be in *Class* 2. A graph G satisfying the inequality $m > \Delta(G)\lfloor n/2 \rfloor$, is said to be an *overfull* graph [9]. A graph G is *subgraph-overfull* [9] when it has an overfull subgraph H with $\Delta(H) = \Delta(G)$. When the overfull subgraph H can be chosen to be a *neighbourhood*, we say that G is *neighbourhood-overfull* [3]. Overfull, subgraph-overfull, and neighbourhood-overfull graphs are in Class 2.

It is well known that the recognition problem for the set of graphs in Class 1 is NP-complete [10]. The problem remains NP-complete for several classes, including comparability graphs [1]. On the other hand, the problem remains unsolved for *indifference graphs*: graphs whose vertices can be linearly ordered so that the vertices contained in the same maximal clique are consecutive in this order [12]. We call such an order an *indifference order*. Given an indifference graph, for each maximal clique A, we call *maximal edge* an edge whose endpoints are the first and the last vertices of A with respect to an indifference order. Indifference graphs form an important subclass of interval graphs: they are also called unitary interval graphs or proper interval graphs. The reduced graph of an indifference graph is an indifference graph with a unique indifference order (except for its reverse). This uniqueness property was used to describe solutions for the recognition problem and for the isomorphism problem for the class of indifference graphs [2].

It has been shown that every odd maximum degree indifference graph is in Class 1 [3] and that every subgraph-overfull indifference graph is in fact neighbourhood-overfull [5]. It has been conjectured that every Class 2 indifference graph is neighbourhood-overfull [3,5]. Note that the validity of this conjecture implies that the edge-colouring problem for indifference graphs is in P.

The goal of this paper is to investigate this conjecture by giving another positive evidence for its validity. We describe a structural characterization for neighbourhood-overfull indifference graphs. This structural characterization implies that no reduced indifference graph is neighbourhood-overfull. We prove that all reduced indifference graphs are in Class 1 by exhibiting edge colourings with Δ colours for these graphs. In order to construct such edge colourings we define two decompositions for reduced indifference graphs. First, we decompose an even maximum degree indifference graph with no twin Δ-vertices into two indifference graphs: a matching covering all Δ-vertices and an odd maximum degree indifference graph. Second, we decompose an arbitrary indifference graph into a bipartite graph and a smaller indifference graph.

The characterization for neighbourhood-overfull indifference graphs is described in Section 2. The decomposition and the edge colouring of indifference graphs with no twin Δ-vertices is in Section 3. The second decomposition which gives an alternative edge colouring for special reduced indifference graphs is in Section 4. Our conclusions are in Section 5.

2. Neighbourhood-overfull indifference graphs

In this section, we study the overfull Δ-neighbourhoods of an indifference graph. Since it is known that every odd maximum degree indifference graph is in Class 1 [3], an odd maximum degree indifference graph contains no overfull Δ-neighbourhoods. We consider the case of even maximum degree indifference graphs. A nontrivial complete graph with even maximum degree Δ is always an overfull Δ-neighbourhood. We characterize the structure of an overfull Δ-neighbourhood obtained from a complete graph by removal of a set of edges.

Theorem 1. *Let $K_{\Delta+1}$ be a complete graph with even maximum degree Δ. Let $F = K_{\Delta+1} \backslash R$, where R is a nonempty subset of edges of $K_{\Delta+1}$. Then, the graph F is an overfull indifference graph with maximum degree Δ if and only if $H = G[R]$ is a $2K_2$-free bipartite graph with at most $\Delta/2 - 1$ edges.*

The proof of Theorem 1 is divided into two lemmas.

Lemma 1. *Let $K_{\Delta+1}$ be a complete graph with even maximum degree Δ. Let R be a nonempty subset of edges of $K_{\Delta+1}$. If $F = K_{\Delta+1} \backslash R$ is an overfull indifference graph with maximum degree Δ, then $H = G[R]$ is a $2K_2$-free bipartite graph and $|R| < \Delta/2$.*

Proof. Let R be a nonempty subset of edges of $K_{\Delta+1}$, a complete graph with even maximum degree Δ such that $F = K_{\Delta+1} \backslash R$ is an overfull indifference graph with maximum degree Δ. Note that $\Delta > 2$.

Because F is an overfull graph, $|V(F)|$ is odd and there are at most $\Delta/2-1$ missing edges joining vertices of F. Hence $|R|<\Delta/2$.

Suppose, by contradiction, that the graph $H=G[R]$ contains a $2K_2$ as an induced subgraph. Then F contains a chordless cycle C_4 as an induced subgraph, a contradiction to F being an indifference graph. Since H is a graph free of $2K_2$, we conclude that the graph H does not contain the chordless cycle C_k, $k\geqslant 6$. We show that the graph H contains no C_5 and no C_3 as an induced subgraph. Assume the contrary. If H contains a C_5 as an induced subgraph, then F contains a C_5 as an induced subgraph, since C_5 is a self-complementary graph, a contradiction to F being an indifference graph. If H contains a C_3 as an induced subgraph, then F contains a $K_{1,3}$ as an induced subgraph, since, by hypothesis, F has at least one vertex of degree Δ, a contradiction to F being an indifference graph. Therefore, H is a bipartite graph, without $2K_2$, and $|R|<\Delta/2$. □

We study in Lemma 2 special vertex-orders of a $2K_2$-free bipartite graph. We refer to [7] for a study of vertex-orders of $2K_2$-free bipartite graphs in the context of difference graphs.

Lemma 2. *Let $K_{\Delta+1}$ be a complete graph with even maximum degree Δ. If $H=G[R]$ is a $2K_2$-free bipartite graph induced by a nonempty set R of edges of size $|R|<\Delta/2$, then $F=K_{\Delta+1}\backslash R$ is an overfull indifference graph with maximum degree Δ.*

Proof. By definition of F and because $|R|<\Delta/2$, we have that F is an overfull graph and it has vertices of degree Δ. We shall prove that F is an indifference graph by exhibiting an indifference order on the vertex set $V(F)$ of F.

Since $H=G[R]$ is a $2K_2$-free bipartite graph, H is connected with unique bipartition of its vertex set into sets X and Y. Now, label the vertices $x_1,x_2,\dots,x_k$ of X and label the vertices $y_1,y_2,\dots,y_\ell$ of Y according to *degree ordering*: labels correspond to vertices in nonincreasing vertex degree order, i.e., $\deg(x_1)\geqslant\deg(x_2)\geqslant\cdots\geqslant\deg(x_k)$ and $\deg(y_1)\geqslant\deg(y_2)\geqslant\cdots\geqslant\deg(y_\ell)$, respectively.

This degree ordering induces the following properties on the vertices of the adjacency of each vertex of X and Y:

- The adjacency of a vertex of H defines an interval on the degree order, i.e., $\mathrm{Adj}_H(x_i)=\{y_j\colon\ 1\leqslant p\leqslant j\leqslant p+q\leqslant\ell\}$ and $\mathrm{Adj}_H(y_j)=\{x_i\colon\ 1\leqslant r\leqslant i\leqslant r+s\leqslant k\}$.
  Indeed, let a be a vertex of H such that $\mathrm{Adj}_H(a)$ is not an interval. Then $\mathrm{Adj}_H(a)$ has at least two vertices b and d, and there is a vertex c such that $ac\notin R$ between b and d. Without loss of generality, suppose that $\deg(b)\geqslant\deg(c)\geqslant\deg(d)$. Since $\deg(c)\geqslant\deg(d)$, there is a vertex e such that e is adjacent to c but is not adjacent to d. It follows that, when either $\deg(e)\leqslant\deg(a)$ or $\deg(a)\leqslant\deg(e)$, H has an induced $2K_2$ (ec and ad), a contradiction.
- The adjacency-sets of the vertices of H are ordered with respect to set inclusion according to the following *containment property*: $\mathrm{Adj}_H(x_1)\supseteq\mathrm{Adj}_H(x_2)\supseteq\cdots\supseteq\mathrm{Adj}_H(x_k)$ and $\mathrm{Adj}_H(y_1)\supseteq\mathrm{Adj}_H(y_2)\supseteq\cdots\supseteq\mathrm{Adj}_H(y_\ell)$.
  For, suppose there are a and b in X with $\deg(a)\geqslant\deg(b)$ and $\mathrm{Adj}_H(a)\not\supseteq\mathrm{Adj}_H(b)$. Hence, there are vertices c and d such that c is adjacent to a but not to b, and d

is adjacent to b but not to a. The edges ac and bd induce a $2K_2$ in H, a contradiction.

- x_1y_1 is a *dominating edge* of H, i.e., every vertex of H is adjacent to x_1 or to y_1. This is a direct consequence of the two properties above.

When H is not a complete bipartite graph, let i and j be the smallest indices of vertices of X and Y, respectively, such that x_iy_j is not an edge of H. Note that, because x_1y_1 is a dominating edge of H, we have i and j greater than 1. Define the following partition of $V(H)$:

$$A := \{x_1, x_2, \ldots, x_{i-1}\}, \qquad S := \{x_i, x_{i+1}, \ldots, x_k\},$$

$$B := \{y_1, y_2, \ldots, y_{j-1}\}, \quad T := \{y_j, y_{j+1}, \ldots, y_\ell\}.$$

Note that S and T can be empty sets and that the graph induced by $A \cup B$ is a complete bipartite subgraph of H.

Now we describe a total ordering on $V(F)$ as follows. We shall prove that this total ordering gives the desired indifference order.

- First, list the vertices of $X = A \cup S$ as $x_1, x_2, \ldots, x_k$.
- Next, list all the Δ-vertices of F, $v_1, v_2, \ldots, v_s$.
- Finally, list the vertices of $Y = T \cup B$ as $y_l, y_{l-1}, \ldots, y_1$.

The ordering within the sets A, S, D, T, B, where D denotes the set of the Δ-vertices of F, is induced by the ordering of $V(F)$.

By the containment property of the adjacency-sets of the vertices of H and because each adjacency defines an interval, the consecutive vertices of the same degree are twins in F. Hence, it is enough to show that the ordering induced on the reduced graph F' of F is an indifference order.

For simplicity, we use the same notation for vertices of F and F', i.e., we call vertices in F' corresponding to vertices of X by $x'_1, x'_2, \ldots, x'_{k'}$, and we call vertices corresponding to vertices of Y by $y'_1, y'_2, \ldots, y'_{\ell'}$. Note that the set D contains only one representative vertex in F', and we denote this unique representative vertex by v'_1.

By definition of F', $x'_1v'_1, x'_2y'_{\ell'}, \ldots, x'_{i-1}y'_{j+1}, x'_iy'_j, \ldots, x'_{k'}y'_2, v'_1y'_1$ are edges of F'. Since vertex v'_1 is a representative vertex of a Δ-vertex of F, it is also a Δ-vertex of F'. Thus v'_1 is adjacent to each vertex of F'. Each edge listed above, distinct from $x'_1v'_1$ and $v'_1y'_1$, has form $x'_py'_q$. We want to show that x'_p is adjacent to all vertices from x'_{p+1} up to y'_q with respect to the order. For suppose, without loss of generality, that x'_p is not adjacent to some vertex z between x'_p and y'_q with respect to the order. Now by the definition of the graphs F and H, every edge of $K_{\Delta+1}$ not in F belongs to graph H. Since H is a bipartite graph, with bipartition of its vertex set into sets X and Y, we have in F all edges linking vertices in X, and so we have $z \neq x_s$, $p \leqslant s \leqslant k$. Vertex z is also distinct from y_s, $q \leqslant s \leqslant l$, by the properties of the adjacency in H. Hence, x'_p is adjacent to all vertices from x'_{p+1} up to y'_q with respect to the order. It follows that each edge listed above defines a maximal clique of F'. Hence, this ordering satisfies the property that vertices belonging to the same maximal clique are consecutive and we conclude that this ordering on $V(F')$ is the desired indifference order. This conclusion completes the proofs of both Lemma 2 and Theorem 1. □

Corollary 1. *Let G be an indifference graph. A Δ-neighbourhood of G with at most $\Delta/2$ vertices of maximum degree is not neighbourhood-overfull.*

Proof. Let F be a Δ-neighbourhood of G with at most $\Delta/2$ vertices of degree Δ. If Δ is odd, then F is not neighbourhood-overfull. If Δ is even, then we use the notation of Lemmas 1 and 2. The hypothesis implies $|X| + |Y| \geqslant (\Delta/2) + 1$. Since vertex x_1 misses every vertex of Y and since vertex y_1 misses every vertex of X, there are at least $|X|+|Y|-1$ missing edges having as endpoints x_1 or y_1. Hence, there are at least $|X| + |Y| - 1 \geqslant \Delta/2$ missing edges in F, and F cannot be neighbourhood-overfull. □

Corollary 2. *An indifference graph with no twin Δ-vertices is not neighbourhood-overfull.*

Proof. Let G be an indifference graph with no twin Δ-vertices. The hypothesis implies that every Δ-neighbourhood F of G contains precisely one vertex of degree Δ. Now Corollary 1 says F is not neighbourhood-overfull and therefore G itself cannot be neighbourhood-overfull. □

Corollary 3. *A reduced indifference graph is not neighbourhood-overfull.*

3. Decomposing indifference graphs with no twin Δ-vertices

We have established in Corollary 2 of Section 2 that an indifference graph with no twin Δ-vertices is not neighbourhood-overfull, a necessary condition for an indifference graph with no twin Δ-vertices to be in Class 1. In this section, we prove that every indifference graph with no twin Δ-vertices is in Class 1. We exhibit a Δ-edge colouring for an even maximum degree indifference graph with no twin Δ-vertices. Since every odd maximum degree indifference graph is in Class 1, this result implies that all indifference graphs with no twin Δ-vertices, and in particular that all reduced indifference graphs are in Class 1.

Let $E_1, \ldots, E_k$ be a partition of the edge set of a graph G. It is clear that if the subgraphs $G[E_i]$, $1 \leqslant i \leqslant k$, satisfy $\Delta(G) = \sum_i \Delta(G[E_i])$ and, if for each i, $G[E_i]$ is in Class 1, then G is also in Class 1. We apply this decomposition technique to our given indifference graph with even maximum degree and no twin Δ-vertices.

We partition the edge set of an indifference graph G with even maximum degree Δ and with no twin Δ-vertices into two sets E_1 and E_2, such that $G_1 = G[E_1]$ is an odd maximum degree indifference graph and $G_2 = G[E_2]$ is a matching.

Let G be an indifference graph and $v_1, v_2, \ldots, v_n$ an indifference order for G. By definition, an edge $v_i v_j$ is maximal if there does not exist another edge $v_k v_\ell$ with $k \leqslant i$ and $j \leqslant \ell$. Note that an edge $v_i v_j$ is maximal if and only if the edges $v_{i-1} v_j$ and $v_i v_{j+1}$ do not exist. In addition, every maximal edge $v_i v_j$ defines a maximal clique having v_i as its first vertex and v_j as its last vertex. Thus, every vertex is incident to zero, one, or two maximal edges. Moreover, given an indifference graph with an indifference order and an edge that is maximal with respect to this order, the removal of this edge gives a smaller indifference graph: the original indifference order is an indifference order for the smaller indifference graph.

Based on Lemma 3, we shall formulate an algorithm for choosing a matching of an indifference graph with no twin Δ-vertices that covers every Δ-vertex of G.

Lemma 3. *Let G be a nontrivial graph. If G is an indifference graph with no twin Δ-vertices, then every Δ-vertex of G is incident to precisely two maximal edges.*

Proof. Let G be a nontrivial indifference graph without twin Δ-vertices and let v be a Δ-vertex of G. Consider $v_1, v_2, \ldots, v_n$ an indifference order for G. Because v is a Δ-vertex of G and G is not a clique, we have $v = v_j$, with $j \neq 1, n$. Let v_i and v_k be the leftmost and the rightmost vertices with respect to the indifference order that are adjacent to v_j, respectively. Suppose that $v_i v_j$ is not a maximal edge. Then $v_{i-1} v_j$ or $v_i v_{j+1}$ is an edge in G. The existence of $v_{i-1} v_j$ contradicts v_i being the leftmost neighbour of v_j. Because v_j and v_{j+1} are not twins, the existence of $v_i v_{j+1}$ implies $\deg(v_{j+1}) \geqslant \Delta + 1$, a contradiction. Analogously, we have that $v_j v_k$ is also a maximal edge. □

We now describe an algorithm for choosing a set of maximal edges that covers all Δ-vertices of G.

Input: an indifference graph G with no twin Δ-vertices with an indifference order $v_1, \ldots, v_n$ of G.

Output: a set of edges M that covers all Δ-vertices of G.

1. For each Δ-vertex of G, say v_j, in the indifference order, put in a set $\mathscr{E}$ the edge $v_i v_j$, where v_i is its leftmost neighbour with respect to the indifference order. Each component of the graph $G[\mathscr{E}]$ is a path. (Each component H of $G[\mathscr{E}]$ has $\Delta(H) \leqslant 2$ and none of the components is a cycle, by the maximality of the chosen edges.)
2. For each path component P of $G[\mathscr{E}]$, number each edge with consecutive integers starting from 1. If a path component P_i contains an odd number of edges, then form a matching M_i of $G[\mathscr{E}]$ choosing the edges numbered by odd integers. If a path component P_j contains an even number of edges, then form a matching M_j choosing the edges numbered by even integers.
3. The desired set of edges M is the union $\bigcup_k M_k$.

We claim that the matching M above defined covers all Δ-vertices of G. For, if a path component of $G[\mathscr{E}]$ contains an odd number of edges, then M covers all of its vertices. If a path component of $G[\mathscr{E}]$ contains an even number of edges, then the only vertex not covered by M is the first vertex of this path component. However, by definition of $G[\mathscr{E}]$, this vertex is not a Δ-vertex of G.

Theorem 2. *If G is an indifference graph with no twin Δ-vertices, then G is in Class* 1.

Proof. Let G be an indifference graph with no twin Δ-vertices. If G has odd maximum degree, then G is in Class 1 [3].

Suppose that G is an even maximum degree graph. Let $v_1,\dots,v_n$ be an indifference order of G. Use the algorithm described above to find a matching M for G that covers all Δ-vertices of G. The graph $G\backslash M$ is an indifference graph with odd maximum degree because the vertex sets of G and $G\backslash M$ are the same and the indifference order of G is also an indifference order for $G\backslash M$. Moreover, since M is a matching that covers all Δ-vertices of G, we have that $\Delta(G\backslash M)=\Delta-1$ is odd. Hence, the edges of $G\backslash M$ can be coloured with $\Delta-1$ colours and one additional colour is needed to colour the edges in the matching M. This implies that G is in Class 1. □

Corollary 4. *All reduced indifference graphs are in Class* 1.

4. Decomposing indifference graphs into a bipartite and a smaller indifference graph

In this section, we show that a general indifference graph can be decomposed into a bipartite graph and a smaller indifference graph. This decomposition gives an alternative edge colouring for a special reduced indifference graph: the so-called red line, defined below.

Let G be an indifference graph. Consider $v_1,v_2,\dots,v_n$ an indifference order for G. Label v_i by i, i.e., for each i, set $label(v_i)=i$. Let S be the set of edges of G that join vertices with labels with equal parity, and let D be the set of edges of G that join vertices with labels with distinct parity.

First we establish in Lemma 4 two properties satisfied by a general indifference graph G decomposed into graphs $G[S]$ and $G[D]$.

Lemma 4. *Let G be a nontrivial indifference graph. Then $G[S]$ is an indifference graph and $G[D]$ is a bipartite graph.*

Proof. Let G be a nontrivial indifference graph. Consider $v_1,v_2,\dots,v_n$ an indifference order for G and a labeling for the vertices of G defined as above.

First define the following indifference order on $G[S]$: order the vertices of $G[S]$ by placing first and consecutively the vertices with even labels with respect to the indifference order for G, and then placing consecutively the vertices with odd labels with respect to the indifference order for G. Let $v_iv_j\in S$. Then $i\neq j$, and i,j have equal parity. Since G is an indifference graph, for each k with the same parity of i, with $i<k<j$, we have that v_iv_k and v_kv_j are edges of G. Notice that $G[S]$ is disconnected. Thus this ordering is an indifference ordering for $G[S]$.

On the other hand, if $G[D]$ contained an odd cycle, then D would contain an edge joining vertices with equal parity. □

The equality $\Delta(G)=\Delta(G[S])+\Delta(G[D])$ does not hold for the indifference graph G depicted in Fig. 1. The graph G is depicted as follows. We arrange the vertices of G into an indifference order and depict only the maximal edges of G. For the graphs $G[S]$ and $G[D]$, we depict all edges. By Lemma 4, $G[S]$ is an indifference graph and

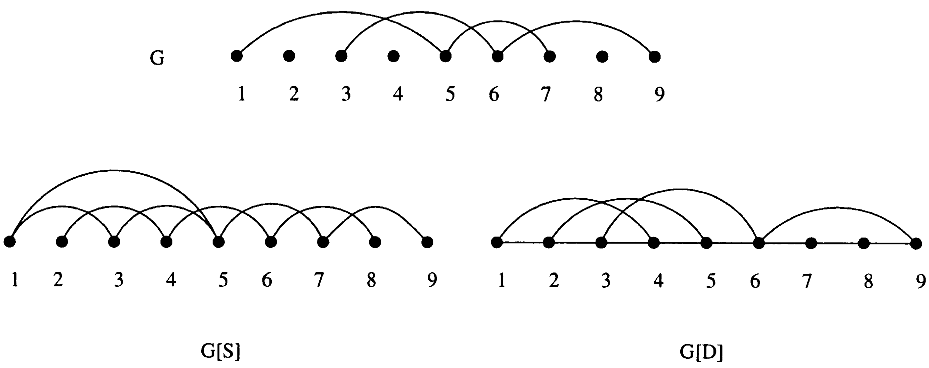

Fig. 1. An indifference graph G with no twin Δ-vertices such that $\Delta(G) \neq \Delta(G[S]) + \Delta(G[D])$.

$G[D]$ is a bipartite graph. Note that $v_2, v_4, v_6, v_8, v_1, v_3, v_5, v_7, v_9$ is an indifference order for the disconnected indifference graph $G[S]$. In this example, $\Delta(G) = 6$, $\Delta(G[S]) = 3$, and $\Delta(G[D]) = 4$. The graph G is an indifference graph with no twin Δ-vertices. We define next a subclass of reduced indifference graphs for which the desired equality $\Delta(G) = \Delta(G[S]) + \Delta(G[D])$ holds.

An indifference graph is a *red line* when, for a given indifference order $v_1, v_2, \ldots, v_n$, there is a constant $x \geqslant 0$ such that $n \geqslant 2x + 1$ and, for each v_i, $1 \leqslant i \leqslant n - x$, its rightmost neighbour is v_{i+x}. The constant x is called the *size* of the red line. The maximal cliques of a red line are in one-to-one correspondence with the maximal edges $v_i v_{i+x}$, $1 \leqslant i \leqslant n - x$. A red line is a reduced graph since each vertex belongs to a different set of maximal cliques. In addition, every vertex of a red line is the first or the last vertex of a maximal clique with respect to its unique indifference order.

Corollary 5. *Let G be a red line with size $x = 2i$ or $x = 2i + 1$, with $i \geqslant 1$. Then $G[S]$ is formed by two connected components, each being a red line with size i.*

Proof. Let G be a red line. By Lemma 4, $G[S]$ is an indifference graph. The vertices with even labels form one connected component of $G[S]$ and the vertices with odd labels form another component. If $x = 2i$, then $v_j v_{j+x} \in S$, $1 \leqslant j \leqslant n - x$. Hence, the maximal edges of the red line G have endpoints of equal parity and are in fact maximal edges of $G[S]$. If $v_j v_{j+x}$ is an edge joining two vertices of odd (even) labels, then there are $i + 1$ vertices v_k of odd (even) labels such that $j \leqslant k \leqslant j + x$. Therefore, the size of the red line $G[S]$ is i. If $x = 2i + 1$, then the maximal edges of G have distinct parity. Hence, for each $v_j \in G$, we have that $v_j v_{j+x-1} \in S$, for $1 \leqslant j \leqslant n - x$, and $G[S]$ is a red line of size i. $\square$

Lemma 5. *Let G be a red line decomposed into $G[S]$ and $G[D]$. Then $\Delta(G) = \Delta(G[S]) + \Delta(G[D])$.*

Proof. The maximum degree of a red line G with size x is $\Delta(G)=2x$. Corollary 5 says the size of the red line $G[S]$ is $\lfloor x/2 \rfloor$. Hence, $\Delta(G[S])=x$, if x is even; and $\Delta(G[S])=x-1$, if x is odd. On the other hand, by the definition of $G[D]$, the maximum degree $\Delta(G[D])=x$, if x is even; and $\Delta(G[D])=x+1$, if x is odd. Therefore, if G is a red line, then $\Delta(G)=\Delta(G[S])+\Delta(G[D])$. □

Now Theorem 3 uses the decomposition of a red line graph G into graphs $G[S]$ and $G[D]$ to prove that a red line graph G is in Class 1. The proof of Theorem 3 defines a recursive edge colouring of a red line graph G with $\Delta(G)$ colours.

Note that Theorems 2 and 3 give two distinct optimal edge colourings for a red line graph.

Theorem 3. *If G is a red line, then G is Class* 1.

Proof. We argue by induction on the size x of a red line G. If $x \leqslant 1$, then G is isomorphic to $G[D]$ and G is Class 1, being a bipartite graph. If $x \geqslant 2$, consider the decomposition of G into graphs $G[S]$ and $G[D]$. Since by Lemma 5, we have the equality $\Delta(G)=\Delta(G[S])+\Delta(G[D])$, it is enough to prove that $G[S]$ and $G[D]$ are both Class 1. By Corollary 5, $G[S]$ is a red line of size $\lfloor x/2 \rfloor$. Hence, by induction, $G[S]$ is Class 1. In addition, by Lemma 4, $G[D]$ is a bipartite graph. □

5. Conclusions

We believe our work makes a contribution to the problem of edge-colouring indifference graphs in three respects.

First, our results on the colouring of indifference graphs show that, in all cases we have studied, neighbourhood-overfullness is equivalent to being Class 2, which gives positive evidence to the conjecture that for any indifference graph neighbourhood-overfullness is equivalent to being Class 2. It would be interesting to extend these results to larger classes. We established recently [4] that every odd maximum degree dually chordal graph is Class 1. This result shows that our techniques are extendable to other classes of graphs.

Second, our results apply to a subclass of indifference graphs defined recently in the context of clique graphs. A graph G is a *minimum indifference graph* if G is a reduced indifference graph and, for some indifference order of G, every vertex of G is the first or the last element of a maximal clique of G [6]. Given two distinct minimum indifference graphs, their clique graphs are also distinct [6]. This property is not true for general indifference graphs [8]. Note that our results apply to minimum indifference graphs: no minimum indifference graph is neighbourhood-overfull, every minimum indifference graph is in Class 1, and we can edge-colour any minimum indifference graph with Δ colours.

Third, and perhaps more important, the decomposition techniques we use to show these results are new and proved to be powerful tools.

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