



On the number of update digraphs and its relation with the feedback arc sets and tournaments



J. Aracena^a, J. Demongeot^c, E. Fanchon^b, M. Montalva^{d,*}

^a *CP²MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Av. Esteban Iturra s/n, Casilla 160-C, Concepción, Chile*

^b *UJF-Grenoble 1, CNRS, Laboratoire TIMC-IMAG UMR 5525, F-38041 Grenoble, France*

^c *AGIM FRE 3405, F-38700 La Tronche, France*

^d *Facultad de Ingeniería y Ciencias, Universidad Adolfo Ibáñez, Av. Diagonal las Torres 2640, Peñalolén, Santiago, Chile*

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ABSTRACT

An update digraph corresponds to a labeled digraph that indicates a relative order of its nodes introduced to define equivalence classes of deterministic update schedules yielding the same dynamical behavior of a Boolean network. In Aracena et al. [1], the authors exhibited relationships between update digraphs and the feedback arc sets of a given digraph G . In this paper, we delve into the study of these relations. Specifically, we show differences and similarities between both sets through increasing and decreasing monotony properties in terms of their structural characteristics. Besides, we prove that these sets are equivalent if and only if all the digraph circuits are cycles. On the other hand, we characterize the minimal feedback arc sets of a given digraph in terms of their associated update digraphs. In particular, for complete digraphs, this characterization shows a close relation with acyclic tournaments. For the latter, we show that the size of the associated equivalence classes is a power of two. Finally, we determine exactly the number of update digraphs associated to digraphs containing a tournament.

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1. Introduction

Boolean networks (BN) are the simplest models for genetic regulatory networks, as well as for other simple distributed dynamical systems. Despite their simplicity, they provide a realistic model in which different phenomena can be reproduced and studied, and indeed, many regulatory models published in the biological literature fit within this framework [5,8,7].

A BN is defined by its connection digraph, its local activation functions, and the update schedule used (see [2] for more details). In [6] equivalence classes of deterministic update schedules in BNs were defined according to the associated connection digraph, named *update digraph*, with labels on its arcs, which are of two types: an arc (a, b) has a *negative* label if the state of a is updated strictly before b , and a *positive* label otherwise.

In [6,2], the authors showed that the elements of such equivalence classes yield exactly the same dynamical behavior of the network. This result was the starting point for the algorithmical and combinatorial study of the update digraphs done in [1], where among others, a close relationship between update digraphs and the feedback arc sets of a given digraph was exhibited.

For a given digraph, a feedback arc set is a set of arcs which if removed leaves the resulting digraph free of cycles. The problem of finding a feedback arc set with minimum size has been a subject of extensive research (see [3] for a good survey). One of these works showed that such a minimum arc set is determined by a sequential ordering of the nodes which minimizes the number of positive arcs [9].

* Corresponding author. Tel.: +56 223311642; fax: +56 995437059.

E-mail addresses: jaracena@ing-mat.udec.cl (J. Aracena), Jacques.Demongeot@agim.eu (J. Demongeot), Eric.Fanchon@imag.fr (E. Fanchon), marco.montalva.m@gmail.com (M. Montalva).

In this paper, we delve into the study of the relationships between update digraphs and feedback arc sets showed in [1] as well as their relations with some particular digraphs such as complete digraphs, acyclic digraphs and tournaments in order to determine exactly the number and size of the associated equivalence classes.

2. Preliminaries

We begin this section giving some basic definitions and introducing the necessary notations. Besides, we recall some known results that allow us to develop the following sections.

A **digraph** is an ordered pair of sets $G = (V, A)$, where $V = \{1, \dots, n\}$ is a set of elements called **vertices** (or **nodes**) and A is a set of ordered pairs (called **arcs**) of vertices of V . The **vertex set** of G is referred to as $V(G)$, its **arc set** as $A(G)$. If more than one arc that connects a node i to a node j are admissible, then we will say that G is a **multidigraph**. A **subdigraph** of G is a digraph $G' = (V', A')$ where $V' \subseteq V$ and $A' \subseteq (V' \times V') \cap A$. We write $G' \subseteq G$. If $A' = (V' \times V') \cap A$, we say that G' is **induced** by V' and call G' an **induced subdigraph** of G which is denoted by $G' = G[V']$. If $V' \subsetneq V \vee A' \subsetneq A$, then we write $G' \subsetneq G$. We will often write $G - U$ whose meaning depends on the nature of U . Thus, if $U \subseteq V$, then $G - U$ is the subdigraph of G induced by $V - U$, i.e. $G - U = G[V - U]$. If $U \subseteq A$, then $G - U = (V, A - U)$.

G is a **complete** digraph if $A = \{(u, v) : u, v \in V \wedge u \neq v\}$. G is a **tournament** if $\forall x, y \in V, x \neq y, (x, y) \in A \vee (y, x) \in A$. A **walk** from a vertex v_1 to a vertex v_m in a digraph G is a sequence of vertices v_1, v_2, \dots, v_m of $V(G)$ such that $\forall k = 1, \dots, m - 1, (v_k, v_{k+1}) \in A(G)$ or $(v_{k+1}, v_k) \in A(G)$. The vertices v_1 and v_m are respectively the initial and terminal vertex of the walk. A walk is **elementary** if each vertex in the walk appears only once with the possible exception that the first and last vertex may coincide. A walk is closed if its initial and terminal vertices coincide. A **circuit** is a closed elementary walk. A walk v_1, v_2, \dots, v_m is a **path** if $(v_k, v_{k+1}) \in A(G)$ for all $k = 1, \dots, m - 1$. A **cycle** is a directed circuit, that is a closed elementary path. G is an **acyclic digraph** if G has no cycle.

An arc set $U \subseteq A$ is a **feedback arc set** of G if $G - U$ is an acyclic digraph. U is said to be a **minimal** feedback arc set of G if U is a feedback arc set of G and there is no other feedback arc set $W \subseteq A$ of G such that $W \subsetneq U$. U is said to be a **minimum** feedback arc set of G if U is a feedback arc set of G and there is no other feedback arc set $W \subseteq A$ of G such that $|W| < |U|$. The sets of feedback arc sets, minimal feedback arc sets and non feedback arc sets of G are denoted by $FAS(G)$, $MFAS(G)$ and $NFAS(G)$ respectively.

A digraph G is said to be **connected** if there is a walk between every pair of its vertices, and **strongly connected** if there is a path between every pair of its vertices. $G' \subseteq G$ is a **strongly connected component** of G if G' is strongly connected and it is maximal for this property, i.e., there is no other strongly connected subdigraph G'' of G such that $G' \subsetneq G''$. If G' is a strongly connected component of G , composed by only one vertex, then G' is called a **trivial component**.

An **undirected graph** (or a **graph**) is an ordered pair $G = (V, E)$ where $V = \{1, \dots, n\}$ is a non-empty finite set of elements called **vertices** (or **nodes**) and E is a finite set of unordered pairs of distinct vertices of V called **edges**. The **vertex set** of G is referred to as $V(G)$, its **arc set** as $E(G)$. An **orientation** of G is a digraph $G' = (V', A)$ where $V' = V, |A| = |E|$ and $\forall \{x, y\} \in E$, either $(x, y) \in A$ or $(y, x) \in A$.

Also, in the sequel, for any integers a and b with $a \leq b$, we will write $\llbracket a, b \rrbracket = \{i \in \mathbb{Z} : a \leq i \leq b\}$.

An **update schedule** of the vertices of a digraph $G = (V, A)$, with $|V| = n$, is a function $s : V \rightarrow \llbracket 1, n \rrbracket$ such that $s(V) = \llbracket 1, m \rrbracket$ for some $m \leq n$. We denote by S_n the **set of update schedules** over $\llbracket 1, n \rrbracket$. A **block** of s is the set $B_i = \{v \in V : s(v) = i\}, 1 \leq i \leq m$. The **number of blocks** of s is denoted by $nb(s) \equiv m$. If $nb(s) = 1$, then s is said to be a **parallel** update schedule. In this case, we will write $s = s_p$. If s is a permutation over the set $\llbracket 1, n \rrbracket$, i.e. $nb(s) = n$, s is said to be a **sequential** update schedule. In all other cases, i.e. when $2 \leq nb(s) \leq n - 1$, s is said to be a **block sequential** update schedule. Frequently, s will be denoted by $s = (j \in B_1)(j \in B_2) \dots (j \in B_{nb(s)})$ or more compactly $s = (B_i)_{i=1}^{nb(s)}$ and $P_s = \{B_1, \dots, B_{nb(s)}\}$ the partition associated to s .

As mentioned in [4], the number of update schedules associated to a digraph of n vertices, i.e. $|S_n|$, is equal to the number of ordered partitions of a set of size n , that is

$$|S_n| = T_n \equiv \sum_{k=0}^{n-1} \binom{n}{k} T_k,$$

where $T_0 \equiv 1$.

Let G be a digraph. A function $lab : A(G) \rightarrow \{\ominus, \oplus\}$ is called a **label function** of G . An arc $a \in A(G)$ such that $lab(a) = \oplus$ is called a **positive arc** and an arc $a \in A(G)$ such that $lab(a) = \ominus$ is called a **negative arc**. A cycle C in G such that $\forall a \in A(C), lab(a) = \oplus$ is called a **positive cycle** and a cycle C in G such that $\forall a \in A(C), lab(a) = \ominus$ is called a **negative cycle**. (G, lab) is named a **labeled digraph**.

Let s be an update schedule of $V(G)$, we denote by lab_s the label function defined as follows (see Fig. 1):

$$\forall (j, i) \in A(G), lab_s(j, i) = \begin{cases} \oplus & \text{if } s(j) \geq s(i) \\ \ominus & \text{if } s(j) < s(i). \end{cases}$$

We define **equivalence classes** with respect to labeled digraphs: if s is an update schedule of the vertices of a digraph G , we write $[s]_G$ the set of update schedules s' such that $s \stackrel{G}{\sim} s'$, that is

$$[s]_G = \{s' : (G, lab_s) = (G, lab_{s'})\}.$$

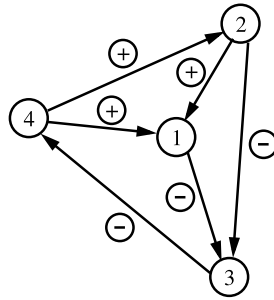


Fig. 1. A digraph $G = (V, A)$ labeled by the function lab_s where $\forall i \in V = \{1, \dots, 4\}, s(i) = i$.

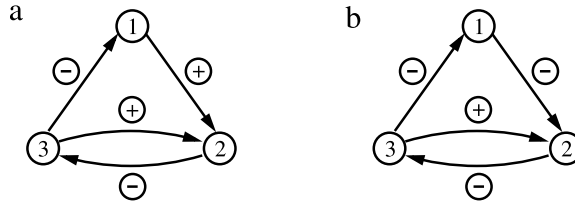


Fig. 2. (a) A labeled digraph (G, lab) which is an update digraph with a maximal (but not maximum) number of negative arcs. (b) A labeled digraph (G, lab') which is not an update digraph.

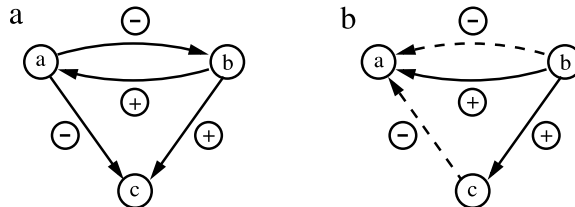


Fig. 3. (a) A labeled digraph $G = (\{a, b, c\}, A)$. (b) (G_R, lab_R) where the arcs drawn in dotted lines are the ones that have been inverted.

In simple words, an equivalence class, $[s]_G$, is a set of update schedules that all yield the same labeled digraph, and consequently the same dynamical behavior of the network [2].

We will say that s has the **maximum number of blocks** if $\forall s' \in [s]_G, nb(s') \leq nb(s)$ and we will denote by S_n^* the set of update schedules having the maximum number of blocks.

A labeled digraph (G, lab) is said to be an **update digraph** if there exists an update schedule s such that $lab = lab_s$, that is $\forall a \in A(G), lab(a) = lab_s(a)$. We denote by $U(G) = \{lab : A(G) \rightarrow \{\ominus, \oplus\} \mid (G, lab) \text{ is an update digraph}\}$ and $NU(G) = \{lab : A(G) \rightarrow \{\ominus, \oplus\} \mid (G, lab) \text{ is a non update digraph}\}$. An update digraph (G, lab) has a **maximal number of negative arcs** if there is no label function $lab^* \in U(G)$ with strictly more negative arcs than lab , and where $lab(u, v) = \ominus$ implies $lab^*(u, v) = \ominus$, for every $(u, v) \in A(G)$ (see example in Fig. 2).

We define the **projection** of (G, lab) onto $G' \subseteq G$ as being the labeled digraph $(G', lab|_{G'})$, where $lab|_{G'}(a) = lab(a), \forall a \in A(G')$.

Given an update digraph $(G, lab), G' \subseteq G$ is said to be a **positive strongly connected component** of (G, lab) if G' is a strongly connected component of $G - \{a \in A(G) : lab(a) = \ominus\}$. In particular, if G' is an isolated vertex, G' is called a **trivial positive component**. Besides, any update digraph (G, lab) has a unique decomposition into positive strongly connected components.

Let (G, lab) be a labeled digraph. The **labeled reoriented multidigraph** associated to (G, lab) , denoted by (G_R, lab_R) , is the labeled multidigraph defined as follows:

- $V(G_R) = V(G)$.
- $\forall (u, v) \in A(G)$ such that $lab(u, v) = \oplus, (u, v) \in A(G_R)$ and $lab_R(u, v) = \oplus$.
- $\forall (u, v) \in A(G)$ such that $lab(u, v) = \ominus, (v, u) \in A(G_R)$ and $lab_R(v, u) = \ominus$.

Observe that if (G, lab) has no cycle of length two with a positive arc and another negative, then (G_R, lab_R) is simply a digraph. An example of labeled reoriented multidigraph is shown in Fig. 3.

A **forbidden cycle** in (G_R, lab_R) is a cycle containing a negative arc.

In this context, the following characterization of update digraphs was proven in [1].

Theorem 1. A labeled digraph (G, lab) is an update digraph if and only if (G_R, lab_R) does not contain any forbidden cycle.

Observe that if (G, lab) has a negative cycle, then (G, lab) is not an update digraph. Besides, if G is a digraph without circuits, then for every label function $lab : A(G) \rightarrow \{\ominus, \oplus\}$, (G, lab) is an update digraph.

On the other hand, in [1] it was also shown that the property of being an update digraph can be extended or projected over update digraphs. Specifically, the authors showed the following Theorem.

Theorem 2. *Let G be a digraph and $G' \subseteq G$. If (G', lab') is an update digraph, then there exists a label function lab of $A(G)$ such that (G, lab) is an update digraph and $lab|_{G'} = lab'$.*

The above theorem will be fundamental in the following section, to be more explicit, in the relationships between the sizes of the sets $U(G)$, $MFAS(G)$ and $FAS(G)$ associated to a given digraph G .

3. Feedback arc sets and update digraphs

In this section we will investigate the relationships that exist between the feedback arc sets and the update digraphs associated to a given digraph, which were first studied in [1].

Next, we first recall a result concerning the sizes of the sets $U(G)$, $MFAS(G)$ and $FAS(G)$ that was shown in [1].

Proposition 3. *Let G be a digraph. Then,*

$$|MFAS(G)| \leq |U(G)| \leq |FAS(G)|.$$

Basically, the proof of the Proposition 3 considers, on the one hand, an injective function $g : U(G) \rightarrow FAS(G)$ such that for every $lab \in U(G)$, $g(lab) = \{a \in A(G) : lab(a) = \oplus\}$ and on the other hand, an injective function $h : MFAS(G) \rightarrow U(G)$ such that for every $F \in MFAS(G)$, $h(F) = lab_F$, where $lab_F(a) = \oplus \Leftrightarrow a \in F$.

Now, being a little bit more specific, it is easy to see that the label function lab_p defined by $lab_p(a) = \oplus, \forall a \in A(G)$ is in $U(G)$ but there does not exist $F \in MFAS(G)$ such that $h(F) = lab_p$, because for each arc in F there exists a cycle in G which does not contain another arc in F . Therefore, the function h cannot be surjective and hence, we have the following result.

Proposition 4. *Let G be a digraph. Then,*

$$|MFAS(G)| < |U(G)| \leq |FAS(G)|.$$

As shown in Proposition 4, there exist relationships between feedback arc sets and update digraphs. Next, we will study in greater depth this relation.

Proposition 5. *Let G be a digraph and $G' \subsetneq G$ where $V(G) = V(G')$. Then $|U(G')| < |U(G)|$ and $|FAS(G')| < |FAS(G)|$.*

Proof. Let G be a digraph with $|V(G)| = n$ and $G' \subsetneq G$ where $V(G) = V(G')$. Due to Theorem 2, there is a function $f : U(G') \rightarrow U(G)$, $lab' \rightarrow f(lab') = lab$, where $lab|_{G'} = lab'$. Thus, it is easy to see that f is injective. Hence, $|U(G')| \leq |U(G)|$.

Besides, since $G' \subsetneq G$ and $V(G) = V(G')$, then $\exists(a, b) \in A(G) - A(G')$. Let $s_1, s_2 \in S_n$ such that $s_1(a) = s_1(b) = s_2(a) = 1, s_2(b) = 2, s_1(c) = 2$ and $s_2(c) = 3, \forall c \in V(G) - \{a, b\}$. Then $lab_{s_1}|_{G'} = lab_{s_2}|_{G'}$, but $lab_{s_1} \neq lab_{s_2}$. Defining $lab' = lab_{s_1}|_{G'}$ and supposing w.l.o.g that $f(lab') = lab_{s_1}$, we deduce that f is not surjective because there is no $lab'' \neq lab', lab'' \in U(G')$ such that $f(lab'') = lab_{s_2}$. Hence, $|U(G')| < |U(G)|$.

On the other hand, the function $g : FAS(G') \rightarrow FAS(G)$, $F' \rightarrow g(F') = F' \cup (A(G) - A(G'))$, is evidently injective and well-defined. Thus, $|FAS(G')| \leq |FAS(G)|$. But for a given $(a, b) \in A(G) - A(G')$, $F = A(G) - \{(a, b)\} \in FAS(G)$, however, there is no $F' \in FAS(G')$ such that $g(F') = F$. Hence, g is not surjective and in this way $|FAS(G')| < |FAS(G)|$. \square

Theorem 6. *Let G be an undirected graph and G_1 and G_2 two orientations of G such that every cycle of G_1 is also a cycle of G_2 . Then $|U(G_1)| \leq |U(G_2)|$ and $|FAS(G_2)| \leq |FAS(G_1)|$.*

Proof. Let G, G_1 and G_2 as in the hypothesis of the proposition. We define the function $f : NU(G_2) \rightarrow NU(G_1)$, $lab \rightarrow f(lab) = lab'$ by:

$$\forall(u, v) \in A(G_1), \quad lab'(u, v) = \begin{cases} lab(u, v), & \text{if } (u, v) \in A(G_1) \cap A(G_2), \\ \overline{lab}(u, v), & \text{otherwise} \end{cases}$$

where $\overline{lab} : A(G_1) \rightarrow \{\ominus, \oplus\}$ is defined by:

$$\overline{lab}(u, v) = \oplus \Leftrightarrow lab(v, u) = \ominus,$$

then f is well defined. First, note from definition of lab' that $(G_2)_R = (G_1)_R$, where $(G_1)_R$ and $(G_2)_R$ are the reoriented multidigraphs associated to (G_1, lab') and (G_2, lab) respectively. In this way, if C is a cycle in $((G_2)_R, lab_R)$ with some negative arc, then C is also a cycle in $((G_1)_R, lab'_R)$ with some negative arc (see Fig. 4).

In fact, suppose on the contrary that C is a cycle in $((G_1)_R, lab'_R)$ with all the arcs positive. This is possible if and only if C is a cycle in (G_1, lab') with all the arcs positive and by hypothesis of the proposition, C is also a cycle in (G_2, lab) with all the arcs positive, which is a contradiction. Therefore, $lab \in NU(G_2) \Rightarrow f(lab) = lab' \in NU(G_1)$.

On the other hand, it is easy to see from the definition of lab' that f is injective. Therefore, $|NU(G_2)| \leq |NU(G_1)|$ and consequently $|U(G_1)| \leq |U(G_2)|$.

Finally, $|NFAS(G_1)| \leq |NFAS(G_2)|$ and consequently $|FAS(G_2)| \leq |FAS(G_1)|$. In fact, if $F \subseteq A(G_1)$ is a non-feedback arc set of G_1 , then $F' = \{(u, v) \in G_2 : (u, v) \in F \vee (v, u) \in F\}$ is a non-feedback arc set of G_2 . \square

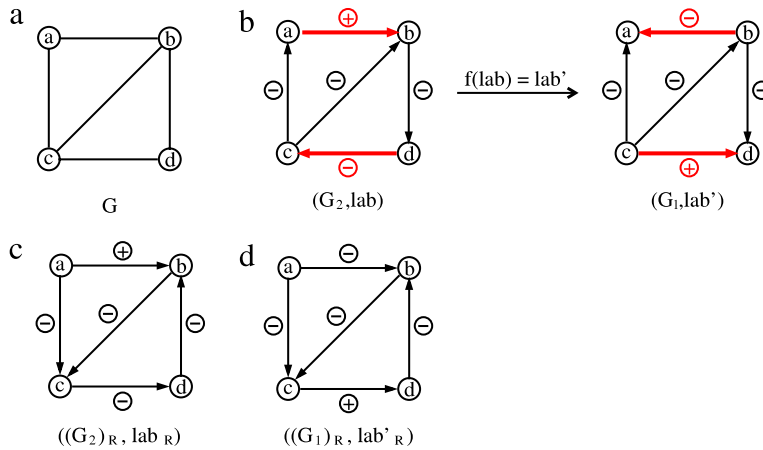


Fig. 4. (a) An undirected graph G . (b) Two orientations G_1 and G_2 of G and the injective function f defined in the proof of Theorem 6. Observe that $|U(G_1)| = 18 < 20 = |U(G_2)|$ and $|FAS(G_2)| = 27 < 32 = |FAS(G_1)|$. (c) The labeled reoriented multigraph $((G_2)_R, lab_R)$ associated to (G_2, lab) . (d) The labeled reoriented multigraph $((G_1)_R, lab'_R)$ associated to (G_1, lab') .

Proposition 7. Let G be a digraph. Then, $|U(G)| = |FAS(G)|$ if and only if all the circuits of G are cycles.

Proof. (\Rightarrow) Let us suppose that G has a circuit C which is not a cycle. We will prove that $|U(G)| < |FAS(G)|$. Let (G, lab) be an update digraph. Then $\{a \in A(G) : lab(a) = \oplus\}$ is a feedback arc set of G induced by the function lab , and thus $|U(G)| \leq |FAS(G)|$. We will show that there is another feedback arc set F' of G not induced by any element of $U(G)$. Indeed, there exists a label function lab' such that the circuit C is a forbidden circuit of (G, lab') with at least a positive and a negative arc in C . Let $F' = A(G) - \{a \in A(G) : lab'(a) = \ominus\}$. It is easy to check that F' is a feedback arc set of G which is not induced by any element in $U(G)$.

(\Leftarrow) Let us suppose that all circuits of G are cycles. We want to prove that $|FAS(G)| \leq |U(G)|$.

We define the following function $T_G : FAS(G) \rightarrow U(G)$ such that for each feedback arc set $F \in FAS(G)$, $T_G(F)$ is the label function:

$$T_G(F)(a) = \begin{cases} \oplus, & \text{if } a \in F, \\ \ominus, & \text{if } a \notin F. \end{cases}$$

Then, $T_G(F) \in U(G)$. Indeed, if there exists a forbidden circuit C in $(G, T_G(F))$, there is at least an arc $a \in A(C)$ such that $T_G(F)(a) = \oplus$. Since, if for all $a \in C$, $T_G(F)(a) = \ominus$ then $A(C) \cap F = \emptyset$, which contradicts the fact: $F \in FAS(G)$. Besides, there exists $a' \in A(C)$, $T_G(F)(a') = \ominus$. Hence, C is a circuit in G which is not a cycle, which is a contradiction. Therefore, T_G is well-defined and obviously injective. \square

From the proof of Proposition 3, we know that to each $F \subseteq A(G)$, a minimal feedback arc set of G , can be associated a label function lab_F such that (G, lab_F) is an update digraph. However, the opposite is not always possible. Next, the following theorem shows a characterization of the update digraphs (G, lab) induced by a minimal feedback arc set.

Theorem 8. Let G be a digraph with $|V(G)| = n$. Then, $F \subseteq A(G)$ is a minimal feedback arc set of G if and only if (G, lab_F) is an update digraph with a maximal number of negative arcs, where $lab_F(u, v) = \oplus \Leftrightarrow (u, v) \in F$.

Proof. (\Rightarrow) Let G be a digraph with $|V(G)| = n$ and F a minimal feedback arc set of G . From the proof of Proposition 3, we know that (G, lab_F) is an update digraph.

Suppose on the contrary that (G, lab_F) does not have a maximal number of negative arcs, i.e., there exists a label function lab with strictly more negative arcs than lab_F such that (G, lab) is an update digraph, and where $lab_F(u, v) = \ominus$ implies $lab(u, v) = \ominus$, for every $(u, v) \in A(G)$. Then $F' = \{(u, v) \in A(G) : lab(u, v) = \oplus\}$ is a feedback arc set of G and verifies $F' \subsetneq F$, which contradicts the minimality of F . Hence, (G, lab_F) is an update digraph with maximal number of negative arcs.

(\Leftarrow) Let lab_F be a label function for which (G, lab_F) is an update digraph with maximal number of negative arcs. We know that $F \subseteq A(G)$ is a feedback arc set of G because (G, lab_F) is an update digraph, which implies that $G - F$ is acyclic.

Now, suppose on the contrary that F is not minimal, i.e., there exists a minimal feedback arc set $F' \subseteq F$ of G and consequently a label function $lab_{F'}$ with which $(G, lab_{F'})$ is also an update digraph, but having the same negative arcs as (G, lab_F) and more. This contradicts the maximality of (G, lab) . Therefore, F is a minimal feedback arc set of G . \square

Remark 9. It is easy to see that a labeled digraph (G, lab) with all its arcs negative is an update digraph if and only if G is an acyclic digraph (see [9]). However, in the general case, we have the following result.

Proposition 10. An update digraph (G, lab) with maximum number of negative arcs has at least $\frac{|A(G)|}{2}$ negative arcs.

Proof. Let (G, lab) be an update digraph and $na(G, lab) = |\{(u, v) \in A(G) : lab(u, v) = \ominus\}|$ the number of negative arcs of (G, lab) .

If G is an acyclic digraph, then (G, lab) with all its arcs negative is an update digraph, i.e. $na(G, lab) = |A(G)|$.

If G has a cycle, then let $s_{q_1} = (v_1)(v_2) \cdots (v_n)$ and $s_{q_2} = (v_n)(v_{n-1}) \cdots (v_1)$ two sequential update schedules with $v_i \in V(G), \forall i \in \llbracket 1, n \rrbracket$. Then, $\forall i, j \in \llbracket 1, n \rrbracket$ such that $(v_i, v_j) \in A(G)$,

$$lab_{s_{q_1}}(v_i, v_j) = \ominus \Leftrightarrow lab_{s_{q_2}}(v_i, v_j) = \oplus.$$

Thus, since (G, lab) has the maximum number of negative arcs, $na(G, lab) \geq \max\{na(G, lab_{s_{q_1}}), na(G, lab_{s_{q_2}})\} \geq \frac{|A(G)|}{2}$. \square

Proposition 11. Let (G, lab) be an update digraph with a maximal number of negative arcs. Then, there is no positive cycle.

Proof. Let (G, lab) be an update digraph with a maximal number of negative arcs, i.e, there is no positive arc that can be changed by a negative arc holding the property of update digraph. Let s be an update schedule such that $(G, lab) = (G, lab_s)$ and suppose on the contrary that there is a positive cycle C , then there exists $(i, j) \in A(C)$ such that $s(i) = s(j)$. Defining s' by:

$$\begin{aligned} s'(k) &= s(k), \quad \forall k, s(k) < s(i) \\ s'(i) &= s(i) \\ s'(k) &= s(k) + 1, \quad \forall k \neq i, s(k) \geq s(i) \end{aligned}$$

we have $lab_s(u, v) = \ominus \Rightarrow lab_{s'}(u, v) = \ominus$, but $lab_s(i, j) = \oplus \neq lab_{s'}(i, j)$ which contradicts the maximality of (G, lab) . \square

Proposition 12. Let (G, lab) be an update digraph. Then, there is a sequential update schedule s_q such that $(G, lab) = (G, lab_{s_q})$ if and only if (G, lab) has no positive cycle.

Proof. (\Rightarrow) Straightforward.

(\Leftarrow) Let (G, lab) be an update digraph with $|V(G)| = n$ and without positive cycles, then (G_R, lab_R) has no cycles. Therefore, there exists a topological order i_1, \dots, i_n of the vertices, i.e., $(i_j, i_k) \in A(G_R) \Rightarrow k > j$, such that $s_q = (i_1)(i_2) \dots (i_n)$ is a sequential schedule satisfying $(G, lab) = (G, lab_{s_q})$. \square

In general, the necessary condition of Proposition 11 is not sufficient. As a counter-example, we can consider a simple labeled digraph (G, lab) composed by only one positive arc (a, b) which is obviously an update digraph without positive cycles, but not maximal. Nevertheless, Theorem 16 of the next section shows that in update complete digraphs, the sufficiency holds.

3.1. Feedback arc sets in update complete digraphs

In the particular case of complete digraphs, we will see how many of the properties of update digraphs and feedback arc sets previously mentioned are equivalent, establishing an explicit form for determining the set $MFAS(G)$.

Remark 13. Let G be a complete digraph with $|V(G)| = n$ and F a minimal feedback arc set of G . Then, $|F| = \binom{n}{2}$ and a feedback arc set is minimal (with respect to set inclusion) if and only if it is minimum (with respect to size). In fact, suppose that $|F| > \binom{n}{2}$, then necessarily there exist u, v in $V(G)$ such that $\{(u, v), (v, u)\} \subseteq F$. Since F is a feedback arc set, there exists path $P_1 = u_1, \dots, u_k$ with $u_1 = v, u_k = u$ which does not contain any arc of F (in particular the arc (v, u)), for some $k \in \llbracket 2, n \rrbracket$ and $u_i \in V(G), \forall i \in \llbracket 1, k \rrbracket$. Analogously, there exists path $P_2 = v_1, \dots, v_j$ with $v_1 = u, v_j = v$ which does not contain any arc of F (in particular the arc (u, v)), for some $j \in \llbracket 2, n \rrbracket$ and $v_i \in V(G), \forall i \in \llbracket 1, j \rrbracket$. Thus, joining the paths P_1 and P_2 we deduce the existence of a cycle which does not contain any arc of F , a contradiction because F is a feedback arc set. Hence $|F| \leq \binom{n}{2}$. On the other hand, $|F| \geq \binom{n}{2}$ because G has $\binom{n}{2}$ cycles of length two.

The following proposition is a simple characterization of update complete digraphs.

Proposition 14. Let G be a complete digraph. Then, (G, lab) is a non update digraph if and only if there exists a forbidden cycle of length either two or three in G_R .

Proof. (\Leftarrow) It is straightforward.

(\Rightarrow) Let (G, lab) be a non-update complete digraph. Then, there exists a forbidden cycle C_R of smallest length in (G_R, lab_R) . Let suppose that the length of C_R is strictly greater than three.

Let C be the forbidden circuit associated to C_R in G . If C has a path of the form $(a, b), (b, c) \oplus$ -labeled, then necessarily (a, c) is also positive in G because otherwise, there would exist a forbidden cycle C' of length three strictly smaller than C_R in G_R . Analogously, if C has a path of the form $(a, b), (b, c) \ominus$ -labeled, then necessarily (a, c) is negative (see Fig. 5: (a) and (b)).

Therefore, we can suppose that C does not have two consecutive arcs with the same label. But when, for example, $lab(b, a) = \ominus$ and $lab(b, c) = \oplus$, necessarily the arc (c, a) is negative in G (i.e., (a, c) is negative in G_R), because on the contrary, there would exist a forbidden cycle C' of length three smaller than C_R in G_R (see Fig. 5: (c) and (d)).

In this way, it is always possible to reduce the length of the forbidden cycle C_R up to a length of three. \square

Proposition 15. Let G be a complete digraph with $|V(G)| = n$. Then, F is a minimal feedback arc set of G if and only if $G' = G - F$ is an acyclic tournament.

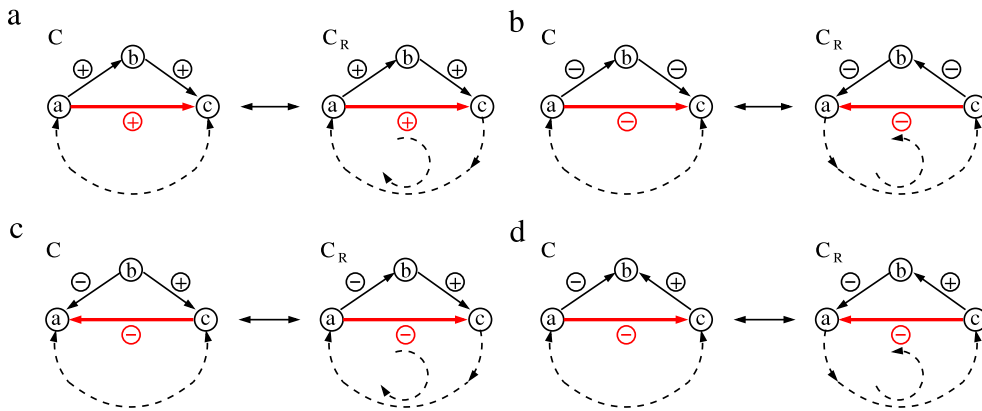


Fig. 5. Proof idea of Proposition 14. (a) When the arcs (a, b) and (b, c) are positive in C , then necessarily (a, c) is also positive in C . (b) When the arcs (a, b) and (b, c) are negative in C , then necessarily (a, c) is also negative in C . (c) If $lab(b, a) = \ominus$ and $lab(b, c) = \oplus$ in C , then necessarily $lab(c, a) = \ominus$. (d) If $lab(a, b) = \ominus$ and $lab(c, b) = \oplus$ in C , then necessarily $lab(a, c) = \ominus$.

Proof. (\Rightarrow) Let F be a minimal feedback arc set of G , then $G' = G - F$ represents an acyclic digraph. Besides, G' is a tournament. In fact, suppose on the contrary that G' is not a tournament, i.e., there is a pair of vertices a, b in $V(G')$ such that $\{(a, b), (b, a)\} \cap A(G') = \emptyset$. That means $\{(a, b), (b, a)\} \subseteq F$ and since F must have at least one arc for each pair of vertices of $V(G)$, we have $|F| \geq \binom{n}{2} + 1$, which is a contradiction because by Remark 13, $|F| = \binom{n}{2}$.

(\Leftarrow) Let G' be an acyclic tournament, then $F = A(G) - A(G')$ is a minimal feedback arc set of G , because $|F| = |A(G) - A(G')| = \binom{n}{2}$. \square

Theorem 16. Let (G, lab) be an update complete digraph with $|V(G)| = n$. The following statements are equivalent:

- (i) (G, lab) has a maximal number of negative arcs.
- (ii) The digraph induced by the negative arcs of (G, lab) is an acyclic tournament.
- (iii) (G, lab) has no positive cycle.

Proof. Let (G, lab) be an update complete digraph with $|V(G)| = n$.

(i) \Rightarrow (ii). If (G, lab) has a maximal number of negative arcs, then by Theorem 8, $F = \{(u, v) \in A(G) : lab(u, v) = \oplus\}$ is a minimal feedback arc set of G . Hence, by Proposition 15, $G' = G - F$ is an acyclic tournament, i.e., the digraph induced by the negative arcs of (G, lab) is an acyclic tournament.

(ii) \Rightarrow (iii). If the digraph induced by the negative arcs of (G, lab) is an acyclic tournament, then the digraph induced by the positive arcs of (G, lab) is also an acyclic tournament because G is a complete digraph, i.e. (G, lab) has no positive cycle.

(iii) \Rightarrow (i). If (G, lab) has no positive cycle, then all its cycles of length two have only one negative arc. Since a complete digraph has $\binom{n}{2}$ cycles of length two, (G, lab) has at least $\binom{n}{2}$ negative arcs, but this is the maximum number, and thus the maximal number of negative arcs that an update complete digraph can have. \square

Corollary 17. Let G be a complete digraph with $|V(G)| = n$. Then, $|MFAS(G)| = |\{lab : (G, lab) \text{ is an update digraph with maximal number of negative arcs}\}| = n!$

Proof. It is easy to see that the number of acyclic tournaments of size n is $n!$. Then, the Corollary 17 follows from Theorem 8 and Proposition 15. \square

Note also due to Theorem 8, Proposition 12 and Theorem 16 that finding all possible minimal feedback arc sets of a complete digraph can be made considering the update digraphs associated with the $n!$ different sequential update schedules, where a minimal feedback arc set will consist of their positive arcs. In other words, if G is a complete digraph, then,

$$MFAS(G) = \{F \subseteq A(G) : lab_{s_q}(a) = \oplus \Leftrightarrow a \in F \wedge s_q \text{ sequential}\}.$$

4. Tournaments and update digraphs

In the previous sections, we found bounds for the sizes of the schedule equivalence classes associated with a given update digraph G as well as relationships between the sets $U(G)$, $MFAS(G)$ and $FAS(G)$. In this section, we will restrict to classical families such as acyclic digraphs, complete digraphs and tournaments in order to determine exactly the number and size of their schedule equivalence classes. Also we will show how the number of negative arcs of a given update digraph (G, lab) is related to these families and the feedback arc sets.

Theorem 18. Let G be an acyclic digraph with $|V(G)| = n$. Then,

- (a) $|U(G)| \leq n!$.
- (b) $|U(G)| = n! \Leftrightarrow G$ is a tournament.
- (c) If G is a tournament, then for each update schedule s of G , $|[s]_G| = 2^k$, for some $k \in \mathbb{N} \cup \{0\}$.

Proof. (a) For every $lab \in U(G)$, (G, lab) has no positive cycle. Hence, by Proposition 12 there exists a sequential schedule s_q such that $(G, lab) = (G, lab_{s_q})$.

Therefore, since the number of different sequential update schedules is $n!$, we have:

$$|U(G)| \leq n!$$

(b) (\Leftarrow) Let G be a tournament with $|V(G)| = n$ and let s_1, s_2 be two different sequential update schedules, then by tournament definition there is $(i, j) \in A(G)$ such that $lab_{s_1}(i, j) \neq lab_{s_2}(i, j)$. Since every sequential update schedule s defines a different update digraph (G, lab_s) and the number of different sequential update schedules is $n!$, we have:

$$|U(G)| \geq n!$$

and due to (a), we conclude that $|U(G)| = n!$.

(\Rightarrow) Let G be an acyclic digraph with $|V(G)| = n$ and $|U(G)| = n!$. Suppose on the contrary that G is not a tournament, then necessarily there exist u, v in $V(G)$ without arcs between them (to have two arcs between them is not possible because G is acyclic) which implies that there are two different sequential update schedules with the same update digraph. This contradicts the fact that $|U(G)| = n!$. Therefore, G is also a tournament.

(c) Let T be an acyclic tournament. We proceed by induction on $m = |V(T)|$.

Basis Step, $m = 2$. There are two possibilities: the equivalence classes of the parallel update schedule s_p , where it is easy to see that $|[s_p]_T| = 2^1 = 2$, and those of the sequential update schedule $s_q \notin [s_p]_T$, where clearly $|[s_q]_T| = 2^0 = 1$.

Induction Hypothesis. T an acyclic tournament with $m \leq n - 1$. Hence, for each update schedule s over $V(T)$, $|[s]_T| = 2^k$, for some $k \in \mathbb{N} \cup \{0\}$.

Let G be an acyclic tournament with $|V(G)| = n$. First, note that the proof of (b) implies the existence of a bijection $f : U(G) \rightarrow \{s \in S_n : s \text{ sequential update schedule on } V(G)\}$. Let (G, lab_s) be an update digraph with the sequential update schedule $s = (i_1)(i_2) \cdots (i_j)(n)(i_{j+1}) \cdots (i_{n-1})$, where $\{i_1, \dots, i_{n-1}\} = \{1, \dots, n - 1\}$.

Let $G' = G - \{n\}$. Clearly, G' is an acyclic tournament with $|V(G')| = n - 1$ and $s' = (i_1) \cdots (i_j)(i_{j+1}) \cdots (i_{n-1})$ is a sequential update schedule such that $(G', lab_{s'})$ is the labeled subdigraph of (G, lab) induced by $\{i_1, \dots, i_{n-1}\}$. Defining $S_{dif} = \{s^* \in [s']_{G'} : s^*(i_j) = s^*(i_{j+1}) - 1\} \neq \emptyset$ and $S_{eq} = \{s^* \in [s']_{G'} : s^*(i_j) = s^*(i_{j+1})\}$, we can see that if $S_{eq} \neq \emptyset$, then $|S_{dif}| = |S_{eq}| = 2^{k-1}$ for some $k \in \mathbb{N} \cup \{0\}$.

There are the following $2^3 = 8$ cases:

Case 1. $lab_s(i_j, n) = \ominus$ and $lab_s(i_{j+1}, n) = lab_s(i_{j+1}, i_j) = \oplus$ (see Fig. 6(a)).

To obtain all the elements of $[s]_G$, for every $s^* \in S_{dif}$, it is enough to generate s_1 and s_2 in $[s]_G$ where $s_1(p) = s_2(p) = s^*(p)$, $\forall p \in V(G')$ such that $s^*(p) \leq s^*(i_j)$; $s_1(n) = s_2(n) = s^*(i_j) + 1$ and $s_2(q) = s_1(q) - 1 = s^*(q)$, $\forall q \in V(G')$ satisfying $s^*(q) \geq s^*(i_{j+1})$. In this way, $|[s]_G| = 2 \cdot |S_{dif}| = 2^k$ for some $k \in \mathbb{N} \cup \{0\}$.

Case 2. $lab_s(n, i_j) = lab_s(i_{j+1}, n) = lab_s(i_{j+1}, i_j) = \oplus$ (see Fig. 6(b)).

To obtain all the elements of $[s]_G$, for every $s^*_1 \in S_{dif}$, we generate s_1, s_2 and s_3 in $[s]_G$ where $s_1(p) = s_2(p) = s_3(p) = s^*_1(p)$, $\forall p \in V(G')$ such that $s^*_1(p) \leq s^*_1(i_j)$; $s_1(n) = s_3(n) = s_2(n) + 1 = s^*_1(i_j) + 1$ and $s_2(q) = s_3(q) = s_1(q) - 1 = s^*_1(q)$, $\forall q \in V(G')$ satisfying $s^*_1(q) \geq s^*_1(i_{j+1})$. And for every $s^*_2 \in S_{eq}$, we generate $s_4 \in [s]_G$ defined by $s_4(p) = s^*_2(p)$, $\forall p \in V(G')$, $p \neq n$ and $s_4(n) = s^*_2(i_j)$. Therefore, $|[s]_G| = 3 \cdot |S_{dif}| + |S_{eq}| = 3 \cdot 2^{k-1} + 2^{k-1} = 2^{k+1}$ for some $k \in \mathbb{N} \cup \{0\}$.

Case 3. $lab_s(i_j, n) = lab_s(i_j, i_{j+1}) = lab_s(n, i_{j+1}) = \ominus$ (see Fig. 6(c)).

Here, for each $s^*_1 \in S_{dif}$, we generate $s_1 \in [s]_G$ where $s_1(p) = s^*(p)$, $\forall p \in V(G')$ such that $s^*(p) \leq s^*(i_j)$; $s_1(n) = s^*(i_j) + 1$ and $s_1(q) = s^*(q) + 1$, $\forall q \in V(G')$ satisfying $s^*(q) \geq s^*(i_{j+1})$. Therefore, $|[s]_G| = |S_{dif}| = 2^{k-1}$ for some $k \in \mathbb{N} \cup \{0\}$.

Case 4. $lab_s(i_j, n) = lab_s(i_j, i_{j+1}) = \ominus$ and $lab_s(i_{j+1}, n) = \oplus$.

Case 5. $lab_s(n, i_j) = lab_s(i_{j+1}, i_j) = \oplus$ and $lab_s(n, i_{j+1}) = \ominus$.

Case 6. $lab_s(n, i_{j+1}) = lab_s(i_j, i_{j+1}) = \ominus$ and $lab_s(n, i_j) = \oplus$.

Cases 4, 5 and 6 are similar to case 1. \square

Observe that the assertion (c) of Theorem 18 is not true for tournaments in general (see Fig. 7 as a counter-example).

Corollary 19. Let (G, lab) be an update digraph where G is an acyclic tournament. Then, all arcs of (G, lab) are negative, if and only if there exists a sequential update schedule s_q such that $(G, lab) = (G, lab_{s_q})$ and $|[s_q]_G| = 1$.

Proof. Let G be an acyclic tournament. First, observe that every acyclic digraph (G, lab) with all its arcs negative is an update digraph.

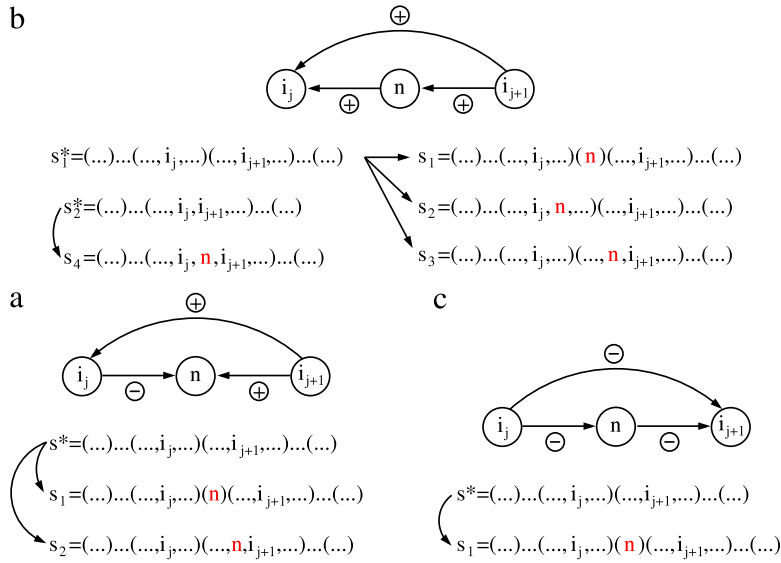


Fig. 6. Proof idea of Theorem 18. (a) Case 1. (b) Case 2. (c) Case 3.

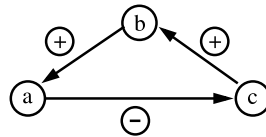


Fig. 7. An update digraph (G, lab_s) where $||s||_G = |\{(a)(b)(c), (a)(b, c), (a, b)(c)\}| = 3$.

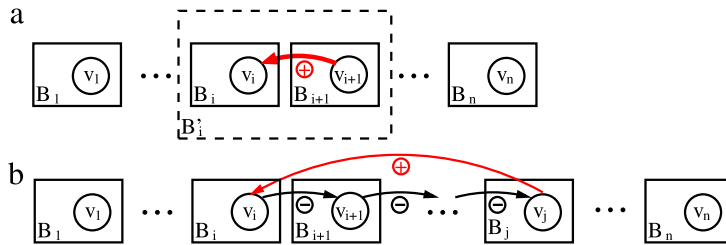


Fig. 8. Proof idea of Corollary 19. (a) Case $j = i + 1$. If there is a positive arc from the B_{i+1} to B_i , it is possible to construct a new update schedule $s \neq s_q$ with block $B'_i = B_i \cup B_{i+1}$. (b) Case $j > i + 1$. There is necessarily a cycle $(v_i, v_{i+1}), (v_{i+1}, v_{i+2}), \dots, (v_{j-1}, v_j), (v_j, v_i)$.

(\Rightarrow) Suppose that all $|A(G)| = \binom{n}{2}$ arcs of (G, lab) are negative. Then, by Proposition 12, there exists a sequential update schedule s_q such that $(G, lab) = (G, lab_{s_q})$. Now, suppose that there exists another update schedule $s \neq s_q$ such that $s \in [s_q]_G$. By Theorem 18, we know that s cannot be a sequential schedule. If s is an update schedule but not sequential, then there is a block B_i of s , $1 \leq i \leq nb(s) < |V(G)|$, such that $|B_i| > 1$, i.e., there exists u, v in B_i such that the arc $(u, v) \in A(G)$ is positive in (G, lab_s) , but the same arc is negative in (G, lab_{s_q}) , which is a contradiction. Therefore, $||s_q||_G = 1$.

(\Leftarrow) Let $s_q = (v_1)(v_2) \cdots (v_n)$ be a sequential update schedule such that $(G, lab) = (G, lab_{s_q})$, $||s_q||_G = 1$ and $V(G) = \{v_1, \dots, v_n\}$. Since (G, lab) is an update tournament, for every pair of vertices v_i, v_j in $V(G)$, there is only one arc between them, either the negative arc (v_i, v_j) or the positive arc (v_j, v_i) with $i, j \in \{1, \dots, n\}$. Observe that $\forall i = 1, \dots, n - 1$, $(v_i, v_{i+1}) \in A(G)$, since otherwise there would be another update schedule $s \neq s_q$, $s = (v_1)(v_2) \cdots (v_{i-1})(v_i, v_{i+1})(v_{i+2}) \cdots (v_n)$ such that $(G, lab) = (G, lab_{s_q}) = (G, lab_s)$, which contradicts that $||s_q||_G = 1$ (see Fig. 8(a)). Besides, for every $i \neq j \in \{1, \dots, n\}$ such that $j > i + 1$, $(v_i, v_j) \in A(G)$ because if there exists the positive arc (v_j, v_i) , from the above mentioned, we would have the existence of the negative arcs $(v_i, v_{i+1}), (v_{i+1}, v_{i+2}), \dots, (v_{j-1}, v_j)$ which implies the existence of a cycle (see Fig. 8(b)). It would be a contradiction, because G is acyclic. Therefore, all arcs of (G, lab) must be negative. \square

Observe from the previous proof, that if G is a tournament with at least one cycle, then there exists an update schedule s such that $|s|_G$ does not contain a sequential schedule. Hence, $|U(G)| > n!$.

Theorem 20. Let G be a digraph with $|V(G)| = n$, $P = \{B_i\}_{i=1}^m$ a partition of $V(G)$, $m \leq n$, and $s_\pi = (B_{\pi(i)})_{i=1}^m \in S_n$, where π is a permutation over $\llbracket 1, m \rrbracket$. Then, the following statements are equivalents:

- (1) $s_\pi \in S_n^*$.
- (2) $\forall i \in \llbracket 1, m \rrbracket$, $G[B_i]$ is a positive strongly connected component of (G, lab_{s_π}) .
- (3) $\forall i \in \llbracket 1, m \rrbracket$, $G[B_i]$ is a strongly connected subdigraph of G .

Proof. Let G be a digraph with $|V(G)| = n$, $P = \{B_i\}_{i=1}^m$ a partition of $V(G)$, $m \leq n$, and $s_\pi = (B_{\pi(i)})_{i=1}^m \in S_n$, where π is a permutation over $\llbracket 1, m \rrbracket$.

(1) \Rightarrow (2) Suppose that $s_\pi = (B_{\pi(i)})_{i=1}^m \in S_n^*$, i.e., s_π has the maximum number of blocks. We note that P is the partition associated to s_π . Now, let us prove that $\forall i \in \llbracket 1, m \rrbracket$, $G[B_i]$ is a positive strongly connected component of (G, lab_{s_π}) , i.e. $G[B_i]$ is a strongly connected component of $G' = G - \{a \in A(G) : lab_{s_\pi}(a) = \emptyset\}$. If $|B_i| = 1$, then the proof is done. Hence, we can assume w.l.o.g. that $|B_i| \geq 2$. Suppose on the contrary that $G[B_i]$ is not a positive strongly connected component of (G, lab_{s_π}) . Then, either $G[B_i]$ is not a subdigraph of G' , which means that $\exists a \in A(G') \cap A(G)$, $lab_{s_\pi}(a) = \emptyset$, but this is impossible because (G, lab_{s_π}) is an update digraph, or $G[B_i]$ is a strongly connected subdigraph of G' , but not maximal for this property. i.e., $\exists u \notin B_i$ strongly connected with the nodes of B_i by positive arcs, this means that $s_\pi(u) = s_\pi(j)$, $\forall j \in B_i$, which is a contradiction because $u \notin B_i$, or $G[B_i]$ is not strongly connected, then there exists a strongly connected component $U \subseteq B_i$ which is a source, i.e. without incoming arc to U , having initial vertex in $B_i - U$. Thus, since $G[B_i] = G[B_{\pi(j)}]$ for some $j \in \llbracket 1, m \rrbracket$, there is another update schedule $s' = (k \in B_{\pi(1)}) \cdots (k \in B_{\pi(j)} - U)(k \in U)(k \in B_{\pi(j+1)}) \cdots (k \in B_{\pi(m)})$ on $V(G)$ such that $s' \in [s_\pi]_G$ and $nb(s') = nb(s_\pi) + 1 = m + 1$, which contradicts that s_π has the maximum number of blocks.

(2) \Rightarrow (3) If $G[B_i]$ is a positive strongly connected component of (G, lab_{s_π}) , then $G[B_i]$ is a strongly connected component of $G' = G - \{a \in A(G) : lab_{s_\pi}(a) = \emptyset\} \subseteq G$, i.e., $G[B_i]$ is a strongly connected subdigraph of G .

(3) \Rightarrow (1) Suppose that $\forall i \in \llbracket 1, m \rrbracket$, $G[B_i]$ is a strongly connected subdigraph of G . Let $s' \in [s_\pi]_G$, then $(G, lab_{s'}) = (G, lab_{s_\pi})$. Thus, s' must satisfy that for every $i \in \llbracket 1, m \rrbracket$, $s'(j) = s'(k)$, $\forall j, k \in B_i$, because s' is an update schedule on $V(G)$ and $\forall i \in \llbracket 1, m \rrbracket$, $G[B_i]$ is a strongly connected subdigraph of G with all its arcs positive in (G, lab_{s_π}) . Hence, $P = \{B_i\}_{i=1}^m$ is also the partition of $V(G)$ associated to s' , then necessarily $nb(s') = nb(s) = m$. Therefore, we have shown that, $\forall s' \in [s_\pi]_G$, $nb(s') \leq nb(s_\pi)$, i.e., $s_\pi \in S_n^*$. \square

Lemma 21. Let G be a digraph with $|V(G)| = n$, containing a tournament of n vertices as a subdigraph. Then, for every $s_1, s_2 \in S_n^*$ such that $s_1 \neq s_2$, we have: $(G, lab_{s_1}) \neq (G, lab_{s_2})$.

Proof. Let G be a digraph with $|V(G)| = n$ containing a tournament of n vertices as a subdigraph. Let $s_1, s_2 \in S_n^*$, $s_1 \neq s_2$ and suppose on the contrary that $(G, lab_{s_1}) = (G, lab_{s_2})$. Let $s_1 = (B_i^1)_{i=1}^{nb(s_1)}$, $s_2 = (B_i^2)_{i=1}^{nb(s_2)}$ and $j = \min\{i : B_i^1 \neq B_i^2\}$. Hence, there are two cases:

Case 1. $B_j^1 \cap B_j^2 \neq \emptyset$. Then, from (2) of Theorem 20, B_j^1 and B_j^2 are positive strongly connected components of (G, lab_{s_1}) . On the other hand, because the decomposition of (G, lab_{s_1}) in positive strongly connected components is unique, then $B_j^1 = B_j^2$, which is a contradiction.

Case 2. $B_j^1 \cap B_j^2 = \emptyset$. Then, $\exists p \in B_j^1, \exists q \in B_j^2, p \neq q: s_1(p) < s_1(q) \wedge s_2(q) < s_2(p)$. Since G has a tournament of n vertices, $(p, q) \in A(G) \vee (q, p) \in A(G)$, i.e., $lab_{s_1}(p, q) \neq lab_{s_2}(p, q) \vee lab_{s_1}(q, p) \neq lab_{s_2}(q, p)$, which contradicts that $(G, lab_{s_1}) = (G, lab_{s_2})$.

Therefore, $(G, lab_{s_1}) \neq (G, lab_{s_2})$. \square

Theorem 22. Let G be a digraph with $|V(G)| = n$, containing a tournament of n vertices as a subdigraph. Then, $|U(G)| = |S_n^*|$.

Proof. Let G be a digraph with $|V(G)| = n$, containing a tournament of n vertices as a subdigraph.

Evidently, due to Lemma 21, we have $|U(G)| \geq |S_n^*|$.

On the other hand, $\forall lab \in U(G), \exists s \in S_n: (G, lab) = (G, lab_s)$. Hence, $\exists s' \in [s]_G$ with the maximum number of blocks. Thus, $|U(G)| \leq |S_n^*|$. \square

Observe from Theorem 20 that:

$$S_n^* = \{(B_{\pi(i)})_{i=1}^m : P = \{B_i\}_{i=1}^m \in \mathbb{P}_n^* \wedge \pi \text{ is a permutation over } \llbracket 1, m \rrbracket\},$$

where \mathbb{P}_n^* is the set of partitions $P = \{B_i\}_{i=1}^m$ of $V(G)$ such that $\forall i \in \llbracket 1, m \rrbracket$, $G[B_i]$ is a strongly connected subdigraph of G . Thus, we obtain straightforwardly the following corollary:

Corollary 23. Let G be a digraph with $|V(G)| = n$ containing a tournament of n vertices as a subdigraph. Then:

$$|U(G)| = \sum_{P \in \mathbb{P}_n^*} card(P)!$$

Note that if G is an acyclic tournament of n vertices, the only strongly connected subdigraphs are the trivial ones, i.e., $\mathbb{P}_n^* = \{\{\{1\}, \dots, \{n\}\}\}$ and therefore,

$$|U(G)| = \sum_{P \in \mathbb{P}_n^*} card(P)! = card(\{\{1\}, \dots, \{n\}\})! = n!,$$

giving the result (b) of Theorem 18.

On the other hand, if G is a complete digraph with $|V(G)| = n$, then every subset of vertices defines a strongly connected subdigraph of G . In this way, \mathbb{P}_n^* is the set of all the possible partitions over $V(G)$ and therefore, $|U(G)| = T_n = \sum_{k=0}^{n-1} \binom{n}{k} T_k$, with $T_0 \equiv 1$, as was obtained in [4,1]. Moreover, if G_1 and G_2 are two orientations of a complete undirected graph satisfying the conditions of Theorem 6, then a strongly connected subdigraph of G_1 is also a strongly connected subdigraph of G_2 . Therefore, Corollary 23 implies that $|U(G_1)| \leq |U(G_2)|$, as established in Theorem 6.

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