

A construction of integrated vertex operator in the pure spinor sigma-model in $AdS_5 \times S^5$

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ABSTRACT: Vertex operators in string theory come in two varieties: integrated and unintegrated. Understanding both types is important for the calculation of the string theory amplitudes. The relation between them is a descent procedure typically involving the b -ghost. In the pure spinor formalism vertex operators can be identified as cohomology classes of an infinite-dimensional Lie superalgebra formed by covariant derivatives. We show that in this language the construction of the integrated vertex from an unintegrated vertex is very straightforward, and amounts to the evaluation of the cocycle on the generalized Lax currents.

KEYWORDS: Superspaces, Non-Commutative Geometry, Conformal Field Models in String Theory, AdS-CFT Correspondence

ARXIV EPRINT: [1306.0145](https://arxiv.org/abs/1306.0145)

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1 Introduction and notations

1.1 Introduction

It is a crucial fact, that string worldsheet theories come in families. In other words, they are deformable. The infinitesimal deformations of the worldsheet action S are parametrized by *integrated vertex operators* U :

$$S \mapsto S + \varepsilon \int d\tau d\sigma U. \tag{1.1}$$

For every integrated vertex operator, there is the corresponding unintegrated one, usually called V . The unintegrated vertices of the Type II theory carry ghost number two. They correspond to the cohomology of the BRST operator Q_{BRST} :

$$Q_{\text{BRST}}V = 0 \tag{1.2}$$

$$V \simeq V + Q_{\text{BRST}}\Phi. \tag{1.3}$$

In the pure spinor formalism, the unintegrated vertex operators are, schematically, of the form:

$$V = A_{\alpha\dot{\alpha}}^{LR}(x, \theta)\lambda_L^\alpha\lambda_R^{\dot{\alpha}} + A_{\alpha\beta}^{LL}(x, \theta)\lambda_L^\alpha\lambda_L^\beta + A_{\dot{\alpha}\dot{\beta}}^{RR}(x, \theta)\lambda_R^{\dot{\alpha}}\lambda_R^{\dot{\beta}} \tag{1.4}$$

where x, θ are the spacetime coordinates and λ_L, λ_R the pure spinor ghosts.

Understanding both integrated and unintegrated versions of the vertex operators is important for the string perturbation theory [1]. The unintegrated vertices are usually somewhat simpler. They correspond to the cohomology classes of (1.2), (1.3). Their construction does not require the use of the worldsheet equations of motion.

The relation between U and V was explained in [2, 3]. Given the integrated vertex $V = A_{\alpha\dot{\alpha}}(x, \theta)\lambda_L^\alpha\lambda_R^{\dot{\alpha}}$, one can construct the integrated vertex as a linear combination of the expressions like $A_{\alpha\dot{\alpha}}\partial_+\theta_L^\alpha\partial_-\theta_R^{\dot{\alpha}}$, $A_{mn}\Pi_+^m\Pi_-^n$, $\Omega_{mn}\hat{\alpha}N_+^{mn}\partial_-\theta_R^{\dot{\alpha}}$, $S_{mnpq}N_+^{mn}N_-^{pq}$ etc., where A, Ω, S, \dots are superfields (i.e. functions of x and θ) constructed from the $A_{\alpha\dot{\alpha}}$ by evaluating various derivatives. The precise relation is given by the “descent procedure”:

$$dV = QV^{(1)} \tag{1.5}$$

$$dV^{(1)} = QU. \tag{1.6}$$

However, this descent procedure is somewhat mysterious¹ and to the best of our knowledge there was no formal proof that it always works.

In this note we will give some mathematical interpretation of this descent procedure. We will consider both flat spacetime \mathbf{R}^{1+9} and $AdS_5 \times S^5$, but mostly concentrate on $AdS_5 \times S^5$ (because flat spacetime is a limit of AdS). The unintegrated vertices were interpreted in [8] as elements of the relative Lie algebra cohomology for some infinite-dimensional Lie superalgebra \mathcal{L}_{tot} . It turns out that in this language, the relation between U and V is rather straightforward. The generalization of the Lax pair considered in [9] plays the key role in the construction. Our construction proves the existence of the chain of operators participating in the descent procedure (i.e., given the unintegrated operator V , always exist $V^{(1)}$ and U satisfying (1.5) and (1.6)). It implies that the relation between the unintegrated and integrated vertex operators can be understood as a manifestation of the Koszul duality.²

1.2 Plan of the paper

We will start in section 2 by constructing the integrated vertex operator for the closed Type IIB superstring in $AdS_5 \times S^5$ (our most general case). Then in section 3 we give a simple example (the β -deformation) where everything can be done explicitly. The flat space limit is considered in section 4. Finally, in section 5 we will consider the construction of the integrated vertex for the open superstring.

1.3 Notations

Let $\mathfrak{g} = \mathfrak{psu}(2, 2|4)$ denote the supersymmetry algebra of $AdS_5 \times S^5$ and $U\mathfrak{g}$ its universal enveloping algebra. The space of formal Taylor series of complex-valued functions at the unit $\mathbf{1} \in \text{PSU}(2, 2|4)$ can be identified with the dual space $(U\mathfrak{g})' = \text{Hom}(U\mathfrak{g}, \mathbf{C})$. We

¹The descent procedure should be controlled by the b -ghost [4]. But the pure spinor b -ghost is somewhat problematic, being a composite field with denominators (see [5–7] for the recent progress and references therein).

²An excellent review of the Koszul duality in the context of pure spinor formalism can be found in the introductory sections of [10].

denote $\mathfrak{g}_{\bar{0}} = so(1,4) \oplus so(5)$ the subalgebra leaving invariant a fixed point in $AdS_5 \times S^5$. Then the space of Taylor series of functions in $AdS_5 \times S^5$ is $\text{Hom}_{U\mathfrak{g}_{\bar{0}}}(U\mathfrak{g}, \mathbf{C})$.

The infinite-dimensional Lie superalgebra \mathcal{L}_{tot} was introduced in [8]. It contains an ideal $I \subset \mathcal{L}_{\text{tot}}$ such that $\mathcal{L}_{\text{tot}}/I = \mathfrak{g}$. Let π denote the projection:

$$\pi : \mathcal{L}_{\text{tot}} \rightarrow \mathfrak{g}. \tag{1.7}$$

2 Integrated vertex for closed string

In this section we will review the interpretation of the unintegrated vertex in terms of the relative cohomology of \mathcal{L}_{tot} , and then explain how to use this language to describe the integrated vertex and the descent procedure.

2.1 Unintegrated vertices from the relative cohomology

Unintegrated vertices correspond to the elements of $H^2(\mathcal{L}_{\text{tot}}, \mathfrak{g}_{\bar{0}}, (U\mathfrak{g})')$ [8]. The 2-cocycle representing such an element is a bilinear function of two elements of \mathcal{L}_{tot} , which we will denote ξ and η . We have to remember that the 2-cocycle takes values in $(U\mathfrak{g})'$ which is identified with the space of Taylor series at the unit of the group manifold. Let $g \in \text{PSU}(2, 2|4)$ denote the group element. The value of the 2-cocycle is a Taylor series of g , not necessarily convergent.³ Therefore, the element $\psi \in H^2(\mathcal{L}_{\text{tot}}, \mathfrak{g}_{\bar{0}}, (U\mathfrak{g})')$ is a function of two ghosts ξ and η taking values in the functions of g :

$$\psi(\xi, \eta)(g). \tag{2.1}$$

To understand the cocycle condition on ψ , we introduce the left derivative. For $a \in \mathfrak{g}$ and $f \in (U\mathfrak{g})'$:

$$(L(a)f)(g) = \left. \frac{d}{ds} \right|_{s=0} f(e^{sa}g). \tag{2.2}$$

We have $[-L(a), -L(b)] = -L([a, b])$. With this definition, the cocycle condition on ψ , or vanishing of $(Q_{\text{Lie}}\psi)(\zeta, \xi, \eta)$, is:

$$L(\pi(\zeta))\psi(\xi, \eta) + \psi([\zeta, \xi], \eta) + \text{cycl}(\zeta, \xi, \eta) = 0 \tag{2.3}$$

where π is the projection from \mathcal{L}_{tot} to \mathfrak{g} .

2.2 Ansatz for integrated vertex

Let ψ be the cocycle representing an element of $H^2(\mathcal{L}_{\text{tot}}, \mathfrak{g}_{\bar{0}}, (U\mathfrak{g})')$. As we have explained, it corresponds to a function of two ghosts ξ, η and a group element g :

$$\psi(\xi, \eta)(g). \tag{2.4}$$

³The $\mathfrak{g}_{\bar{0}}$ -covariance condition in the definition of the relative cohomology implies that this is actually a section of a bundle over $AdS_5 \times S^5$.

The Lax operator L_{\pm} was introduced in [8]. Let us define $J_{\pm} \in \mathcal{L}_{\text{tot}}$ starting from the Lax operator:

$$L_{\pm} = \frac{\partial}{\partial \tau^{\pm}} + J_{\pm}. \quad (2.5)$$

The BRST transformation of J_{\pm} was calculated in [8]:

$$\epsilon Q_{\text{BRST}} J_{\pm} = -[L_{\pm}, \epsilon \lambda_L^{\alpha} \nabla_{\alpha}^L + \epsilon \lambda_R^{\dot{\alpha}} \nabla_{\dot{\alpha}}^R]. \quad (2.6)$$

We will denote:

$$\Lambda = \lambda_L^{\alpha} \nabla_{\alpha}^L + \lambda_R^{\dot{\alpha}} \nabla_{\dot{\alpha}}^R. \quad (2.7)$$

Let us consider the following ansatz for the integrated vertex:

$$U = \psi(J_+, J_-(g)). \quad (2.8)$$

In the rest of this section we will prove that this U satisfies the descent equations (1.6), (1.5). Therefore it is the integrated vertex, i.e. the deformation of the action (1.1). It would be interesting to explicitly confirm that the deformed action is conformally invariant in perturbation theory; we leave this for the future work [11].

2.3 Ascent: the first step

We have:

$$\begin{aligned} (\epsilon Q_{\text{BRST}} \psi)(J_+, J_-) &= -\psi \left(\left[\frac{\partial}{\partial \tau^+} + J_+, \epsilon \Lambda \right], J_- \right) - \\ &\quad - \psi \left(J_+, \left[\frac{\partial}{\partial \tau^-} + J_-, \epsilon \Lambda \right] \right) + \\ &\quad + L(\pi(\epsilon \Lambda)) \psi(J_+, J_-) = \\ &= -\frac{\partial}{\partial \tau^+} \psi(\epsilon \Lambda, J_-) + \frac{\partial}{\partial \tau^-} \psi(\epsilon \Lambda, J_+) + \\ &\quad + \psi(\epsilon \Lambda, \partial_+ J_- - \partial_- J_+) - \\ &\quad - L(\pi(J_+)) \psi(\epsilon \Lambda, J_-) + L(\pi(J_-)) \psi(\epsilon \Lambda, J_+) + \\ &\quad + \psi([\epsilon \Lambda, J_+], J_-) + \psi(J_+, [\epsilon \Lambda, J_-]) + \\ &\quad + L(\pi(\epsilon \Lambda)) \psi(J_+, J_-). \end{aligned} \quad (2.9)$$

Taking into account that on-shell $\partial_+ J_- - \partial_- J_+ = -[J_+, J_-]$ and $Q_{\text{Lie}} \psi = 0$, we conclude that $(\epsilon Q_{\text{BRST}} \psi)(J_+, J_-)$ on-shell is a total derivative:

$$\begin{aligned} (\epsilon Q_{\text{BRST}} \psi)(J_+, J_-) &= - (Q_{\text{Lie}} \psi)(\epsilon \Lambda, J_+, J_-) - \partial_+ \psi(\epsilon \Lambda, J_-) + \partial_- \psi(\epsilon \Lambda, J_+) = \\ &= -\partial_+ \psi(\epsilon \Lambda, J_-) + \partial_- \psi(\epsilon \Lambda, J_+). \end{aligned} \quad (2.11)$$

In other words:

$$\epsilon Q_{\text{BRST}} (\psi(J_+, J_-) d\tau^+ \wedge d\tau^-) = -d\psi(\epsilon \Lambda, J). \quad (2.12)$$

2.4 Ascent: the second step

At the first step we have seen that:

$$\epsilon Q_{\text{BRST}} U = -d\psi(\epsilon\Lambda, J). \quad (2.13)$$

Continuing the ascent, we get:

$$\begin{aligned} \epsilon' Q_{\text{BRST}} (\psi(\epsilon\Lambda, J)) &= -\psi(\epsilon\Lambda, [d+J, \epsilon'\Lambda]) + (L(\pi(\epsilon'\Lambda))\psi(\epsilon\Lambda, J)) = \\ &= -\frac{1}{2}d(\psi(\epsilon\Lambda, \epsilon'\Lambda)) - \frac{1}{2}(L(\pi(J))\psi(\epsilon\Lambda, \epsilon'\Lambda)) - \\ &\quad -\psi(\epsilon\Lambda, [J, \epsilon'\Lambda]) + \\ &\quad + (L(\pi(\epsilon'\Lambda))\psi(\epsilon\Lambda, J)) = \\ &= -\frac{1}{2}d(\psi(\epsilon\Lambda, \epsilon'\Lambda)). \end{aligned} \quad (2.14)$$

Here we used that:

$$\psi(\{\Lambda, \Lambda\}, J)(g) = 0 \quad (2.15)$$

— this is because ψ represents a *relative* (w.r.to \mathfrak{g}_0) cohomology, and $\{\Lambda, \Lambda\} \in \mathfrak{g}_0$.
corresponding to $\psi \in H^2(\mathcal{L}_{\text{tot}}, \mathfrak{g}_0, (U\mathfrak{g})')$.

2.5 If ψ is exact, then U is a total derivative

On the other hand, consider the case when $\psi = Q_{\text{Lie}}\phi$. In this case:

$$\psi(\xi, \eta) = -L(\pi(\xi))\phi(\eta) + L(\pi(\eta))\phi(\xi) - \phi([\xi, \eta]). \quad (2.16)$$

And the corresponding integrated vertex is:

$$\begin{aligned} U &= -L(\pi(J_+))\phi(J_-) + L(\pi(J_-))\phi(J_+) - \phi([J_+, J_-]) = \\ &= -L(\pi(J_+))\phi(J_-) + L(\pi(J_-))\phi(J_+) + \phi(\partial_+ J_- - \partial_- J_+) = \\ &= \partial_+ \phi(J_-) - \partial_- \phi(J_+). \end{aligned} \quad (2.17)$$

3 Example: β -deformation

The unintegrated vertex is:

$$V(\epsilon, \epsilon') = (g^{-1}(\epsilon\lambda_3 - \epsilon\lambda_1)g)_a B^{ab} (g^{-1}(\epsilon'\lambda_3 - \epsilon'\lambda_1)g)_b \quad (3.1)$$

where B^{ab} is a constant antisymmetric tensor: $B \in (\mathfrak{g} \wedge \mathfrak{g})'$ [12, 13].

The corresponding element of $H^2(\mathcal{L}_{\text{tot}}, \mathfrak{g}_0, (U\mathfrak{g})')$ is:

$$\psi(\xi, \eta)(g) = (g^{-1}\pi([\text{deg}, \xi])g)_a B^{ab} (g^{-1}\pi([\text{deg}, \eta])g)_b. \quad (3.2)$$

Here deg is the degree operator, which is an outer derivation of \mathcal{L}_{tot} :

$$\begin{aligned} [\text{deg}, t_{[mn]}^0] &= 0 \\ [\text{deg}, \nabla_\alpha^L] &= \nabla_\alpha^L \\ [\text{deg}, \nabla_\alpha^R] &= -\nabla_\alpha^R. \end{aligned} \quad (3.3)$$

In particular,

$$[\text{deg}, \lambda_3^\alpha \nabla_\alpha^L + \lambda_1^{\dot{\alpha}} \nabla_{\dot{\alpha}}^R] = \lambda_3^\alpha \nabla_\alpha^L - \lambda_1^{\dot{\alpha}} \nabla_{\dot{\alpha}}^R \quad (3.4)$$

— this “explains” the minus sign in (3.1). It is straightforward to verify that $\psi(\xi, \eta)(g)$ defined by (3.2) satisfies the cocycle condition (2.3). In fact, the relative cohomology has a multiplicative structure, and our ψ is the product of 1-cocycles of the form $\xi \mapsto g^{-1}[\text{deg}, \xi]g$.

In order to obtain the integrated vertex, we substitute $\xi \mapsto J_+$ and $\eta \mapsto J_-$:

$$U = (g^{-1}\pi([\text{deg}, J_+])g)_a B^{ab} (g^{-1}\pi([\text{deg}, J_-])g)_b. \tag{3.5}$$

This is equivalent to the expression $j_{a+} B^{ab} j_{b+}$ of [13].

4 Flat space limit

4.1 The İnönü-Wigner contraction

The flat space limit of \mathcal{L}_{tot} is by the İnönü-Wigner contraction, which is essentially a change of variables:

$$\overset{\text{flat}}{\nabla}_{\dot{\alpha}}^L = \varepsilon \nabla_{\dot{\alpha}}^L, \quad \overset{\text{flat}}{\nabla}_m^L = \varepsilon^2 \nabla_m^L, \quad \overset{\text{flat}}{W}_{\dot{\alpha}}^L = \varepsilon^3 W_{\dot{\alpha}}^L, \quad \dots \tag{4.1}$$

$$\overset{\text{flat}}{\nabla}_{\dot{\alpha}}^R = \varepsilon \nabla_{\dot{\alpha}}^R, \quad \overset{\text{flat}}{\nabla}_m^R = \varepsilon^2 \nabla_m^R, \quad \overset{\text{flat}}{W}_{\dot{\alpha}}^R = \varepsilon^3 W_{\dot{\alpha}}^R, \quad \dots \tag{4.2}$$

$$\text{but: } \overset{\text{flat}}{t}_{[mn]}^0 = t_{[mn]}^0 \tag{4.3}$$

where $\varepsilon \rightarrow 0$ is a small parameter. In the limit $\varepsilon \rightarrow 0$: $\{\overset{\text{flat}}{\nabla}_{\dot{\alpha}}^L, \overset{\text{flat}}{\nabla}_{\dot{\alpha}}^R\} = 0$. Therefore, in this limit \mathcal{L}_{tot} becomes a semidirect sum:

$$(\mathcal{L}_L \oplus \mathcal{L}_R) + \mathfrak{g}_{\bar{0}} \tag{4.4}$$

where the ideal is the direct sum $\mathcal{L}_L \oplus \mathcal{L}_R$. The cohomology group $H(\mathcal{L}_{\text{tot}}, \mathfrak{g}_{\bar{0}}, (U\mathfrak{g})')$ becomes $H(\mathcal{L}_L \oplus \mathcal{L}_R, (U(\mathfrak{g}^{\text{flat}}/\mathfrak{g}_{\bar{0}}))')$. Notice that $\mathfrak{g}^{\text{flat}}/\mathfrak{g}_{\bar{0}}$ is the flat space supersymmetry algebra **susy**. We conclude that the flat space vertex operators are identified with $H^2(\mathcal{L}_L \oplus \mathcal{L}_R, (U(\mathbf{susy}))')$.

Notice that $U(\mathcal{L}_L \oplus \mathcal{L}_R)$ is a quadratic algebra, the Koszul dual to the algebra of functions of two pure spinors λ_L and λ_R . In this case the identification of the BRST cohomology with $H^2(\mathcal{L}_L \oplus \mathcal{L}_R, (U(\mathbf{susy}))')$ is a straightforward consequence of the central fact in the theory of Koszul quadratic algebras, namely that the following complex:

$$\begin{aligned} \dots &\longrightarrow \text{Hom}_{\mathbf{C}}(A_n^!, A) \longrightarrow \text{Hom}_{\mathbf{C}}(A_{n-1}^!, A) \longrightarrow \dots \\ \dots &\longrightarrow \text{Hom}_{\mathbf{C}}(A_1^!, A) \longrightarrow A \longrightarrow \mathbf{C} \longrightarrow 0 \end{aligned} \tag{4.5}$$

provides a free resolution⁴ of the trivial A -module \mathbf{C} . This means, that for any module V the cohomology $H^n(A, V)$ can be computed as $\text{Ext}_A^n(\mathbf{C}, V)$, i.e. as the cohomology of the complex:

$$\begin{aligned} \dots &\longrightarrow \text{Hom}_A(A, V) \longrightarrow A_1^! \otimes \text{Hom}_A(A, V) \longrightarrow \dots \\ \dots &\longrightarrow A_{n-1}^! \otimes \text{Hom}_A(A, V) \longrightarrow A_n^! \otimes \text{Hom}_A(A, V) \longrightarrow \dots \end{aligned} \tag{4.6}$$

⁴we are grateful to the JHEP referee for pointing out an error in the original version of this paragraph.

In our case $A^!$ is the algebra of functions of λ_L, λ_R , $A = U(\mathcal{L}_L \oplus \mathcal{L}_R)$, $V = (U(\mathbf{susy}))'$. Also notice that $\text{Hom}_A(A, V) \simeq V$. The case of $AdS_5 \times S^5$ is more subtle, because \mathcal{L}_{tot} is not a quadratic algebra.

4.2 Lax pair in flat space

Flat space can be described by the coset sPoincaré/Lorentz. An element of the AdS coset is $g = e^{X^m t_m^2 + \theta_L^\alpha t_\alpha^3 + \theta_R^{\dot{\alpha}} t_{\dot{\alpha}}^1}$. To go to the flat space limit we do the field redefinition:

$$\begin{aligned} \theta_L^\alpha &= \varepsilon \vartheta_L^\alpha, & \theta_R^{\dot{\alpha}} &= \varepsilon \vartheta_R^{\dot{\alpha}}, & \lambda_L^\alpha &= \varepsilon \tilde{\lambda}_L^\alpha, & \lambda_R^{\dot{\alpha}} &= \varepsilon \tilde{\lambda}_R^{\dot{\alpha}} \\ X^m &= \varepsilon^2 x^m \\ \frac{\partial L}{\partial(\partial_+ \theta_L)} &= \varepsilon^3 p_{L\alpha-}, & \frac{\partial L}{\partial(\partial_- \theta_R)} &= \varepsilon^3 p_{R\alpha+}, & w_{1+} &= \varepsilon^3 \tilde{w}_{1+}, & w_{3-} &= \varepsilon^3 \tilde{w}_{3-}. \end{aligned} \quad (4.7)$$

Let us study the behaviour of the Maurer-Cartan currents under such a rescaling. They can be decomposed according to the \mathbf{Z}_4 -grading

$$\begin{aligned} -K_+ &= \partial_+ g g^{-1} = \\ &= \varepsilon (\partial_+ \vartheta_L^\alpha + O(\varepsilon^2)) t_\alpha^3 + \varepsilon^2 (\Pi_+^m + O(\varepsilon^2)) t_m^2 + \varepsilon^3 d_+^\alpha t_\alpha^1 - \varepsilon^4 K_{\bar{0}+}^{\text{flat}} \end{aligned} \quad (4.8)$$

$$\begin{aligned} -K_- &= \partial_- g g^{-1} = \\ &= \varepsilon (\partial_- \vartheta_R^{\dot{\alpha}} + O(\varepsilon^2)) t_{\dot{\alpha}}^1 + \varepsilon^2 (\Pi_-^m + O(\varepsilon^2)) t_m^2 + \varepsilon^3 d_-^{\dot{\alpha}} t_{\dot{\alpha}}^3 - \varepsilon^4 K_{\bar{0}-}^{\text{flat}} \end{aligned} \quad (4.9)$$

where⁵

$$\Pi_+^m = \partial_+ x^m + (\vartheta_L \gamma^m \partial_+ \vartheta_L) \quad (4.10)$$

$$\Pi_-^m = \partial_- x^m + (\vartheta_R \gamma^m \partial_- \vartheta_R) \quad (4.11)$$

$$\varepsilon^3 d_+^{\dot{\alpha}} = J_{1+}^{\dot{\alpha}} = C^{\dot{\alpha}\alpha} d_{\alpha+} \quad (4.12)$$

$$\varepsilon^3 d_-^{\alpha} = J_{3-}^{\alpha} = C^{\alpha\dot{\alpha}} d_{\dot{\alpha}-}. \quad (4.13)$$

These K_\pm satisfy the Maurer-Cartan identities:

$$\partial_+ K_- - \partial_- K_+ - [K_+, K_-] = 0.$$

To construct J_\pm , we start with K_\pm and do three things:

1. In the K_+ of (4.8), replace the $\mathbf{psu}(2, 2|4)$ generators $t_\alpha^3, t_m^2, t_\alpha^1$ with $\nabla_\alpha^L, \nabla_m^L$ and W_α^L . Similarly, in the K_- of (4.9) replace $t_{\dot{\alpha}}^1, t_m^2$ and $t_{\dot{\alpha}}^3$ with $\nabla_{\dot{\alpha}}^R, \nabla_m^R$ and $W_{\dot{\alpha}}^R$.
2. Add $\{\lambda_L^\alpha \nabla_\alpha^L, w_{L+}^\beta W_\beta^L\} - \lambda_L^\alpha w_{L+}^\beta f_{\alpha\beta}^{[mn]} t_{[mn]}^0$ to the K_+ and similarly add $\{\lambda_R^{\dot{\alpha}} \nabla_{\dot{\alpha}}^R, w_{R-}^\beta W_\beta^R\} - \lambda_R^{\dot{\alpha}} w_{R-}^\beta f_{\dot{\alpha}\beta}^{[mn]} t_{[mn]}^0$ to the K_- .
3. Finally, replace the generators of \mathcal{L}_{tot} with the rescaled generators using (4.1), (4.2) and (4.3) and drop all the terms containing positive powers of ε .

⁵We are using the convention where there structure constants of the \mathbf{susy} algebra do not have 1/2 and i .

This results in the following flat space expressions:

$$J_+ = \partial_+ \vartheta_L^\alpha \overset{\text{flat}}{\nabla}_\alpha^L + \Pi_+^m \overset{\text{flat}}{\nabla}_m^L + d_{\alpha+} \overset{\text{flat}}{W}_L^\alpha + \lambda_L^\alpha w_{\beta+} \{ \overset{\text{flat}}{\nabla}_\alpha^L, \overset{\text{flat}}{W}_L^\beta \} \quad (4.14)$$

$$J_- = \partial_- \vartheta_R^{\dot{\alpha}} \overset{\text{flat}}{\nabla}_{\dot{\alpha}}^R + \Pi_-^m \overset{\text{flat}}{\nabla}_m^R + d_{\dot{\alpha}-} \overset{\text{flat}}{W}_R^{\dot{\alpha}} + \lambda_R^{\dot{\alpha}} w_{\dot{\beta}-} \{ \overset{\text{flat}}{\nabla}_{\dot{\alpha}}^R, \overset{\text{flat}}{W}_R^{\dot{\beta}} \}. \quad (4.15)$$

Notice that in the flat space limit the left generators $\overset{\text{flat}}{\nabla}_\alpha^L, \overset{\text{flat}}{\nabla}_m^L, \dots$ commute with the right generators $\overset{\text{flat}}{\nabla}_{\dot{\alpha}}^R, \overset{\text{flat}}{\nabla}_m^R, \dots$ (see eq. (4.4)). Therefore the zero curvature conditions become $\partial_+ J_- - \partial_- J_+ = 0$ which is equivalent to

$$\partial_+ J_- = \partial_- J_+ = 0. \quad (4.16)$$

This is simply the statement that $\partial_+ \vartheta_L, \Pi_+, d_+, \lambda_L, w_{L+}$ are holomorphic and $\partial_- \vartheta_R, \Pi_-, d_-, \lambda_R, w_{R-}$ are antiholomorphic.

Using the flatspace BRST transformations it can also be shown that the generalized Lax pair has the following BRST transformation:

$$\epsilon Q_{BRST}^{\text{flat}} J_\pm = [\epsilon \Lambda, L_\pm] = [\epsilon \Lambda, \partial_\pm + J_\pm],$$

where $\Lambda = \lambda_L^\alpha \overset{\text{flat}}{\nabla}_\alpha^L + \lambda_R^{\dot{\alpha}} \overset{\text{flat}}{\nabla}_{\dot{\alpha}}^R$.

5 Construction of integrated vertex for the open string

In the case of Maxwell theory, the Koszul duality implies that unintegrated vertices can be identified as elements of $H^1(\mathcal{L}_{\text{YM}}, (U\text{susy}_{N=1})')$, see [8]. The $(U\text{susy}_{N=1})'$ is identified with the space of Taylor series at the unit of the group manifold (the super Minkowski space). Therefore an element of $H^1(\mathcal{L}_{\text{YM}}, (U\text{susy}_{N=1})')$ is represented by a linear function of $\xi \in \mathcal{L}_{\text{YM}}$ taking values in the Taylor series of $g = (x, \theta) \in \text{SUSY}_{N=1}$. As in section 2, we will simply write:

$$\psi(\xi)(g). \quad (5.1)$$

Consider the Maxwell theory living on a D -brane. Consider the flat space Lax current $J_+ d\tau^+ + J_- d\tau^-$ from (4.14), (4.15), restricted on the string boundary. On the boundary there is some relation of the form:

$$d\theta_L^\alpha = A_\beta^\alpha d\theta_R^{\dot{\beta}} \quad (5.2)$$

and a similar relation for x . To construct the open string integrated vertex we substitute in (5.1) in place of ξ the restricted current J_\pm with the replacement:

$$\overset{\text{flat}}{\nabla}_\alpha^L \mapsto \nabla_\alpha, \quad \overset{\text{flat}}{\nabla}_{\dot{\alpha}}^R \mapsto A_\alpha^\beta \nabla_\beta, \dots \quad (5.3)$$

where ∇_α are the generators of the super-Yang-Mills algebra. We get the integrated boundary vertex operator:

$$U = \psi(J_+)(g) d\tau^+ + \psi(J_-)(g) d\tau^-. \quad (5.4)$$

The cocycle condition on ψ is:

$$L(\pi(\xi))\psi(\eta) - L(\pi(\eta))\psi(\xi) + \psi([\xi, \eta]) = 0. \quad (5.5)$$

For example:

$$L(\pi(\nabla_\alpha))\psi(\nabla_\beta) + L(\pi(\nabla_\beta))\psi(\nabla_\alpha) + \Gamma_{\alpha\beta}^m \psi(\nabla_m) = 0. \quad (5.6)$$

In the literature on pure spinors $L(\pi(\nabla_\alpha))$ is usually denoted D_α . Eq. (5.6) tells us to identify $\psi(\nabla_m)$ with A_m . We can continue writing the cocycle conditions (5.5) for other values of ξ and η . We have:⁶

$$\psi(\nabla_\alpha)(x, \theta) = A_\alpha(x, \theta) \quad (5.7)$$

$$\psi(\nabla_m)(x, \theta) = A_m(x, \theta) \quad (5.8)$$

$$\psi(W^\alpha)(x, \theta) = W^\alpha(x, \theta) \quad (5.9)$$

$$\psi(\{\nabla_\alpha, W^\beta\})(x, \theta) = (\Gamma^{[mn]}F_{[mn]}(x, \theta))_\alpha^\beta \quad (5.10)$$

$$\dots \quad (5.11)$$

This is the well-known chain of identities leading to the construction of the integrated vertex:

$$U = A_\alpha(x, \theta)d\theta^\alpha + A_m(x, \theta)(dx^m + (\theta\Gamma^m d\theta)) + W^\alpha d_\alpha + \left(\lambda \Gamma^{[mn]}F_{[mn]}w\right). \quad (5.12)$$

The new thing here is that we give an interpretation of this chain in terms of the Lie algebra cohomology, and this actually explains why this chain exists and is self-consistent.

Acknowledgments

We would like to thank Nathan Berkovits for discussions, and the JHEP referee for preparing a thorough review and pointing out a mistake in the original version. We thank the FONDECYT grant 1120263 for partial financial support.

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⁶We use the same letter W for the generator of the super-Yang-Mills algebra and for the target space superfield.

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