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Note

Crossing information in two-dimensional Sandpiles

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Abstract

We prove that in a two-dimensional Sandpile automaton, embedded in a regular infinite planar cellular space, it is impossible to cross information, if the bit of information is the presence (or absence) of an avalanche. This proves that it is impossible to embed arbitrary logical circuits in a Sandpile through quiescent configurations. Our result applies also for the non-planar neighborhood of Moore. Nevertheless, we also show that it is possible to compute logical circuits with a two-dimensional Sandpile, if a neighborhood of radius two is used in $\mathbb{Z}^2$; crossing information becomes possible in that case, and we conclude that for this neighborhood the Sandpile is P-complete and Turing universal.

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1. Introduction

Bak et al. [1] introduced a model, mostly known as the Sandpile automaton, which captures important features of many nonlinear dissipative systems. It has been studied in diverse contexts. In particular from an algebraic point of view where several problems remain still open [3].

We are interested in the computational complexity of Sandpiles. The computational complexity of a discrete dynamical system (DDS) is the amount of time steps, memory or other computational resources needed to *predict* its behavior. More precisely, we can define the complexity of a d-dimensional cellular automaton (CA) as the computational complexity of the PRED problem:

> (PRED) Given the initial configuration c of a finite part of the space, a cell v, a state s and a natural number n, decide whether after n steps, s will or will not be the state of v, when starting with c.

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The problem PRED can be solved by explicitly simulating the CA. This can be done in $o(n^{d+1})$. Off course, the state of v at iteration n must be well defined, which implies that n must be on the order of the size of c. Then PRED is a polynomial problem.

But there are some particular CAs for which PRED is simpler: for the identity CA the PRED problem is $o(1)$. A less trivial case is the one-dimensional Sandpile, for which PRED is in NC [7]. More interesting CAs are those for which the PRED problem is P-complete, that is to say, those to which any other polynomial problem can be reduced by a logarithmic space algorithm.

In order to prove P-completeness of PRED, one usually tries to reduce CIRCUITVALUE[2] to PRED. This is frequently worked out by emulating the physical form of a circuit. This means that the movement of a *signal* over a *wire* is emulated. The presence or the absence of a signal is interpreted as the *logical values TRUE or FALSE*, respectively. The logical gates (*OR*, *AND*, *NOT*, etc.) are implemented through finite configurations to which the wires can be connected. In order to construct the circuit it is also necessary to have a way to bifurcate a wire and to cross wires without intersection. Then devices called, respectively, *fanout* and *cross-over* are also developed. The computation is made by the automaton dynamics and the result can be read from the state of some specific cells—defined as outputs—after a certain number of steps. This method was first used by Banks [2].

Suppose now that we have a CA in a two-dimensional regular grid (2DCA) that can compute circuits in the previous way. The behavior of a Turing Machine (TM) over its tape in one time step can be computed with a boolean circuit with multiple outputs. By concatenating these circuits, it is possible to compute several steps of a TM, and if we concatenate an infinite number of circuits, the complete behavior of a TM can be reproduced. Then, if a 2DCA can compute circuits, it is both P-complete and Turing universal.

As Moore remarks [7], it is not necessary to emulate all of the former devices. NOT-gate and cross-overs are not simultaneously needed, but at least one of them must be constructed.

Goles et al. [4,5] proved the Turing universality of Sandpiles over a particular infinite graph. Moore [7] showed that the construction of Goles can be embedded in the three-dimensional cubic grid, and asserted that this would not be the case in the two-dimensional square grid, essentially because he thought it impossible to construct a finite configuration allowing signal cross-over.

In this paper, we prove that cross-overs are, in fact, not constructable. This implies that it is not possible to compute arbitrary logical circuits by using the Banks's approach. Our result can be generalized for the three regular planar grids: the hexagonal, triangular and square grids; and also for the Moore neighborhood of radius one in $\mathbb{Z}^2$.

On the other hand, we prove that previous results are sharp. In fact, by considering the von Neumann neighborhood of radius two (which implies non-planar connections), we prove that the Sandpile is P-complete and Turing universal.

2. Definitions and previous results

Let us give a general definition of a Sandpile over an arbitrary undirected graph.

Definition 1 (*Sandpile*). We consider a set of cells, not necessarily finite, such that each cell, i, is connected to a finite set of neighbors denoted V_i. A finite number of tokens, $x_i(0) \geqslant 0$, is assigned to each cell. For $t > 0$ the system evolves under the following rule applied synchronously to each cell.

> If $x_i(t)$ exceeds the number of neighbors of i: $d_i = |V_i|$, then the cell "fires" and all its neighbors increase its number of tokens by the number of times that d_i divides $x_i(t)$, while the number of tokens of i decreases by d_i times this number.

The evolution equation is

$$x_i(t+1) = x_i(t) - d_i \left\lfloor \frac{x_i(t)}{d_i} \right\rfloor + \sum_{j \in V_i} \left\lfloor \frac{x_j(t)}{d_j} \right\rfloor,$$

where $\lfloor r \rfloor$ denotes the largest integer smaller than r.

Fig. 1 exhibit an example of the evolution of a Sandpile over a graph of four vertices.

[2] CIRCUITVALUE is the problem of determining the value of a given logical circuit with a given assignment of values to its variables, i.e., the problem of evaluating a circuit (see for example [6]).

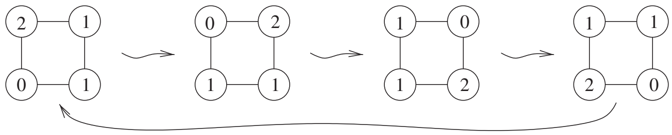


Fig. 1. Example of the dynamics of a Sandpile over a simple graph.

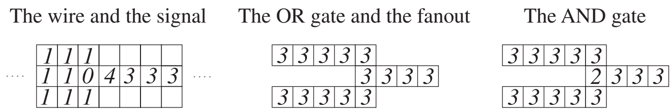


Fig. 2. Some logical devices for a two-dimensional Sandpile with the von Neumann neighborhood.

A first observation about Sandpiles defined over an undirected graph is that if at a given step each cell i has less than $2d_i - 1$ tokens, then at future steps each cell will have no more than $2d_i - 1$ tokens and the system can be defined as a CA.

If at a given step each cell has less than $d_i - 1$ tokens, then the system is over a fixed point. In this case we say that the configuration is stable or *quiescent*.

3. Cross-over impossibility in $\mathbb{Z}^2$ with a planar regular grid

In the Banks approach, a device as a logical gate or a cross-over is defined by a configuration over a finite portion of $\mathbb{Z}^2$ and it is embedded over a given background. In order for the construction to work, the devices cannot modify the background. In the present article, we will suppose that the background is a quiescent configuration. It is not difficult to see that any finite configuration over such a background will either converge to a fixed point or modify the background. Thus devices must be also quiescent configurations. As an example consider the devices developed by Moore for the square grid in Fig. 2.

We do not know the nature of signals, but the effect of a signal will always be to drop tokens in the cross-over. We also suppose that the cross-over neither interact with the background nor with other devices during its action, i.e., it receives tokens only from the inputs, and that tokens that falls outside the cross-over are lost.

In the following we will suppose that we are in the square grid (with von Neumann neighborhood), but the analysis for the other grids is analogous. In the Sandpile over $\mathbb{Z}^2$ with von Neumann neighborhood each cell (i, j) has four neighbors which corresponds to the cells $(i+1, j)$, $(i-1, j)$, $(i, j+1)$ and $(i, j-1)$. Then a configuration is *quiescent* if and only if each cell has strictly less than four tokens.

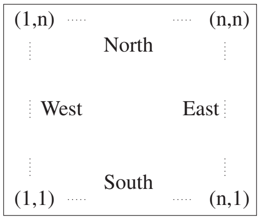
In order to study the cross-over one may consider, without loss of generality, that it is defined over a finite $n \times n$ square. A configuration over a $n \times n$ square is an assignment of tokens to each of its cells. Such a configuration is said to be a *transporter from West to East* if when adding tokens on the west column one gets tokens on the east column. In an analogous way we define a *transporter from North to South*.

A transporter from West to East is said to be *isolated to the South* if no cell fires on the southern row when tokens are added on the western column, during the whole evolution of the Sandpile. The analogous definition applies for a transporter from North to South. Fig. 3 illustrates our coordinate system and the previous concepts.

A quiescent configuration is said to be a *cross-over* if it is a transporter from West to East with isolation to the South and it is also a transporter from North to South with isolation to the East. We do not care about the isolation with the West and North sides which are the input sides of the cross-over.

Such a configuration works even if the tokens are added to both the West and the North sides, independently of the order of arrival of the tokens.

A cross-over satisfies our intuition about crossing information in the sense that a token appears in the East side if and only if a token is added on a given cell of the West side, and a token appears in the South side if and only if a token is added on a given cell of the North side. The "square" orientation of the cross-over may seem arbitrary, but if there exists a cross-over with another orientation one can add wires in order to construct a cross-over oriented as a "square."



(i)

0	0	0	0	0	0
0	3	3	3	3	0
3	3	0	0	2	3
0	3	3	3	3	3
0	0	2	0	0	0

(ii)

0	0	0	0	0	0
0	3	3	3	3	0
3	3	0	0	2	3
0	3	3	3	3	3
0	0	3	0	0	0

(iii)

Fig. 3. (i) Coordinate system. (ii) A configuration that transports from West to East isolated to the South. (iii) A non-isolated configuration that transports from West to East.

0	0	0	2	1	3
1	3	3	2	2	3
3	3	3	3	3	1
3	1	3	2	0	1
1	2	3	0	1	0

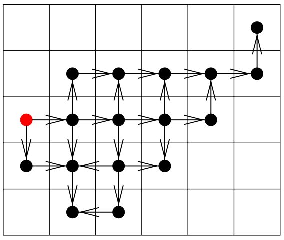


Fig. 4. A configuration that transports from West to East and its Firing Graph.

Lemma 1. *If a cell fires k times either it has a neighbor that fires k times before it or it started with more than three tokens. If the cell is at the boundary of the square and* $k \geqslant 2$, *it either has two neighbors that fires k times, or some neighbor fires* $k+1$ *or it started with more than* $k+2$ *tokens.*

Proof. In order to fire k times, a cell with i tokens must receive at least $4k-i$ tokens. If each of its four neighbors fires $k-1$ times, it receives only $4k-4$ tokens and if it is a boundary cell it receives only $3k-3$ tokens. The lemma concludes directly from this. □

When tokens are added to a square, the Sandpile evolves to a quiescent configuration. This phenomenon is called an *avalanche*. We will initially study cross-overs that needs at most one token by cell to work. Lemma 1 implies that in this case no cell fires more than once, then we can define the following concept.

Definition 2 (*Firing Graph*). Let us consider a quiescent configuration $c : \{1, .., n\}^2 \to \{0, .., 3\}$ and a set of cells $\mathcal{O}$ on the boundary of c. We define its Firing Graph as the directed graph $G = (V, E)$ where:

V is the set of cells in $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ that fire if a token is added to each cell from $\mathcal{O}$ in c, and

E is defined by $(u, v) \in E \Leftrightarrow u$ and v are neighbors and u fires before v.

Fig. 4 shows an example of a Firing Graph. Some direct properties of a Firing Graph G are:

- It has no cycles.
- The sources of G belong to $\mathcal{O}$ (vertices with in-degree equal to 0).
- If the in-degree of u is k then u has at least $4-k$ tokens in the initial configuration (i.e., $c(u) \geqslant 4-k$).

Now we can state our main theorem.

Theorem 1. *There does not exist a cross-over stimulated with only one token by cell for the Sandpile in* $\mathbb{Z}^2$.

Proof. Suppose that there exists a cross-over c. Let $G_{\mathrm{we}} = (V_{\mathrm{we}}, E_{\mathrm{we}})$ and $G_{\mathrm{ns}} = (V_{\mathrm{ns}}, E_{\mathrm{ns}})$ be the Firing Graphs corresponding to the West to East avalanche and the North to South avalanche, respectively. Let us consider the subgraph of G_{we}, S_{we}, generated by the set $V_{\mathrm{we}} \setminus V_{\mathrm{ns}}$. Fig. 5 shows an example.

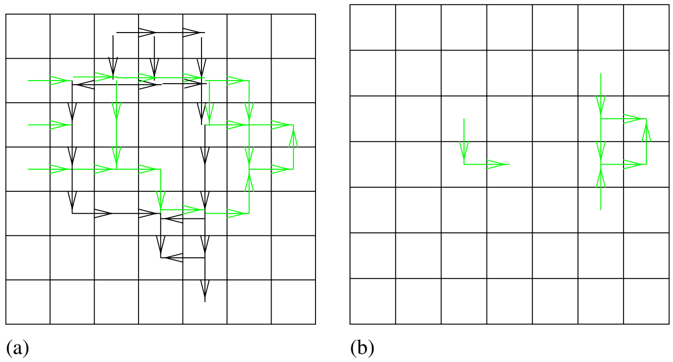


Fig. 5. (a) The overlapping of the Firing Graph of both the *ns* and the *we* avalanches. (b) The graph S_{we}.

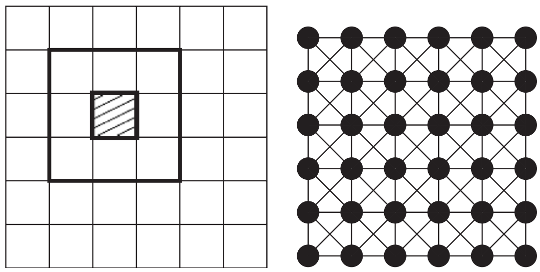


Fig. 6. The Moore neighborhood and the induced non-planar graph in $\mathbb{Z}^2$.

Like G_{we}, S_{we} has no cycles. Then it must have some vertices with in-degree equal to 0, other than those on the West border. Otherwise, this would imply that G_{ns} do not connect North to South, due to the planarity of $\mathbb{Z}^2$.

Let u be a vertex with in-degree equal to 0 in S. All the predecessors of u in G_{we} belong to V_{ns}. Then, when the North to South avalanche is fired, u also fires, which is a contradiction since $u \notin V_{\mathrm{ns}}$. □

The previous result as well as the definitions and the Firing Graph apply also for Sandpiles defined over any of the three planar regular grids.

Remark 1. If we want to study cross-overs that needs more tokens to work, a different—but similar—proof of Theorem 1 can be made. The idea is to remark that if a cell fires more times during an avalanche than during the other one, it must have a neighboring cell with the same property. Then there must exists a chain of cells, from West to East borders, that fire more times during the West to East avalanche than during the North to South avalanche. Of course the analogous chain exists from North to South. But this produces a contradiction with the planarity of the grid, because these chains cannot intersect.

4. Cross-over impossibility in $\mathbb{Z}^2$ with the Moore neighborhood

The previous result proved that in a Sandpile with planar interactions we cannot cross information. But what happens with non-planar neighborhoods as it is the case of the Moore neighborhood? (see Fig. 6). In this case the Sandpile has a critical threshold equal to 8. We will prove that it is not possible to define a cross-over, as in the previous case, but in this case we will study only cross-overs that work with the addition of at most one token by cell. The notions introduced in the previous section can be directly generalized for this case.

Theorem 2. *There does not exist a cross-over stimulated with only one token by cell for the Sandpile in $\mathbb{Z}^2$ with Moore neighborhood.*

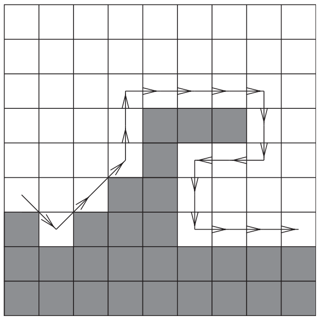


Fig. 7. The set V_{south} is gray colored.

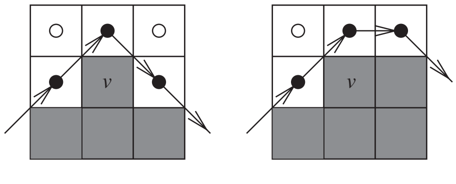


Fig. 8. Whatever the shape of p, v has always more neighbors in p than outside $V_{\text{south}} \cup p$.

Proof. Suppose that there exists one. Let c be a cross-over. Let $G_{\text{we}} = (V_{\text{we}}, E_{\text{we}})$ and $G_{\text{ns}} = (V_{\text{ns}}, E_{\text{ns}})$ be their West to East and North to South Firing Graphs, respectively. As before, let us consider the subgraph S_{we} generated by the set $V_{\text{we}} \backslash V_{\text{ns}}$.

Since $\mathbb{Z}^2$ with the Moore neighborhood is not planar, S_{we} may have no vertex with in-degree equal to 0 (maximal vertex) outside the West column of c. If this was the case, an argument analogous to that applied in the former theorem carries to a contradiction. So let us suppose that every maximal vertex of S_{we} is on the West column of c. Then there exists a path in S_{we} from the West column to the East column, $p = (u_1, u_2, \dots, u_k)$, where $u_i \in V_{\text{we}} \setminus V_{\text{ns}}$.

Let us define inductively the set V_{south} by the following two assertions. See Fig. 7 for an example.

(1) The South row of c belongs to V_{south}.
(2) If $u \notin p$ and it is a neighbor of a vertex v in V_{south} by the von Neumann neighborhood, then u also belongs to V_{south}.

Now let us consider the subgraph H generated by the set $(V_{\text{ns}} \cap V_{\text{south}}) \setminus V_{\text{we}}$. Since V_{south} is defined by using the von Neumann neighborhood and the path p separates c in two parts, H must have a maximal vertex. Let v be a maximal vertex of H. All the predecessors of v in G_{ns} are not in V_{south} nor in p.

In order for v not to fire when the West to East avalanche is activated, it must have more predecessors in G_{ns} than neighbors in p. But this is not geometrically possible, as Fig. 8 shows. □

5. Universality in $\mathbb{Z}^2$ with a neighborhood of radius two

Previous results are not extensible to arbitrary neighborhoods in $\mathbb{Z}^2$, as the following theorem shows.

Theorem 3. *The Sandpile over $\mathbb{Z}^2$ with the von Neumann neighborhood of radius $k \geqslant 2$ is P-complete and Turing universal.*

Proof. Let us consider the von Neumann neighborhood with radius $r = 2$. Fig. 9 shows the neighborhood, the construction of the wire, the logical gates and the cross-over. For $r > 2$ the construction is analogous.

Then the Sandpile can compute any monotone circuit and then it is P-complete.

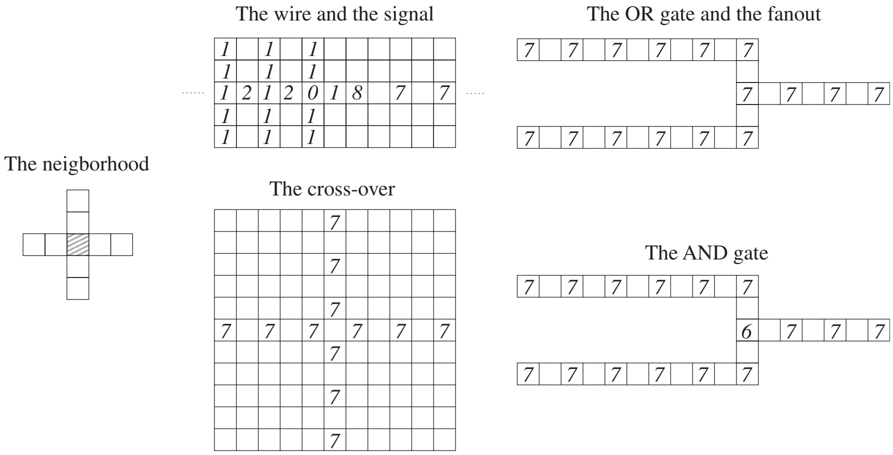


Fig. 9. The devices that show the P-completeness of the Sandpile on a von Neumann neighborhood of radius 2. The wire is a path of cells with seven tokens at distance two. The signal is a cell with eight tokens. When the signal propagates, the wire is distorted.

As we remarked in the introduction, this P-complete Sandpile can simulate any one-dimensional CA, and thus it is Turing universal. □

6. Conclusions

We have proved that, within the Banks's approach, it is impossible to cross information in $\mathbb{Z}^2$ with the Moore and von Neumann neighborhood. This implies that the Banks's approach cannot be used to simulate logical circuits. We suppose that signals are perturbations that propagates over a quiascent background, and that devices are finite configurations that do not modify the background.

It may be possible to conceive signals propagating over an unstable but periodic background. Our tools do not apply for that case. It is also possible to try other methods like the used to prove universality of one-dimensional cellular automata. The reader who wants to prove P-completeness of two-dimensional Sandpiles must discard the Banks method and consider one of these or other approaches.

We introduced the notion of Firing Graph. A similar notion was already used by Dhar [3] to study the recurrent configurations. It helped us to understand the relation between the initial configuration and the avalanche that it produces. It was defined for initial configurations where the cells with four tokens are only on the boundary, but it can be generalized to other kinds of configurations in which some cells fire more than once. For this, one can consider several copies of each cell, one for each of its firings.

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