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Pre-Schwarzian and Schwarzian Derivatives of Harmonic Mappings

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Abstract In this paper we introduce a definition of the pre-Schwarzian and the Schwarzian derivatives of *any* locally univalent harmonic mapping f in the complex plane without assuming any additional condition on the (second complex) dilatation ω_f of f. Using the new definition for the Schwarzian derivative of harmonic mappings, we prove theorems analogous to those by Chuaqui, Duren, and Osgood. Also, we obtain a Becker-type criterion for the univalence of harmonic mappings.

Keywords Pre-Schwarzian derivative · Schwarzian derivative · Harmonic mappings · Univalence · Becker's criterion · Convexity

Mathematics Subject Classification Primary 30C55 · Secondary 31A05 · 30C45

1 Introduction

Let f be a locally univalent analytic function defined on a simply connected domain $\Omega \subset \mathbb{C}$. The *pre-Schwarzian derivative* Pf of f is the logarithmic derivative of f'.

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That is,

$$Pf = \frac{f''}{f'}. \tag{1}$$

The *Schwarzian derivative* of such a function f is defined by

$$Sf = (Pf)' - \frac{1}{2}(Pf)^2. \tag{2}$$

It is well known that $P(A \circ f) = Pf$ for all linear (or affine) transformations $A(w) = aw + b$, $a \neq 0$. Also, that the Schwarzian derivative is invariant under (non-constant) linear fractional (or Möbius) transformations. In other words, $S(T \circ f) = S(f)$ for every function T of the form

$$T(w) = \frac{aw+b}{cw+d}, \quad ad - bc \neq 0.$$

These are just particular cases of the *chain rule* for the pre-Schwarzian and Schwarzian derivatives:

$$P(g \circ f) = (Pg \circ f)f' + Pf \quad \text{and} \quad S(g \circ f) = (Sg \circ f)(f')^2 + Sf, \tag{3}$$

respectively, which hold whenever the composition $g \circ f$ is defined.

The pre-Schwarzian and the Schwarzian derivatives of locally univalent analytic mappings f are widely used tools in the study of geometric properties of such functions. For instance, they can be used to get either necessary or sufficient conditions for the global univalence, or to obtain certain geometric conditions on the range of f. More specifically, it is well known that any *univalent* analytic transformation f in the unit disk $\mathbb{D}$ satisfies the sharp inequality

$$|Pf(z)| \leq \frac{6}{1-|z|^2}, \quad z \in \mathbb{D}.$$

Also, *Becker's univalence criterion* [2] states that if f is locally univalent (and analytic) in $\mathbb{D}$ and

$$\sup_{z \in \mathbb{D}} |zPf(z)|(1-|z|^2) \leq 1,$$

then f is univalent in the unit disk. Becker and Pommerenke [3] proved later that the constant 1 is sharp.

In terms of the Schwarzian derivative, we have the sharp inequality

$$|Sf(z)| \leq \frac{6}{(1-|z|^2)^2} \quad (|z| < 1)$$

that holds for all univalent analytic functions f in $\mathbb{D}$. This result is due to Krauss [19] and was rediscovered by Nehari [22] who also proved that if a locally univalent analytic function f in the unit disk satisfies

$$|Sf(z)| \leq \frac{2}{(1-|z|^2)^2}, \quad |z| < 1, \tag{4}$$

then f is univalent. The constant $C = 2$ is sharp. Moreover, whenever the Schwarzian derivative of a function f as above satisfies $|Sf(z)|(1-|z|^2)^2 \le C$ in the unit disk for some constant $C < 2$, then not only is f univalent in $\mathbb{D}$ but it also maps the disk onto a Jordan domain on the Riemann sphere (see [1] and [15]).

It is possible to prove (4) using the connection of the Schwarzian derivative with a certain linear differential equation which indeed solves the problem of finding the most general analytic function with prescribed Schwarzian derivative: A function f has Schwarzian derivative $Sf = 2p$ if and only if it has the form $f = u_1/u_2$ for some pair of independent solutions u_1 and u_2 of the linear differential equation $u'' + pu = 0$. As a corollary, it is easy to obtain that if $Sf = Sg$ then $g = T \circ f$ for some linear fractional transformation T. In particular, the only analytic functions with $Sf = 0$ are the Möbius transformations. There is still another consequence: any analytic function is the Schwarzian derivative of some locally univalent function f.

For more properties and results related to the pre-Schwarzian and Schwarzian derivatives of locally univalent analytic mappings, we refer the reader to the monographs [13, 25], or [17].

A definition of the Schwarzian derivative for a more general class of complex-valued *harmonic* functions was presented by Chuaqui, Duren, and Osgood in [5]. The formula was derived by passing to the minimal surface associated locally with a given harmonic function and appealing to a definition given in [23] for the Schwarzian derivative of a conformal mapping between arbitrary Riemannian manifolds. Specifically, recall that a complex-valued harmonic function in a simply connected domain has a canonical representation $f = h + \overline{g}$ (where h and g are analytic functions) that is unique up to an additive constant. Any harmonic mapping f with $|h'| + |g'| \neq 0$ lifts to a mapping $\widetilde{f}$ onto a minimal surface defined by conformal parameters if and only if the dilatation $\omega = g'/h'$ equals the square of an analytic function q. In other words, $\omega = q^2$ for some analytic function q. For such mappings f, the Schwarzian derivative presented in [5] (which we denote by $\mathbb{S}$) is defined by the formula

$$\mathbb{S}f = 2(\sigma_{zz} - \sigma_z^2),$$

where $\sigma = \log(|h'| + |g'|)$ and

$$\sigma_z = \frac{\partial \sigma}{\partial z} = \frac{1}{2}\left(\frac{\partial \sigma}{\partial x} - i\frac{\partial \sigma}{\partial y}\right), \qquad z = x + iy.$$

In terms of the canonical representation $f = h + \overline{g}$ and the dilatation $\omega = q^2$,

$$\mathbb{S}f = Sh + \frac{2\overline{q}}{1+|q|^2}\left(q'' - q'\frac{h''}{h'}\right) - 4\left(\frac{q'\overline{q}}{1+|q|^2}\right)^2, \tag{5}$$

where Sh is the classical Schwarzian derivative of the analytic function h. The Schwarzian derivative $\mathbb{S}f$ is well defined in a neighborhood of any point where ω has a zero of odd order. If q has a zero of order at least 2 at a point z_0, then $\mathbb{S}f$ tends to Sh as $z \to z_0$ (see [5]). The Schwarzian derivative $\mathbb{S}f$ is well defined for some harmonic mappings f that are not locally univalent. For instance, as an extreme example of this fact, we can consider the harmonic mapping $f(z) = z + \overline{z} = 2\,\mathrm{Re}\{z\}$, $z \in \mathbb{D}$. Since $|h'| + |g'| \neq 0$, we can define $\mathbb{S}f$. Indeed, it is easy to see that $\mathbb{S}f(z) \equiv 0$ in $\mathbb{D}$.

Chuaqui, Duren, and Osgood, using their definition (5) for the Schwarzian derivative of harmonic mappings, have obtained many interesting results related to different questions as some curvature properties, the univalence of the Weierstrass–Enneper lift, or a characterization of the uniform local univalence (with respect to the hyperbolic metric), among others (see [6–10]).

The Schwarzian derivative (5) has one disadvantage that arises from the fact that the dilatation is required to fulfill an extra condition. This causes that in many cases one cannot define (globally) the Schwarzian derivative of a univalent harmonic mapping f in a simply connected domain Ω. For instance, although it is known [9] that any univalent harmonic mapping f in the unit disk $\mathbb{D}$ with dilatation $\omega_f = q^2$ satisfies

$$|\mathbb{S}f(z)| \le \frac{C}{(1-|z|^2)^2}, \quad |z| < 1,$$

where C is a constant with unknown sharp value, it is not possible to state an analogous result for the whole family of harmonic mappings in the unit disk in terms of $\mathbb{S}$.

It should perhaps be pointed out that also, even in the case when f is a univalent harmonic mapping in Ω with dilatation $\omega_f = q^2$, the Schwarzian derivative of the composition $F = A \circ f$, where A is an *affine harmonic mapping* given by $A(w) = aw + b\overline{w} + c$ $(a, b, c \in \mathbb{C})$, may fail to be defined at some point $z_0 \in \Omega$ because of the fact that the dilatation ω_F of F can have a simple zero at z_0. Keeping in mind that the correspondence between the well-known families of univalent harmonic mappings suitably normalized, S_H (which is a normal family, but non-compact), and S_H^0 (normal and compact) is realized via affine harmonic mappings (see [14, Sects. 5.1 and 5.2]), the fact that $\mathbb{S}F$ is not defined at z_0 results in the loss of normality argument tools to solve certain extremal problems involving the Schwarzian derivative (and related to univalent harmonic mappings).

The main purpose of this paper is to present another definition of the Schwarzian derivative (which will be denoted by S_f to differentiate it from the definitions of the Schwarzian derivatives already mentioned) for *any* locally univalent harmonic mapping f in a simply connected domain in the complex plane. We will justify the new definition in two different analytic ways: the first one consists in applying a "harmonic analogue" of the classical argument of approximating by Möbius transformations that seems to go back to E. Cartan [4] (see also [16, p. 113]). The second one is based on a relation between S_f and the classical formula (2) for locally univalent analytic functions (in terms of the Jacobian). This relation will be used to define the *pre-Schwarzian derivative* P_f of a locally univalent harmonic mapping.

From the geometrical point of view, we will see that S_f equals the Schwarzian tensor of a certain conformal metric relative to the Euclidean metric (just like $\mathbb{S}$).

We will prove that not only S_f satisfies all the properties that $\mathbb{S}f$ does (characterization of the harmonic Möbius transformations, chain rule, ...), but also others that seem to be natural in the context of the theory of harmonic mappings. For instance, that $S_{(A\circ f)} = S_f$ for any affine harmonic mapping A as above and any locally univalent harmonic mapping f in a simply connected domain. Moreover, although $\mathbb{S}f$ and

S_f are different (in the cases when $\mathbb{S}f$ is defined), the quantities

$$\|\mathbb{S}f\| = \sup_{|z|<1} |(\mathbb{S}f)(z)|(1-|z|^2)^2 \quad \text{and} \quad \|S_f\| = \sup_{|z|<1} |S_f(z)|(1-|z|^2)^2$$

are simultaneously finite or infinite. Hence, we are able to prove the theorems by Chuaqui, Duren, and Osgood involving $\|\mathbb{S}f\|$ and related to uniform local univalence in terms of the norm $\|S_f\|$ of the "new" Schwarzian derivative (in many cases, using the same arguments as they did). The statements of the corresponding theorems in this paper hold for the full class of locally univalent harmonic mappings in the unit disk with no extra assumption on the dilatation.

We will also prove other additional results. For instance, we will show a criterion of univalence for sense-preserving harmonic mappings that generalizes the classical Becker's criterion.

In terms of the canonical representation of a sense-preserving harmonic mapping $f = h + \overline{g}$ in a simply connected domain Ω with dilatation $\omega = g'/h'$, we define S_f (in Ω) by

$$S_f = Sh + \frac{\overline{\omega}}{1-|\omega|^2}\left(\frac{h''}{h'}\omega' - \omega''\right) - \frac{3}{2}\left(\frac{\omega'\overline{\omega}}{1-|\omega|^2}\right)^2.$$

2 Background

2.1 Harmonic Mappings in the Complex Plane

A straightforward calculation in terms of the well-known Wirtinger operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad z = x + iy,$$

shows that the Laplacian of f is

$$\triangle f = 4\frac{\partial^2 f}{\partial z \partial \overline{z}}. \tag{6}$$

Therefore, for functions f with continuous second partial derivatives, we have that f is harmonic if and only if $\partial f/\partial z$ is analytic. This fact proves that a complex-valued harmonic function f in a simply connected domain $\Omega \in \mathbb{C}$ has the representation $f = h + \overline{g}$ (where h and g are analytic functions in Ω) that is unique up to an additive constant. For a harmonic mapping f of the unit disk $\mathbb{D}$, it is convenient to choose the additive constant so that $g(0) = 0$. The representation $f = h + \overline{g}$ is therefore unique and is called the *canonical representation* of f.

It is a consequence of the inverse mapping theorem that if the Jacobian of a C^1 mapping $f : \mathbb{C} \to \mathbb{C}$ does not vanish, then f is locally univalent. A theorem of Lewy [20] asserts that the converse is also true for harmonic mappings. Specifically, a harmonic mapping $f = h + \overline{g}$ in a simply connected domain Ω is locally univalent if and only if its Jacobian, $J_f = |h'|^2 - |g'|^2$, does not vanish in Ω. By Lewy's theorem, harmonic mappings are either *sense-preserving* or *sense-reversing* depending on the conditions $J_f > 0$ and $J_f < 0$ throughout the domain Ω where f is locally univalent, respectively. If f is sense-preserving, then $\overline{f}$ is sense-reversing. Locally univalent analytic functions are sense-preserving. Notice that if $f = h + \overline{g}$ is sense-preserving, its analytic part h is locally univalent: $h' \neq 0$ in Ω, and the second complex dilatation of f, $\omega_f = g'/h'$, is an analytic function in Ω with $|\omega_f| < 1$.

2.2 Univalent Harmonic Mappings in $\mathbb{D}$. The Shear Construction

A harmonic mapping f in Ω is *univalent* if $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in \Omega$ with $z_1 \neq z_2$. Let f be a univalent harmonic mapping in the unit disk and $f = h + \overline{g}$ its canonical representation (with $g(0) = 0$). If f is sense-preserving, then h is locally univalent, so there is no loss of generality in assuming that $h(0) = 1 - h'(0) = 0$. The class of all sense-preserving univalent harmonic mappings in the unit disk with the normalizations $g(0) = h(0) = 1 - h'(0) = 0$ is denoted by S_H. This class S_H is not compact (see [14, p. 78]). To produce a compact normal family it is enough to use one further normalization: the family $S_H^0 = \{f \in S_H : g'(0) = 0\}$ is compact. Note that if $f = h + \overline{g} \in S_H$ and $g'(0) = b_1 \in \mathbb{D}$, the composition $f_0 = \tau \circ f$, where

$$\tau(w) = \frac{w - \overline{b_1}\overline{w}}{1 - |b_1|^2},$$

belongs to S_H^0. Conversely, given any value of $b_1 \in \mathbb{D}$, there is a unique function $f = f_0 + \overline{b_1 f_0} \in S_H$ with $g'(0) = b_1$ that corresponds to a given function $f_0 \in S_H^0$. In other words, the correspondence between functions in the families S_H and S_H^0 is realized using affine harmonic mappings of the form $A(w) = aw + b\overline{w}$ with $a \neq 0$ and $|b| < |a|$.

An effective way of constructing sense-preserving univalent harmonic mappings in $\mathbb{D}$ is the so-called *shear construction* introduced in the paper [12] by Clunie and Sheil-Small. Recall that a function f is convex in the θ direction ($0 \le \theta < 2\pi$) if the intersection of the domain $f(\mathbb{D})$ with every line parallel to the line through 0 and $e^{i\theta}$ is either the empty set or an interval. In [12], it was proved that given a locally univalent harmonic mapping $f = h + \overline{g}$ in the unit disk, then f is univalent and convex in the θ direction if and only if the analytic function $h - e^{2i\theta} g$ is univalent and convex in the θ direction. This result was used in [12] to construct two explicit examples of univalent harmonic mappings in $\mathbb{D}$: the *harmonic Koebe function* and the *half-plane harmonic mapping*. Many other examples of univalent harmonic mappings constructed using the shear construction are shown in [14, Sect. 3.4]. We review some of them.

The *harmonic Koebe function* is defined by $K = h + \overline{g}$, where

$$h(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}, \qquad g(z) = \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} \quad (z \in \mathbb{D}). \tag{7}$$

This function K maps $\mathbb{D}$ onto the full plane minus the part of the negative real axis from $-1/6$ to infinity and can be obtained as the horizontal shear (that is, the shear in the 0 direction) of the Koebe mapping $k(z) = z/(1-z)^2$ with dilatation $\omega(z) = z$. For many reasons, it is thought that the harmonic Koebe mapping K is the probable analogue of the Koebe function k for the class S_H^0 of harmonic univalent functions.

The *half-plane harmonic mapping* $L = h + \overline{g}$ was defined in [12] as the vertical shear (the shear in the $\pi/2$ direction) of the function $\ell(z) = z/(1-z)$ with dilatation $\omega(z) = -z$. The function L maps the disk onto the half-plane $\{z \in \mathbb{C}\colon \operatorname{Re}\{z\} > -1/2\}$. The explicit formulas for h and g are

$$h(z) = \frac{2z - z^2}{2(1-z)^2}, \qquad g(z) = \frac{-z^2}{2(1-z)^2}. \tag{8}$$

Two other univalent harmonic mappings will be important for our purposes: The *half-strip harmonic mapping* S_1 and the *strip harmonic mapping* S_2. Both functions are horizontal shears of the conformal mapping

$$s(z) = \frac{1}{2} \log \frac{1+z}{1-z} \tag{9}$$

that maps $\mathbb{D}$ onto the domain $\Omega = \{z \in \mathbb{C}\colon |\operatorname{Im}\{z\}| < \pi/4\}$. The mapping S_1 has dilatation $w(z) = z$. The dilatation of S_2 is $w(z) = z^2$. Concretely, it can be seen that

$$S_1 = h_1 + \overline{g_1}, \quad \text{where } h_1 = \frac{1}{2}(\ell + s) \quad \text{and} \quad g_1(z) = \frac{1}{2}(\ell - s) \tag{10}$$

in the unit disk. Moreover,

$$S_1(\mathbb{D}) = \left\{ w \in \mathbb{C}\colon \operatorname{Re}\{w\} > -\frac{1}{2}, |\operatorname{Im}\{w\}| < \frac{\pi}{4} \right\}.$$

The harmonic mapping $S_2 = h_2 + \overline{g}_2$ is defined by

$$h_2 = \frac{1}{2}(q + s) \quad \text{and} \quad g_2(z) = \frac{1}{2}(q - s), \tag{11}$$

where $q(z) = \sqrt{k(z^2)} = z/(1-z^2)$, $|z| < 1$. Note that $S_2(\mathbb{D}) = s(\mathbb{D})$.

Each function L, S_1, and S_2 maps $\mathbb{D}$ onto a convex domain.

3 Two Analytic Derivations of the Formula for the Schwarzian Derivative of Locally Univalent Harmonic Mappings

In this section, we derive the formula for the Schwarzian derivative S_f mentioned in the Introduction in two different analytic ways. We also see in Sect. 3.3 that S_f equals the Schwarzian tensor of a certain conformal metric (relative to the Euclidean metric).

3.1 Approximation by Harmonic Möbius Transformations

In the analytic case, the Schwarzian derivative appears as the third Taylor coefficient of a certain function that comes up by approximating by a certain Möbius transformation. This natural idea already appears in [4] and was applied in Tamanoi's paper [27] to define Schwarzian derivatives of higher order.

In this section, we review this classical argument and show how to use it to propose our definition for the Schwarzian derivative of locally univalent harmonic mappings.

For a locally univalent analytic function f in the simply connected domain Ω with $0 \in \Omega$, the *best Möbius approximation of* f *at the origin* is the linear fractional transformation

$$T_f(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0,$$

which approximates $f(z)$ at $z = 0$ up to the second order: $T_f(0) = f(0)$, $T_f'(0) = f'(0)$, and $T_f''(0) = f''(0)$. Since the family of Möbius transformations is the complex 3-dimensional group $PSL_2(\mathbb{C})$, the three conditions given above determine T_f uniquely.

The *deviation of a locally univalent analytic function* f *from its best Möbius transformation* is the function $F_f = T_f^{-1} \circ f$. Let the Taylor expansion of F_f at the origin be

$$F_f(z) = \sum_{n=0}^{\infty} s_n \frac{z^n}{n!} = z + \frac{1}{3!} Sf(0) z^3 + \cdots,$$

where $s_n = s_n(f)$ are the Taylor coefficients of F_f and S is the classical Schwarzian derivative. The Taylor coefficients s_n are invariant under pre-composition with Möbius transformations (see [27, Lemma 3-1]). In other words, $s_n(T \circ f) = s_n(f)$ for all (non-constant) Möbius transformations T and each non-negative integer n. Note that $SF_f(0) = 6(s_3 - s_2{}^2) = Sf(0)$ since, in this case, $s_2 = 0$.

For a point $w \in \Omega \setminus \{0\}$, consider the analytic function $f_w = f(w+z)$ which is well defined in the domain $\Omega_w = \{z \in \mathbb{C} \colon z - w \in \Omega\}$ (which contains the origin). Let T_{f_w} be the best Möbius approximation of f_w at $z = 0$. It can be checked (see [27, p. 136]) that

$$F_{f_w}(z) = T_{f_w}^{-1} \circ f_w(z) = \sum_{n=0}^{\infty} s_n(f)(w) \frac{z^n}{(n)!} = z + \frac{1}{3!} Sf(w) z^3 + \cdots.$$

Therefore,

$$Sf(w) = SF_f(w) = 6\left(s_3(f)(w) - (s_2(f)(w))^2\right).$$

We will follow the same argument as before to derive a formula for the Schwarzian derivative of a sense-preserving harmonic mapping in the complex plane.

A *sense-preserving harmonic Möbius transformation* M is a transformation of the form

$$M = T + \alpha \overline{T},$$

where T is a (classical) non-constant Möbius transformation and α is a constant with $|\alpha| < 1$. Let $f = h + \overline{g}$ be a sense-preserving harmonic mapping in a simply connected domain $\Omega \subset \mathbb{C}$ that contains the origin. We define the *best harmonic Möbius approximation of f at the origin* as the harmonic Möbius transformation $M_f(z)$ that satisfies $M_f(0) = f(0)$,

$$\frac{\partial M_f}{\partial z}(0) = \frac{\partial f}{\partial z}(0) = h'(0), \qquad \frac{\partial M_f}{\partial \overline{z}}(0) = \frac{\partial f}{\partial \overline{z}}(0) = \overline{g'(0)},$$

and

$$\frac{\partial^2 M_f}{\partial z^2}(0) = \frac{\partial^2 f}{\partial z^2}(0) = h''(0).$$

Notice that these four conditions determine M_f uniquely.

We define the *deviation of a sense-preserving harmonic function f from its best harmonic Möbius transformation* as the function $F_f = (M_f)^{-1} \circ f$. The Taylor expansion of F in terms of z and $\overline{z}$ has the form

$$\begin{aligned} F_f(z) = z &- \frac{1}{2!}\left(\frac{\overline{\omega(0)}\omega'(0)}{1-|\omega(0)|^2}\right) z^2 + \frac{1}{2!}\left(\frac{\overline{h'(0)\omega'(0)}}{h'(0)(1-|\omega(0)|^2)}\right)\overline{z}^2 \\ &+ \frac{1}{3!}\left(Sh(0) + \frac{\overline{\omega(0)}}{1-|\omega(0)|^2}\left(\omega'(0)\frac{h''}{h'}(0) - \omega''(0)\right)\right) z^3 \\ &- \frac{1}{3!}\left(\frac{h''(0)\overline{h'(0)\omega'(0)}}{(h'(0))^2(1-|\omega(0)|^2)}\right) z\overline{z}^2 \\ &- \frac{1}{3!}\left(\frac{1}{1-|\omega(0)|^2}\left(\overline{\omega''(0) + 2\omega'(0)\frac{h''}{h'}(0)}\right)\right)\overline{z}^3 + \cdots . \end{aligned}$$

For a point $w \in \Omega \setminus \{0\}$, we consider the function $f_w = f(w+z)$, which is well defined in the domain $\Omega_w = \{z \in \mathbb{C} \colon z - w \in \Omega\}$ as before, and we define the *Schwarzian derivative S_f of f at $w \in \Omega$* as 6 times the difference between the coefficient of z^3 and the square of the coefficient of z^2 in the Taylor series of $F_w = (M_{f_w})^{-1} \circ f_w$. That is,

$$S_f = Sh + \frac{\overline{\omega}}{1-|\omega|^2}\left(\frac{h''}{h'}\omega' - \omega''\right) - \frac{3}{2}\left(\frac{\omega'\overline{\omega}}{1-|\omega|^2}\right)^2, \quad \text{in } \Omega, \tag{12}$$

which is a generalization of the classical Schwarzian derivative (2). (Note that, for f analytic, $\omega \equiv 0$.)

3.2 The Classical Schwarzian Derivative in Terms of the Jacobian

The Jacobian J_f of a locally univalent harmonic mapping $f = h + \overline{g}$ in the simply connected domain Ω is defined by $J_f = |h'|^2 - |g'|^2$. When f is analytic (that is, when $g \equiv 0$ in Ω), $J_f = |h'|^2$. In this case, the classical formulas for the pre-Schwarzian and the Schwarzian derivatives ((1) and (2), respectively) can be written as

$$Pf = \rho_z \quad \text{and} \quad Sf = \rho_{zz} - \frac{1}{2}\rho_z^2, \tag{13}$$

where $\rho = \log(J_f) = \log|h'|^2$ and $\rho_z = \partial\rho/\partial z$.

It is just a matter of a straightforward calculation to check that for a sense-preserving harmonic mapping $f = h + \overline{g}$ with Jacobian J_f, the formula (12) for the Schwarzian derivative of f equals

$$S_f = \delta_{zz} - \frac{1}{2}\delta_z^2, \tag{14}$$

where $\delta = \log(J_f) = \log\left(|h'|^2 - |g'|^2\right)$ in this case. In other words, we see again that S_f generalizes the classical formula (2).

In view of (14) and (13), we define the *pre-Schwarzian derivative* P_f of a sense-preserving harmonic mapping f in Ω by $P_f = (\log(J_f))_z$. Then, we get that

$$S_f = (P_f)_z - \frac{1}{2}(P_f)^2.$$

In terms of the canonical decomposition of $f = h + \overline{g}$, we have

$$P_f = \frac{h''}{h'} - \frac{\overline{\omega}\omega'}{(1 - |\omega|^2)}, \quad z \in \Omega, \tag{15}$$

where $\omega = g'/h'$ is the dilatation of f.

Notice that using the definition $P_f = (\log(J_f))_z$ for the pre-Schwarzian derivative of a sense-preserving harmonic mapping f, it is easy to conclude that $P_{\overline{f}} = P_f$ (just recall that f is sense-preserving if and only if $\overline{f}$ is sense-reversing and check that $J_{\overline{f}} = -J_f$). This readily implies that both the pre-Schwarzian derivative (15) and the Schwarzian derivative (12) are well defined for *any locally univalent harmonic mapping* (sense-preserving or sense-reversing) in a simply connected domain $\Omega \subset \mathbb{C}$.

3.3 The Schwarzian Derivative as a Schwarzian Tensor

As was mentioned in the Introduction, the Schwarzian derivative $\mathbb{S}$ presented in [5] is defined by the formula

$$\mathbb{S}f = 2(\sigma_{zz} - \sigma_z^2),$$

where $\sigma = \log(|h'| + |g'|)$. This means that $\mathbb{S}f$ is equal to the Schwarzian tensor $B_{\mathbf{g_0}}(\log\sigma)$ of the conformal metric $\mathbf{g} = e^{2\sigma}\mathbf{g_0}$ relative to the Euclidean metric $\mathbf{g_0}$.

Consider now the metric

$$\mathbf{h} = e^{2\lambda_f}\mathbf{g_0},$$

with $\lambda_f = \sqrt{J_f}$.

Note that since f is sense-preserving, its Jacobian does not vanish. Bearing in mind (14) and the definition of λ_f, it is easy to check that

$$S_f = 2\left[(\log\lambda_f)_{zz} - (\log\lambda_f)_z^2\right].$$

Therefore, S_f is equal to the Schwarzian tensor $B_{\mathbf{g_0}}(\log\lambda_f)$ of the conformal metric $e^{2\lambda_f}|dz|$ relative to the Euclidean metric.

4 Some Properties of the New Pre-Schwarzian and Schwarzian Derivatives

In this section we obtain several properties related to the pre-Schwarzian (15) and Schwarzian (12) derivatives for locally univalent harmonic mappings in a simply connected domain. As we have mentioned in Sect. 3.2, there is no loss of generality by assuming that f is sense-preserving (because $P_{\overline{f}} = P_f$ and $S_{\overline{f}} = S_f$). We will consider these kinds of mappings to state the results that can be found in this section. We would like to point out that some of these results can be proved using the techniques developed in [5] involving the relationship of S_f with the corresponding Schwarzian tensor. The proofs we show here involve only analytical arguments.

4.1 The Chain Rule

Let f be a sense-preserving harmonic mapping in a simply connected domain $\Omega \subset \mathbb{C}$. It is well known that if φ is a locally univalent analytic function for which the composition $f \circ \varphi$ is defined, then the function $f \circ \varphi$ is again a sense-preserving harmonic mapping. A straightforward calculation shows that

$$S_{(f\circ\varphi)} = (S_f \circ \varphi)(\varphi')^2 + S\varphi, \tag{16}$$

which is a direct generalization of the chain rule (3) for the Schwarzian derivative of analytic functions. The same formula (16) holds for the Schwarzian derivative $\mathbb{S}$ defined by (5). It is also easy to check that $P_{f\circ\varphi} = (P_f \circ \varphi)\varphi' + P\varphi$.

If A is an affine mapping of the form $A(z) = az + b\overline{z} + c$ with $a, b, c \in \mathbb{C}$, the composition $A \circ f$ is harmonic (and sense-preserving if $a \neq 0$ and $|b/a| < 1$). In this case, and as was mentioned in [5], the Schwarzian derivative (5) of $A \circ f$ is not equal to that of f unless $b = 0$.

In Proposition 1 below we prove that the pre-Schwarzian and the Schwarzian derivatives of a locally univalent harmonic mapping f are preserved by pre-composition with affine mappings. The proof will make use of the following lemma.

Lemma 1 *Let $f = h + \overline{g}$ be a sense-preserving harmonic mapping in the simply connected domain Ω with dilatation ω. Then, for all $z_0 \in \Omega$,*

$$P_f(z_0) = P(f - \overline{\omega(z_0)}g)(z_0) \quad \textit{and} \quad S_f(z_0) = S(f - \overline{\omega(z_0)}g)(z_0). \tag{17}$$

Proof Recall that for $z_0 \in \Omega$,

$$P_f(z_0) = Ph(z_0) - \frac{\overline{\omega(z_0)}\omega'(z_0)}{1 - |\omega(z_0)|^2}$$

and

$$S_f(z_0) = Sh(z_0) + \frac{\overline{\omega(z_0)}}{1 - |\omega(z_0)|^2}\left(\frac{h''}{h'}(z_0)\omega'(z_0) - \omega''(z_0)\right) - \frac{3}{2}\left(\frac{\omega'(z_0)\overline{\omega(z_0)}}{1 - |\omega(z_0)|^2}\right)^2.$$

For each $|\lambda| < 1$, define $\varphi_\lambda = h + \lambda g$ in Ω. A straightforward calculation shows that

$$P\varphi_\lambda = Ph + \frac{\lambda\omega'}{1 + \lambda\omega} \tag{18}$$

and

$$S\varphi_\lambda = Sh - \frac{\lambda}{1 + \lambda\omega}\left(\frac{h''}{h'}\omega' - \omega''\right) - \frac{3}{2}\left(\frac{\lambda\omega'}{1 + \lambda\omega}\right)^2. \tag{19}$$

Set $\lambda = -\overline{\omega(z_0)}$ and evaluate (18) and (19) at z_0 to get (17). □

Proposition 1 *Let $f = h + \overline{g}$ be a sense-preserving harmonic mapping in Ω with dilatation ω_f. Consider the affine harmonic mapping $A(z) = az + b\overline{z} + c$ with $a, b, c \in \mathbb{C}$, and $|a| \neq |b|$. Then, $P_{(A\circ f)} \equiv P_f$ and $S_{(A\circ f)} \equiv S_f$ in Ω.*

Proof As was mentioned in Sect. 3, $P_{\overline{f}} = P_f$. Since $\overline{A \circ f} = \overline{A} \circ f$, there is no loss of generality if we assume that A is sense-preserving. Moreover, it is easy to see that $P_{(c_1 f + c_2)} = P_f$ for all sense-preserving harmonic mappings f, and all complex numbers c_1 and c_2 with $c_1 \in \mathbb{C} \setminus \{0\}$. Hence, we can suppose that $A(z) = z + \alpha\overline{z}$ for some non-zero complex number $|\alpha| < 1$. Under these assumptions, $F = A \circ f = f + \alpha\overline{f}$, and what we have to prove is that $P_F = P_f$ and $S_f = S_F$.

Notice that the canonical representation of F in Ω equals $H + \overline{G}$, where $H = h + \alpha g$ and $G = \overline{\alpha}h + g$. The dilatation ω_F of F is

$$\omega_F(z) = \frac{\omega_f(z) + \overline{\alpha}}{1 + \alpha\omega_f(z)}.$$

Fix an arbitrary point $z_0 \in \Omega$. By Lemma 1,

$$P_f(z_0) = P(h - \overline{\omega_f(z_0)}g)(z_0) \quad \text{and} \quad P_F(z_0) = P(H - \overline{\omega_F(z_0)}G)(z_0).$$

Since

$$\begin{aligned} H - \overline{\omega_F(z_0)}G &= (h + \alpha g) - \overline{\omega_F(z_0)}(\overline{\alpha}h + g) \\ &= \left(1 - \overline{\alpha\omega_F(z_0)}\right) \cdot \left(h + \frac{\alpha - \overline{\omega_F(z_0)}}{1 - \overline{\overline{\alpha}\omega_F(z_0)}}g\right) \\ &= \left(1 - \overline{\alpha\omega_F(z_0)}\right) \cdot \left(h - \overline{\omega_f(z_0)}g\right), \end{aligned}$$

we obtain that $P_F(z_0) = P_f(z_0)$ for each $z_0 \in \Omega$. The same proof works to show that $S_F(z_0) = S_f(z_0)$. □

We would like to stress the following property for the pre-Schwarzian and Schwarzian derivatives of harmonic mappings which follows from Proposition 1. It allows us to rewrite P_f and S_f at each point $z_0 \in \Omega$ in terms of the (classical) pre-Schwarzian and Schwarzian derivatives of the analytic part of another harmonic function.

Corollary 1 *Let $f = h + \overline{g}$ be a sense-preserving harmonic mapping in Ω. Then, for each $z_0 \in \Omega$, there exists a sense-preserving harmonic mapping $F = H + \overline{G}$ in Ω such that*

$$P_f(z_0) = P_F(z_0) = PH(z_0) \quad \textit{and} \quad S_f(z_0) = S_F(z_0) = SH(z_0).$$

Proof Consider the harmonic mapping $F = f - \overline{\omega_f(z_0) f} = H + \overline{G}$, where ω_f is the dilatation of f. By Proposition 1, $P_f(z_0) = P_F(z_0)$ and $S_f(z_0) = S_F(z_0)$. Since the dilatation ω_F of F vanishes at z_0, we get $P_F(z_0) = PH(z_0)$ and $S_F(z_0) = SH(z_0)$. □

4.2 Analytic Schwarzian Derivative

If a sense-preserving harmonic mapping f in the simply connected domain Ω has constant dilatation, then $f = h + \alpha\overline{h} + \gamma$ for some complex constants α and γ (with $|\alpha| < 1$) and some locally univalent function h that is analytic in Ω. In this case, the pre-Schwarzian derivative P_f of f is analytic. On the other hand, if we assume that the pre-Schwarzian derivative P_f of a harmonic mapping $f = h + \overline{g}$ with dilatation ω is analytic, we get that

$$\frac{\partial P_f}{\partial \overline{z}} = \frac{|\omega'|^2}{(1-|\omega|^2)^2} \equiv 0 \quad \text{in } \Omega,$$

which implies that ω is constant. In other words, P_f is analytic if and only if the dilatation of f is constant. Moreover, it is easy to check that $P_f \equiv 0$ in Ω if and only if f is an affine harmonic mapping. This is, $f = az + b\overline{z} + c$ for certain constants a, b, and c.

The next theorem characterizes the harmonic mappings with analytic Schwarzian derivative.

Theorem 1 *Let $f = h + \overline{g}$ be a sense-preserving harmonic mapping in Ω. Then, S_f is analytic if and only if f has constant dilatation. That is, if and only if $f = h + \alpha\overline{h} + \gamma$, for some locally univalent analytic mapping h, some $|\alpha| < 1$, and some $\gamma \in \mathbb{C}$.*

Proof If $f = h + \alpha\overline{h} + \gamma$ for some locally univalent analytic mapping h, some $|\alpha| < 1$, and some $\gamma \in \mathbb{C}$, then $S_f \equiv Sh$ in Ω and the result follows.

Suppose that a sense-preserving harmonic mapping $f = h + \overline{g}$ with dilatation ω has analytic Schwarzian S_f defined by

$$S_f = Sh + \frac{\overline{\omega}}{1-|\omega|^2}\left(\frac{h''}{h'}\omega' - \omega''\right) - \frac{3}{2}\left(\frac{\omega'\overline{\omega}}{1-|\omega|^2}\right)^2.$$

If $\omega \equiv a$, $a \in \mathbb{D}$, then $f = h + a\overline{h} + \gamma$ for some locally univalent analytic function h and we are done. In order to get a contradiction, assume that ω is not constant in Ω. After multiplying the last equation by $(1-|\omega|^2)^2$, we obtain

$$(Sh - S_f)(1-|\omega|^2)^2 + \overline{\omega}(1-|\omega|^2)\left(\frac{h''}{h'}\omega' - \omega''\right) - \frac{3}{2}\omega'^2\overline{\omega}^2 = 0. \tag{20}$$

Rewriting (20) in terms of 1, $\overline{\omega}$, and $\overline{\omega}^2$, and denoting $u = Sh - S_f$, we have

$$u + \overline{\omega}\left(\frac{h''}{h'}\omega' - \omega'' - 2\omega u\right) + \overline{\omega}^2\left(\omega^2 u - \omega\left(\frac{h''}{h'}\omega' - \omega''\right) - \frac{3}{2}\omega'^2\right) = 0. \tag{21}$$

Differentiate with respect to $\overline{z}$ in (21) to get

$$\overline{\omega'}\left(\frac{h''}{h'}\omega' - \omega'' - 2\omega u\right) + 2\overline{\omega\omega'}\left(\omega^2 u - \omega\left(\frac{h''}{h'}\omega' - \omega''\right) - \frac{3}{2}\omega'^2\right) = 0. \tag{22}$$

Now, since ω is not constant, there exists a disk $D(z_0, R)$ with center z_0 and radius $R > 0$ contained in Ω where $\omega' \neq 0$. We can divide both sides of (22) by $\overline{\omega'}$, and take derivatives with respect to $\overline{z}$ again to obtain

$$\overline{\omega'}\left(\omega^2 u - \omega\left(\frac{h''}{h'}\omega' - \omega''\right) - \frac{3}{2}\omega'^2\right) = 0 \quad \text{in } D(z_0, R).$$

Keeping in mind that $\omega' \neq 0$ in $D(z_0, R)$, we see that

$$\omega^2 u - \omega\left(\frac{h''}{h'}\omega' - \omega''\right) - \frac{3}{2}\omega'^2 \equiv 0. \tag{23}$$

Hence, from (22) we have

$$\frac{h''}{h'}\omega' - \omega'' - 2\omega u = 0, \tag{24}$$

which implies, by (21), that $u = Sh - S_f \equiv 0$ in $D(z_0, R) \subset \Omega$. But in this case, using (24), we obtain

$$\frac{h''}{h'}\omega' - \omega'' = 0,$$

and substituting in (23), we get $\omega' \equiv 0$ in $D(z_0, R)$. This is a contradiction. Hence, $\omega \equiv \alpha \in \mathbb{D}$ and $f = h + \alpha\overline{h} + \gamma$, as we wanted to prove. □

Recall that a locally univalent analytic function T in the simply connected domain Ω is a Möbius transformation if and only if $Sf \equiv 0$ in Ω. A locally univalent harmonic Möbius transformation is a harmonic mapping of the form $f = \alpha T + \beta\overline{T} + \gamma$, where T is a (classical) Möbius transformation and $\alpha, \beta, \gamma \in \mathbb{C}$ are constants. These transformations were characterized in [5] in terms of the Schwarzian derivative (5). The following corollary also characterizes the harmonic Möbius transformation in terms of S_f.

Corollary 2 *A locally univalent mapping $f=h+\overline{g}$ equals a harmonic Möbius transformation if and only if $S_f\equiv 0$.*

Proof Since $S_{\overline{f}}=S_f$, we can assume that f is sense-preserving. If f is a Möbius harmonic transformation, then $S_f=0$. Conversely, if $S_f=0$, by Theorem 1, $f=h+\alpha\overline{h}+\gamma$ for some constants $\alpha\in\mathbb{D}$ and $\gamma\in\mathbb{C}$, and for some locally univalent analytic function h with $Sh=0$. Hence, h is a Möbius transformation. □

4.3 Harmonic Schwarzian Derivative

The (classical) Schwarzian derivative of a locally univalent analytic function φ is also analytic. This can be understood as a consequence of the fact that for such a function φ, its pre-Schwarzian derivative $P\varphi$ is an analytic function.

In the case of harmonic mappings $f=h+\overline{g}$, it is not difficult to check that the pre-Schwarzian derivative P_f is harmonic if and only if the dilatation ω of f is constant (which is also equivalent, as we proved in Sect. 4.2, to the fact that the pre-Schwarzian and the Schwarzian derivatives of f are analytic). In this section we see that this is precisely the case for which S_f is harmonic.

Theorem 2 *Let $f=h+\overline{g}$ be a sense-preserving harmonic mapping in a simply connected domain Ω. If S_f is harmonic, then S_f is analytic.*

Proof By a straightforward calculation,

$$\frac{\partial^2 S_f}{\partial\overline{z}\partial z}=\varphi_2\frac{2\overline{\omega}\overline{\omega'}}{(1-|\omega|^2)^3}+\varphi_1'\frac{\overline{\omega'}}{(1-|\omega|^2)^2}-9\omega'^3\frac{\overline{\omega}^2\overline{\omega'}}{(1-|\omega|^2)^4}, \tag{25}$$

where

$$\varphi_1=\omega'\frac{h''}{h'}-\omega'' \quad \text{and} \quad \varphi_2=\varphi_1\omega'-3\omega'\omega''.$$

Assume that $\Delta S_f\equiv 0$ in Ω. Hence, according to formula (6), the right-hand side of (25) is identically zero in Ω. After multiplying (25) by $(1-|\omega|^2)^4$, we get

$$0\equiv 2\varphi_2\overline{\omega}\overline{\omega'}(1-|\omega|^2)+\varphi_1'\overline{\omega'}(1-|\omega|^2)^2-9\omega'^3\overline{\omega}^2\overline{\omega'}. \tag{26}$$

If the dilatation ω of f is constant, then the Schwarzian derivative of f is analytic. If ω is not constant, there exists a disk $D(z_0,R)\subset\Omega$ where $\omega'\neq 0$. Dividing (26) by $\overline{\omega'}$, we obtain

$$0\equiv 2\varphi_2\overline{\omega}(1-|\omega|^2)+\varphi_1'(1-|\omega|^2)^2-9\omega'^3\overline{\omega}^2,$$

which is equivalent to

$$0\equiv\varphi_1'+\left(2\varphi_2-2\varphi_1'\omega\right)\overline{\omega}+\left(\varphi_1'\omega^2-2\varphi_2\omega-9\omega'^3\right)\overline{\omega}^2. \tag{27}$$

Taking derivatives with respect to $\overline{z}$ in (27) and dividing by $\overline{\omega'}$, we have

$$0\equiv\left(2\varphi_2-2\varphi_1'\omega\right)+2\left(\varphi_1'\omega^2-2\varphi_2\omega-9\omega'^3\right)\overline{\omega}\quad \text{in } D(z_0,R).$$

If $\varphi_1'\omega^2 - 2\varphi_2\omega - 9\omega'^3 \neq 0$, then $\overline{\omega}$ must be constant (in this case ω would be both analytic and anti-analytic). This is a contradiction since we are assuming that $\omega' \neq 0$ in $D(z_0, R)$. Hence, $\varphi_1'\omega^2 - 2\varphi_2\omega - 9\omega'^3 \equiv 0$ (in some disk $D \subset D(z_0, R)$). This fact implies that $\varphi_2 - \varphi_1'\omega = 0$ in D. Therefore, using (27), we obtain that $\varphi_2 = \varphi_1' = 0$ as well and we conclude that ω is constant in D. This proves the theorem. □

5 Locally Univalent Harmonic Mappings with Equal Pre-Schwarzian Derivatives

By exploiting the differential geometry of the associated minimal surface to a locally univalent, sense-preserving harmonic function $f = h + \overline{g}$ with dilatation $\omega = q^2$ (for some analytic function q with $|q| < 1$), Chuaqui, Duren, and Osgood characterized the sense-preserving harmonic functions with equal Schwarzian derivatives $\mathbb{S}$ (see [5]). Their theorem can be stated as follows.

Theorem 1 *Let $f = h + \overline{g}$ and $F = H + \overline{G}$ be sense-preserving harmonic functions defined on a common domain $\Omega \subset \mathbb{C}$. Let $\omega_f = q^2$ and $\omega_F = Q^2$ be the dilatations of f and F, respectively. Then,*

(a) *If ω_f is not constant, then $\mathbb{S}f = \mathbb{S}F$ if and only if $|h'| + |g'| = c\ (|H'| + |G'|)$ for some constant $c > 0$.*
(b) *If ω_f is constant, then $\mathbb{S}f = \mathbb{S}F$ if and only if $f = h + \alpha\overline{h}$, $F = H + \beta\overline{H}$, and $H = T(h)$ for some locally univalent functions h and H, some complex constants α and β with $|\alpha| < 1$ and $|\beta| < 1$, and some analytic Möbius transformation T.*

Using Theorem 1, it is easy to prove that a result analogous to (b) in the last theorem holds for the new Schwarzian derivative (12). Namely, we can see that if ω_f is constant, then $S_f = S_F$ if and only if $\mathbb{S}f = \mathbb{S}F$. Hence, the functions f and F are related as in statement (b) in Theorem 1. In the next theorem, we see that a result analogous to Theorem 1 (a) holds for the *pre-Schwarzian* derivative.

Theorem 3 *Let $f = h + \overline{g}$ and $F = H + \overline{G}$ be sense-preserving harmonic functions defined on a common simply connected domain $\Omega \subset \mathbb{C}$ with non-constant dilatations ω_f and ω_F, respectively. Then $P_f = P_F$ if and only if the Jacobians are homothetic. That is, if and only if $|h'|^2 - |g'|^2 = c\ (|H'|^2 - |G'|^2)$ for some constant $c > 0$.*

Proof Recall that $P_f = (\log J_f)_z$. Therefore, if the Jacobians are homothetic, then $P_f = P_F$.

Assume now that $P_f = P_F$. Since the dilatations are not constant, there exists a point $z_0 \in \Omega$ with $\omega_f'(z_0)$ and $\omega_F'(z_0)$ different from 0. Consider a Riemann mapping ψ from the disk onto Ω with $\psi(0) = z_0$ and define $f_0 = f \circ \psi = h_0 + \overline{g_0}$ and $F_0 = F \circ \psi = H_0 + \overline{G_0}$. The dilatations of f_0 and F_0 are $\omega_{f_0} = \omega_f \circ \psi$ and $\omega_{F_0} = \omega_F \circ \psi$, respectively. Hence, they are non-constant analytic functions in the unit disk with $\omega_{F_0}'(0) \neq 0$ and $\omega_{f_0}'(0) \neq 0$.

Using that $P_{(f\circ\psi)} = (P_f \circ \psi)\psi' + P\psi$, we see that $P_{f_0} = P_{F_0}$ in $\mathbb{D}$.

Now, it is a straightforward calculation to check that the sense-preserving harmonic mappings $f_1 = f_0 - \overline{\omega_{f_0}(0) f_0} = h_1 + \overline{g_1}$ and $F_1 = F_0 - \overline{\omega_{F_0}(0) f_0} = H_1 + \overline{G_1}$ have equal pre-Schwarzian derivatives and that their corresponding dilatations ω_{f_1} and ω_{F_1} fix the origin and have non-zero derivatives at $z = 0$. Notice also that the functions $f_2 = (f_1 - f_1(0))/h_1'(0)$ and $F_2 = (F_1 - F_1(0))/H_1'(0)$ satisfy the additional normalizations $h_2(0) = H_2(0) = g_2(0) = G_2(0) = 0$ and $h_2'(0) = H_2'(0) = 1$.

To sum up, we have constructed two sense-preserving harmonic mappings (that we again denote by $f = h + \overline{g}$ and $F = H + \overline{G}$) in the unit disk with non-constant dilatations ω_f and ω_F, respectively, with $\omega_f(0) = \omega_F(0) = h(0) = H(0) = g(0) = G(0) = 0$, $h'(0) = H'(0) = 1$, $\omega_f'(0) \neq 0$, and $\omega_F'(0) \neq 0$, and such that $P_f = P_F$.

We first need to prove the following lemma.

Lemma 2 *Let f and F be two sense-preserving harmonic mappings normalized as above. If $P_f = P_F$, then $Ph = PH$ and therefore $h = H$.*

Proof Suppose that $P_f = P_F$. By (15), this condition is

$$Ph - \frac{\overline{\omega_f}\omega_f'}{1 - |\omega_f|^2} = PH - \frac{\overline{\omega_F}\omega_F'}{1 - |\omega_F|^2}. \tag{28}$$

Since $P_f = P_F$, we have that for all non-negative integers n

$$\left(\frac{\partial^n P_f}{\partial z^n}\right)(0) = \left(\frac{\partial^n P_F}{\partial z^n}\right)(0).$$

Using that $\omega_f(0) = \omega_F(0) = 0$ and (28), we obtain

$$\left(\frac{\partial^n P_f}{\partial z^n}\right)(0) = Ph^{(n)}(0) \quad \text{and} \quad \left(\frac{\partial^n P_F}{\partial z^n}\right)(0) = PH^{(n)}(0),$$

where $Ph^{(n)}(0)$ and $PH^{(n)}(0)$ denote the n-th derivatives of Ph and PH, respectively. Since both Ph and PH are analytic functions in the unit disk, it follows that $Ph = PH$ in $\mathbb{D}$. Hence, $h = AH + B$ for certain constants A, B with $A \neq 0$. Using that $h(0) = H(0) = 0$ and that $h'(0) = H'(0)$, we get that $h \equiv H$ in the unit disk. □

We now continue with the proof of Theorem 3. Once we know that if $P_f = P_F$ then $h = H$, we have from (28) that

$$\frac{\overline{\omega_f}\omega_f'}{1 - |\omega_f|^2} = \frac{\overline{\omega_F}\omega_F'}{1 - |\omega_F|^2}. \tag{29}$$

Taking derivatives with respect to $\overline{z}$ on both sides of (29), we obtain

$$\frac{\overline{\omega_f}'\omega_f'}{(1 - |\omega_f|^2)^2} = \frac{\overline{\omega_F}'\omega_F'}{(1 - |\omega_F|^2)^2}, \tag{30}$$

which is equivalent to

$$\overline{\omega_f}'\omega_f'\left(1+\frac{1}{(1-|\omega_f|^2)^2}-1\right)=\overline{\omega_F}'\omega_F'\left(1+\frac{1}{(1-|\omega_f|^2)^2}-1\right). \tag{31}$$

Note that (31) can be written as

$$\overline{\omega_f'}\omega_f'+\overline{\omega_f}P=\overline{\omega_F'}\omega_F'+\overline{\omega_F}Q,$$

for certain functions P and Q. Hence,

$$\left(\frac{\partial^n}{\partial z^n}\left(\overline{\omega_f'}\omega_f'+\overline{\omega_f}P\right)\right)(0)=\left(\frac{\partial^n}{\partial z^n}\left(\overline{\omega_F'}\omega_F'+\overline{\omega_F}Q\right)\right)(0),$$

for all non-negative integers n. Using again that both ω_f and ω_F fix the origin, we obtain

$$\overline{\omega_f'(0)}\omega_f^{(n)}(0)=\overline{\omega_F'(0)}\omega_F^{(n)}(0),$$

which implies that

$$\overline{\omega_f'(0)}\omega_f'=\overline{\omega_F'(0)}\omega_F'.$$

Therefore, $\omega_F'=\overline{k}\omega_f'$, with $k=\omega_f'(0)/\omega_F'(0)\neq 0$, and (30) becomes

$$\frac{|\omega_f'|}{1-|\omega_f|^2}=|k|^2\frac{|\omega_f'|}{1-|\overline{k}\omega_f|^2}$$

in the unit disk. In particular, taking $z=0$, we see that $|k|=1$. Hence,

$$J_f=|h'|^2(1-|\omega_f|^2)=J_F,$$

which proves the theorem for functions with the normalizations stated before Lemma 2 (that is, for those functions f_2 and F_2 that we constructed from the initials f and F). Unwinding the definitions back to f and F leads directly to the assertion of the theorem. □

We would like to remark that using the arguments from [5], it is possible to prove that the condition $J_f=cJ_F$ is equivalent to the fact that $(H',\omega_F)=\gamma(h',\omega_f)$ for some element γ in the group $\mathcal{G}$ generated by the following transformations of the pair (h',ω_f):

$$R_p(\lambda):\quad (h',\omega_f)\to(\lambda h',\omega_f),\quad \lambda\neq 0,$$
$$R_q(\mu):\quad (h',\omega_f)\to(h',\mu\omega_f),\quad |\mu|=1,$$
$$I(a):\quad (h',\omega_f)\to\left(\frac{(1+\overline{a}\omega_f)^2}{1-|a|^2}h',\frac{a+\omega_f}{1+\overline{a}\omega_f}\right),\quad a\in\mathbb{D}.$$

Notice that this group $\mathcal{G}$ is different from the one presented in [5].

Except for the case when the dilatations are constant, the action of the elements of the group $\mathcal{G}$ can be identified with the following operators acting on the space of sense-preserving harmonic mappings: Let $f = h + \overline{g}$ be such a function with dilatation ω. Then

$$R_p(\lambda)(f) = \lambda h + \overline{\lambda g}, \quad R_q(\mu)(f) = h + \overline{\mu g}, \quad \text{and} \quad I(a)(f) = H + \overline{G}, \tag{32}$$

where $H' = \lambda h'/\sqrt{\phi_a' \circ \omega}$ for a certain constant $\lambda \neq 0$, and $Q = G'/H' = \phi_a \circ \omega$. The function ϕ_a is defined in the unit disk by

$$\phi_a(z) = \frac{a+z}{1+\overline{a}z}. \tag{33}$$

Note that since f is supposed to be a sense-preserving harmonic mapping, the composition $\phi_a \circ \omega$ is well defined. Also, that $I_a(f) = f + \overline{af}$.

It is not difficult to check that the operations described by (32) are precisely the ones we used in the proof of Theorem 3 to construct f_2 and F_2 from f_0 and F_0, respectively.

We end this section with the following result that follows directly from the proof of Theorem 3.

Corollary 3 *Let $f = h + \overline{g}$ and $F = H + \overline{G}$ be sense-preserving harmonic functions defined on a common domain $\Omega \subset \mathbb{C}$ with non-constant dilatations ω_f and ω_F, respectively. Then $P_f = P_F$ if and only if then there exist $a \in \mathbb{D}$, μ on $\partial\mathbb{D}$, and $\lambda \neq 0$ such that*

$$\omega_F = \mu(\phi_a \circ \omega_f) \quad \text{and} \quad H' = \frac{\lambda h'}{\sqrt{\phi_a \circ \omega_f}},$$

where ϕ_a is the automorphism of $\mathbb{D}$ defined by (33).

6 Norm of the Schwarzian Derivative

Let $f = h + \overline{g}$ be a sense-preserving harmonic mapping in the unit disk $\mathbb{D}$. Using the chain rule (16), it is easy to see that for each $z \in \mathbb{D}$,

$$|S_f(z)|(1-|z|^2)^2 = |S_{(f\circ\phi)}(0)|,$$

where ϕ is any automorphism of the unit disk with $\phi(0) = z$. The *Schwarzian norm* $\|S_f\|$ of f is defined by

$$\|S_f\| = \sup_{z\in\mathbb{D}} |S_f(z)|(1-|z|^2)^2.$$

Krauss [19] proved that if f is analytic and univalent in the unit disk, then $\|Sf\| \leq 6$. Another related result due to Nehari [22] states that if f is *convex* (that is, f is univalent in the unit disk and the domain $f(\mathbb{D})$ is convex), then $\|Sf\| \leq 2$. Both constants 2 and 6 are sharp: The *analytic Koebe mapping*

$$k(z) = \frac{z}{(1-z)^2}, \quad z \in \mathbb{D},$$

is univalent in the unit disk. It maps $\mathbb{D}$ onto the full plane minus the part of the negative real axis from $-1/4$ to infinity and has Schwarzian derivative

$$Sk(z) = -\frac{6}{(1-z^2)^2}.$$

Hence, $\|S_f\| = 6$. Note that $|Sf(r)|(1-r^2)^2 = 6$ for all real numbers $-1 < r < 1$. The (analytic and univalent) strip mapping s, defined in the unit disk by (9) is convex. It has Schwarzian derivative

$$Ss(z) = \frac{2}{(1-z^2)^2}.$$

The same constant 2 also appears in the Nehari criterion for univalence. This criterion states that if the Schwarzian norm of a locally univalent analytic function f in the unit disk is bounded by 2, then f is univalent in $\mathbb{D}$.

Now let $\mathcal{F}$ be the family of univalent sense-preserving harmonic mappings $f = h + \overline{g}$ in the unit disk with dilatation $w = q^2$ for some analytic function q (with $|q| < 1$) in $\mathbb{D}$. Notice that the harmonic Koebe function K as in (7) does not belong to $\mathcal{F}$ since K has dilatation $w(z) = z$. There is a result analogous to that by Krauss for the functions in $\mathcal{F}$ in terms of the Schwarzian derivative $\mathbb{S}$ defined by (5). Namely, it was proved in [10] that there exists a constant C_1 such that

$$\|\mathbb{S}f\| = \sup_{z\in\mathbb{D}} |\mathbb{S}f(z)|(1-|z|^2)^2 \leq C_1 \tag{34}$$

for all $f \in \mathcal{F}$. The sharp value of C_1 is unknown. It is thought to be equal to 16 since the Schwarzian norm of the univalent harmonic mapping K_2 constructed as the horizontal shear of the Koebe function with dilatation $\omega(z) = z^2$ belongs to $\mathcal{F}$ and satisfies $\|\mathbb{S}K_2\| = 16$. The function $K_2 = h + \overline{g}$ is defined by

$$h(z) = \frac{1}{3}\left[\frac{1}{(1-z)^3} - 1\right] \quad \text{and} \quad g(z) = \left[\frac{z^2 - z + \frac{1}{3}}{(1-z)^3} - \frac{1}{3}\right]. \tag{35}$$

We will prove a bound analogous to that in (34) for the norm of the Schwarzian derivative S_f. Concretely, we will see that there exists a constant (with unknown sharp value) C_2 such that *for all* univalent harmonic mappings f in the unit disk (with no extra assumption on the dilatation), the inequality $\|S_f\| \leq C_2$ holds. The proof will use the results obtained in Sect. 6.1 below that are related to convex (univalent) harmonic mappings.

All the results obtained in this section can be stated in terms of the pre-Schwarzian norm defined by

$$\|P_f\| = \sup_{z\in\mathbb{D}} |P_f(z)|(1-|z|^2).$$

The proofs are similar to those presented here (and often easier). This is the reason why we have decided to leave most of the explicit statements involving $\|P_f\|$ out. The exception is Theorem 4, where we obtain a sharp bound for the pre-Schwarzian norm of any convex harmonic mapping.

6.1 The Schwarzian Norm of Convex Harmonic Mappings

Let $f = h + \overline{g}$ be a univalent harmonic mapping in the unit disk. If $f(\mathbb{D})$ is convex, we say that f is a convex (harmonic) mapping. In Sect. 2.2, we have presented three particular examples of these kinds of mappings. We begin this section by computing their Schwarzian derivatives and Schwarzian norms.

Example 1 Consider the half-plane harmonic mapping L defined by (8). Its Schwarzian derivative equals

$$\begin{aligned} S_L(z) &= -\frac{3}{2}\frac{1}{(1-z)^2} + \frac{3\overline{z}}{(1-z)(1-|z|^2)} - \frac{3}{2}\left(\frac{\overline{z}}{1-|z|^2}\right)^2 \\ &= -\frac{3}{2}\left(\frac{1}{1-z} - \frac{\overline{z}}{1-|z|^2}\right)^2 = -\frac{3}{2}\left(\frac{1-\overline{z}}{1-z}\cdot\frac{1}{1-|z|^2}\right)^2. \end{aligned}$$

Note that for all $z \in \mathbb{D}$, $(1-|z|^2)^2\,|S_L(z)| = 3/2$. Therefore, $\|S_L\| = 3/2$.

Example 2 The Schwarzian derivative of the half-strip harmonic mapping S_1 given by (10) is

$$S_{S_1}(z) = \frac{2(1-|z|^2)(1+\overline{z}) + 3(1-\overline{z}^2)(1+z)}{2(1-z^2)(1+z)(1-|z|^2)^2}.$$

Therefore,

$$|S_{S_1}(z)|(1-|z|^2)^2 = \left|\frac{(1+\overline{z})(1-|z|^2)}{(1-z^2)(1+z)} + \frac{3}{2}\cdot\frac{1-\overline{z}^2}{1-z^2}\right| \le \frac{5}{2}.$$

Moreover, $S_{S_1}(r)(1-r^2)^2 = 5/2$ for all real numbers $r \in (-1,1)$ and hence, $\|S_{S_1}\| = 5/2$.

The final example is related to the strip harmonic mapping S_2 defined by (11).

Example 3 The Schwarzian derivative of S_2 is

$$S_{S_2}(z) = 2\frac{2+|z|^4-\overline{z}^2-2|z|^4\overline{z}^2}{(1-z^2)(1-|z|^4)^2}.$$

Hence,

$$\begin{aligned} \left|S_{S_2}(z)\right|(1-|z|^2)^2 &= 2\left|\frac{2+|z|^4-\overline{z}^2-2|z|^4\overline{z}^2}{(1-z^2)(1+|z|^2)^2}\right| \\ &= 2\left|\frac{1-|z|^4+1-\overline{z}^2+2|z|^4(1-\overline{z}^2)}{(1-z^2)(1+|z|^2)^2}\right| \\ &\le 2\frac{2+|z|^2+2|z|^4}{(1+|z|^2)^2} = 4 - 2\frac{|z|^2}{(1+|z|^2)^2} \le 4. \end{aligned}$$

Since $S_{S_2}(0) = 4$, we obtain that $\|S_{S_2}\| = 4$.

As we said in the beginning of this section, it is known that the Schwarzian norm of every convex analytic function is less than or equal to 2. In the next proposition, we prove an analogous theorem for convex harmonic mappings. To do so, we use one of the theorems by Clunie and Sheil-Small (see [12, Theorem 5.7]): If f is a convex harmonic mapping in the unit disk, then for each $\varepsilon \in \overline{\mathbb{D}}$, the analytic function $\varphi_\varepsilon = h + \varepsilon g$ is close-to-convex in $\mathbb{D}$, thus univalent.

Proposition 2 *Let $f = h + \overline{g}$ be a convex harmonic mapping in the unit disk. Then*

$$\|S_f\| \leq 6.$$

Proof Fix an arbitrary point $z_0 \in \mathbb{D}$. Since f is convex, the analytic functions $\varphi_\varepsilon = h + \varepsilon g$ are univalent for all $\varepsilon \in \overline{\mathbb{D}}$. In particular, for $\varepsilon = -\overline{\omega(z_0)}$, where ω is the dilatation of f. Therefore, using Lemma 1 and Krauss's theorem, we get

$$|S_f(z_0)|(1-|z_0|^2)^2 = |S(h - \overline{\omega(z_0)}g)(z_0)|(1-|z_0|^2)^2 \leq 6. \qquad \square$$

We do not know if the constant 6 in the last proposition is sharp. However, by considering the pre-Schwarzian norm instead of $\|S_f\|$, we can prove the following result.

Theorem 4 *Let $f = h + \overline{g}$ be a convex harmonic mapping in the unit disk. Then,*

$$\|P_f\| = \sup_{z\in\mathbb{D}} |P_f(z)|(1-|z|^2) \leq 5.$$

The constant 5 *is sharp.*

Proof Without loss of generality, we can assume that f is sense-preserving.

If the dilatation of f is constant, then $f = ah + b\overline{h} + c$ for some convex analytic function h and certain constants a, b, and c, and we have $P_f = Ph$. In this case, a result due to Yamashita [28] states that $\|Ph\| \leq 4$. Hence, the result follows for convex harmonic functions with constant dilatations.

To prove the general case, we first find the corresponding upper bound for $|P_f(0)|$.

Using that $P_{(A\circ f)} = P_f$ for all affine mappings $A(z) = z + \alpha\overline{z}$, we can suppose that $f \in C_H^0 = \{f \in S_H^0 : f(\mathbb{D}) \text{ is convex}\}$. It is known that if $f = h + \overline{g} \in C_H^0$, then $|h''(0)| \leq 3$ (see [14, p. 50]). Hence, we obtain $|P_f(0)| = |Ph(0)| = |h''(0)| \leq 3$.

The function L defined by (8) satisfies $|P_L(0)| = 3$.

Now, fix an arbitrary point $z_0 \in \mathbb{D}$ and take an automorphism of the disk ϕ with $\phi(0) = z_0$. Then, the function $f \circ \phi$ is convex and

$$|P_f(z_0)|(1-|z_0|) = |P_{f\circ\psi}(0) - 2\overline{z_0}| \leq |P_{f\circ\phi}(0)| + 2|z_0| \leq 5.$$

Therefore, $\|P_f\| \leq 5$.

It is easy to check that $\lim_{r\to 1^-} \|P_L(r)\| = 5$. This proves that the constant 5 is sharp. $\square$

Note that the last theorem is actually a special case of a more general result. The properties we have used are that the family of convex harmonic mappings is affine and linear invariant and that the value of the supremum of $|h''(0)|$ among all functions $f \in C_H^0$ is known. Therefore, the theorem still holds for any subclass of S_H that is invariant under normalized affine transformations and disk automorphisms.

6.2 The Krauss Theorem for Harmonic Mappings

We prove that an analogue of the well-known theorem by Krauss holds for univalent harmonic mappings in the unit disk. To prove our result, we need to use Theorem 9 in [26], which states what follows: There exists a positive real constant R such that if f is a univalent harmonic mapping in $\mathbb{D}$ normalized by $h(0) = g(0) = 1 - h'(0) = 0$ (in other words, if $f \in S_H$), then $f_R(z) = f(Rz)$ is convex (and univalent) in the unit disk.

Theorem 5 *There exists a positive constant C such that $\|S_f\| \le C$ for all univalent harmonic mappings f in the unit disk.*

Proof By Proposition 1, we may assume that $f \in S_H$. Notice that if f is univalent and ϕ is an automorphism of the disk with $\phi(0) = z$, then $f \circ \phi$ is also univalent and $|S_{(f\circ\phi)}(0)| = |S_f(z)|(1 - |z|^2)^2$. Hence, it is not difficult to prove that

$$\sup_{f\in S_H} \|S_f\| = \sup_{f\in S_H} |S_f(0)|.$$

Now, fix an arbitrary function $f \in S_H$. According to [26, Theorem 9], $f_R(z) = f(Rz)$ is a univalent convex harmonic mapping. Therefore, $|S_{f_R}(0)| \le 6$ by Proposition 2. Using the chain rule, we see that

$$R^2|S_f(0)| = |S_{f_R}(0)| \le 6.$$

Therefore, $\sup_{f\in S_H} |S_f(0)| \le 6/R^2$. This proves the theorem. □

We end this section with two examples.

Example 4 Recall that the analytic Koebe function k has Schwarzian norm $\|Sk\| = 6$ and that for each $-1 < r < 1$, $|Sk(r)|(1 - r^2)^2 = \|Sk\|$. The Schwarzian derivative of the harmonic Koebe function K defined by (7) equals

$$S_K(z) = -\frac{19 + 10z + 3z^2 - 44|z|^2 - 10z|z|^2 + 19|z|^4 - 10\overline{z} + 10\overline{z}|z|^2 + 3\overline{z}^2}{2(1 - z^2)^2(1 - |z|^2)^2}.$$

A tedious calculation shows that

$$|S_K(z)| \le \frac{19}{2} \cdot \frac{1}{(1 - |z|^2)^2}.$$

Hence, $\|S_K\| = |S_K(0)| = 19/2$. Notice that $|S_K(r)|(1 - r^2)^2 = \|S_K\|$ for all real numbers $r \in (-1, 1)$.

Example 5 The Schwarzian derivative of the harmonic function $K_2 = h + \overline{g}$, where h and g are defined by (35), is equal to

$$S_{K_2}(z) = \frac{-2}{(1-z)^2(1-|z|^4)^2}\left(2+|z|^4+\overline{z}^2-6|z|^2\overline{z}+2\overline{z}^2|z|^4\right).$$

It can be checked that

$$\|S_{K_2}\| = \lim_{r\to 1}|S_{K_2}(r)|(1-r^2)^2 = 19/2 = \|S_K\|.$$

6.3 Harmonic Mappings in the Unit Disk with Finite Schwarzian Norm

The following theorem asserts that a harmonic mapping has finite Schwarzian norm $\|S_f\|$ if and only if its analytic part does. That the same result holds for the Schwarzian norm $\|\mathbb{S}\|$ (restricted to the family $\mathcal{F}$ defined in the beginning of this section) was proved in [8]. The proof of our theorem is similar to that in [8] but, in this case, we need to prove that there exists a constant C such that for all $z \in \mathbb{D}$

$$\left|\frac{\omega''(z)\omega(z)}{1-|\omega(z)|^2}\right|(1-|z|^2)^2 \le C. \tag{36}$$

The last inequality is a consequence of the results obtained in [21] using boundedness criteria for weighted composition operators between various Bloch-type spaces. We give a direct proof of (36).

Lemma 3 *Let $\omega : \mathbb{D} \to \mathbb{D}$ be an analytic function. Then, there exists a constant C such that* (36) *holds.*

Proof Let φ be any convex analytic mapping in the unit disk with $\varphi(0) = 0$. Consider the harmonic mapping f constructed as the horizontal shear of the function φ with dilatation ω. This is, $f = h + \overline{g}$, where h and g are obtained by solving the linear system of equations

$$\begin{cases} h(z) - g(z) = \varphi(z) \\ g'(z)/h'(z) = \omega(z) \end{cases} \quad (z \in \mathbb{D}),$$

with $h(0) = g(0) = 0$. By Theorem 5.17 in [12], since φ is convex, the harmonic mapping f is univalent and h is close-to-convex. Therefore, using Theorem 5 and the well-known results for univalent analytic functions, we have that $\|S_f\|$, $\|Sh\|$, and $|(h''/h')(z)|(1-|z|^2)$ are bounded in the unit disk. By the Schwarz–Pick lemma, we obtain that

$$\frac{|\omega(z)\omega'(z)|(1-|z|^2)}{1-|\omega(z)|^2}$$

is bounded by 1 in $\mathbb{D}$.

Using the definition of the Schwarzian derivative (12) and the triangle inequality, we see that

$$\frac{\left|\omega(z)\omega''(z)\right|(1-|z|^2)^2}{1-|\omega(z)|^2} \le \|Sh\| + \|S_f\| + \left|\frac{h''}{h'}(z)\right|(1-|z|^2) + \frac{3}{2} \le C < \infty. \qquad \square$$

Now we have all the tools to prove the next theorem.

Theorem 6 *Let $f = h + \overline{g}$ be a locally univalent harmonic mapping in the unit disk. Then $\|S_f\| < \infty$ if and only if $\|Sh\| < \infty$.*

Proof As we mentioned before, since $S_{\overline{f}} = S_f$, there is no loss of generality if we assume that f is sense-preserving. Suppose that $\|Sh\| < \infty$. By the triangle inequality, we see that

$$|S_f| \le |Sh| + \frac{|ww'|}{1-|w|^2}\left|\frac{h''}{h'}\right| + \frac{|ww''|}{1-|w|^2} + \frac{3}{2}\left|\frac{ww'}{1-|w|^2}\right|^2.$$

By hypotheses, there exists a constant C_1 such that

$$|Sh(z)|(1-|z|^2)^2 \le C_1, \quad z \in \mathbb{D}.$$

Since f is sense-preserving, its dilatation ω maps the unit disk into itself. Using the Schwarz–Pick Lemma, we have that

$$\frac{\left|\omega'(z)\omega(z)\right|(1-|z|^2)}{1-|\omega(z)|^2} \le 1.$$

By Lemma 3, there exists a constant C_2 such that

$$\frac{\left|\omega(z)\omega''(z)\right|(1-|z|^2)^2}{1-|\omega(z)|^2} \le C_2.$$

Finally, a result of Pommerenke (see [24]) asserts that

$$(1-|z|^2)\left|\frac{h''}{h'}(z)\right| \le 2 + 2\sqrt{1 + \frac{1}{2}\|Sh\|}. \tag{37}$$

Putting all the estimates together, we obtain

$$\|S_f\| \le C_1 + 2\sqrt{1 + \frac{1}{2}C_1} + C_2 + \frac{7}{2} < \infty.$$

Conversely, suppose that $\|S_f\| < \infty$. Using formula (12) and the preceding estimates we have

$$|Sh(z)| \le |S_f(z)| + \frac{1}{1-|z|^2}\left|\frac{h''}{h'}(z)\right| + \frac{C}{(1-|z|^2)^2}. \tag{38}$$

In order to use (37), we apply inequality (38) to the dilation $f_r(z) = f(rz) = h_r + \overline{g_r}$, where $0 < r < 1$. By the chain rule, we have that $S_{f_r}(z) = r^2 S_f(rz)$. Therefore,

$$|S_{f_r}(z)|(1-|z|^2)^2 \le |S_f(rz)|(1-|rz|^2)^2 \le \|S_f\|.$$

Since $\|Sh_r\|$ is finite for each $r < 1$, we can use (37) to get

$$\|Sh_r\| - 2\sqrt{1+\frac{1}{2}\|Sh_r\|} \le \|S_f\| + C + 2.$$

Let $r \to 1$ to obtain $\|Sh\| < \infty$. □

Recall that the *pseudo-hyperbolic metric* in the unit disk is defined by

$$\rho(\alpha,\beta) = \left|\frac{\alpha-\beta}{1-\overline{\alpha}\beta}\right|, \quad \alpha,\beta \in \mathbb{D}.$$

The pseudo-hyperbolic disk with center α and radius r is

$$\Delta(\alpha, r) = \{z \in \mathbb{D} \colon \rho(z,\alpha) < r\}.$$

A sense-preserving harmonic mapping f is called *uniformly locally univalent* if there exists a fixed $0 < r < 1$ such that f is univalent on every pseudo-hyperbolic disk $\Delta(\alpha, r)$.

To conclude this section, we would like to mention explicitly Theorem 3 from [10]. This theorem states that given any sense-preserving harmonic mapping $f = h + \overline{g}$ whose dilatation $\omega = g'/h'$ is the square of an analytic function in the unit disk, then $\|\mathbb{S}f\| < \infty$ if and only if f is uniformly locally univalent. The proof of this result uses that $\|\mathbb{S}f\| < \infty$ if and only if $\|Sh\| < \infty$. The requirement on the dilatation is only needed to define the Schwarzian derivative $\mathbb{S}f$. By Theorem 6, we know that $\|Sh\| < \infty$ if and only if $\|S_f\| < \infty$. Using this fact and the argument used in the proof of [10, Theorem 3], we obtain the next result (which does not require any extra condition on the dilatation).

Theorem 7 *Let $f = h + \overline{g}$ be a sense-preserving harmonic mapping in the unit disk. Then, $\|S_f\| < \infty$ if and only if f is uniformly locally univalent.*

Proof It was proved in [10] that if $f = h + \overline{g}$ is a sense-preserving harmonic mapping that is uniformly locally univalent in the unit disk, then its analytic part h is uniformly locally univalent as well. Hence, an application of Krauss's theorem shows that $\|Sh\| < \infty$. Using Theorem 6, we get $\|S_f\| < \infty$.

Assume now that $\|S_f\| < \infty$ or, equivalently (by Theorem 6), that $\|Sh\| < \infty$. Then, by a theorem of B. Schwarz, h is uniformly locally univalent. In other words, h is univalent on all pseudo-hyperbolic disks $\Delta(\alpha, r)$ of center at $\alpha \in \mathbb{D}$ and fixed radius $0 < r < 1$. Using the classical result on the radius of convexity [13, p. 44], we see that h maps every disk $\Delta(\alpha, (2-\sqrt{3})r)$ onto a convex domain. This fact implies (by [11, Theorem 1]) that f is univalent in each disk $\Delta(\alpha, (2-\sqrt{3})r)$, and thus uniformly locally univalent. □

7 A Becker-Type Criterion of Univalence

We finish this paper with a criterion of univalence for sense-preserving harmonic mappings.

Theorem 8 *Let $f = h + \overline{g}$ be a sense-preserving harmonic function in the unit disk with dilatation ω. If for all $z \in \mathbb{D}$*

$$|zP_f(z)| + \frac{|z\omega'(z)|}{1-|\omega(z)|^2} \le \frac{1}{1-|z|^2}, \tag{39}$$

then f is univalent. The constant 1 *is sharp.*

Before proving Theorem 8, we would like to remark that when ω is equal to a constant $\alpha \in \mathbb{D}$, say, then (39) becomes

$$\left| z\frac{h''}{h'}(z)\right| \le \frac{1}{1-|z|^2},$$

which implies that h is univalent. Hence, f (which is necessarily of the form $h + \alpha\overline{h}$) is also univalent. In other words, Theorem 8 generalizes the classical Becker's criterion of univalence for analytic functions.

Proof Using (15), the triangle inequality, which ω maps the unit disk into itself, and assuming that (39) holds, we obtain

$$\begin{aligned}\left| z\frac{h''}{h'}(z)\right| &\le |zP_f(z)| + \frac{|z\overline{\omega(z)}\omega'(z)|}{1-|\omega(z)|^2}\\ &\le |zP_f(z)| + \frac{|z\omega'(z)|}{1-|\omega(z)|^2} \le \frac{1}{1-|z|^2}.\end{aligned}$$

Therefore, h is univalent by Becker's criterion.

For an arbitrary $a \in \mathbb{D}$, consider the function $f_a = f - \overline{af} = h_a + \overline{g_a}$. The dilatation ω_a of f_a equals $\phi_a \circ \omega$, where ϕ_a is the automorphism of the disk defined by (33). Then, $P_{f_a} = P_f$ (by Proposition 1) and

$$\frac{|z\omega_a'(z)|}{1-|\omega_a(z)|^2} = \frac{|z\omega'(z)|}{1-|\omega(z)|^2}.$$

Thus, (39) holds for f_a as well. We conclude that the analytic parts $h_a = h + ag$ of f_a are univalent for all $a \in \mathbb{D}$. Using Hurwitz's theorem (and that f is sense-preserving), we get that $h + \lambda g$ are univalent for all $|\lambda| = 1$. This implies that $f_\lambda = h + \lambda\overline{g}$ are univalent for all such λ (see [18]). In particular, we obtain that f is univalent.

Since the constant 1 is sharp for analytic functions, it follows that this constant is sharp for the harmonic mappings, too. □

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