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Stochastic volatility models at $\rho = \pm 1$ as second class constrained Hamiltonian systems

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HIGHLIGHTS

- Stochastic volatility models for extremely correlated cases $\rho = \pm 1$ are analyzed.
- Stochastic volatility models at $\rho = \pm 1$ are constrained Hamiltonian systems.
- Dirac's method of singular Hamiltonian systems is used.
- The propagator is obtained as a path integral over the volatility alone.
- By semi-classical arguments, the propagator is evaluated for the simplest SABR model.

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ABSTRACT

The stochastic volatility models used in the financial world are characterized, in the continuous-time case, by a set of two coupled stochastic differential equations for the underlying asset price S and volatility σ. In addition, the correlations of the two Brownian movements that drive the stochastic dynamics are measured by the correlation parameter ρ $(-1 \leq \rho \leq 1)$. This stochastic system is equivalent to the Fokker–Planck equation for the transition probability density of the random variables S and σ. Solutions for the transition probability density of the Heston stochastic volatility model (Heston, 1993) were explored in Dragulescu and Yakovenko (2002), where the fundamental quantities such as the transition density itself, depend on ρ in such a manner that these are divergent for the extreme limit $\rho = \pm 1$. The same divergent behavior appears in Hagan et al. (2002), where the probability density of the SABR model was analyzed. In an option pricing context, the propagator of the bi-dimensional Black–Scholes equation was obtained in Lemmens et al. (2008) in terms of the path integrals, and in this case, the propagator diverges again for the extreme values $\rho = \pm 1$. This paper shows that these similar divergent behaviors are due to a universal property of the stochastic volatility models in the continuum: all of them are second class constrained systems for the most extreme correlated limit $\rho = \pm 1$. In this way, the stochastic dynamics of the $\rho = \pm 1$ cases are different of the $-1 < \rho < 1$ case, and it cannot be obtained as a continuous limit from the $\rho \neq \pm 1$ regimen. This conclusion is achieved by considering the Fokker–Planck equation or the bi-dimensional Black–Scholes equation as a Euclidean quantum Schrödinger equation. Then, the analysis of the underlying classical mechanics of the quantum model, implies that stochastic volatility models at $\rho = \pm 1$ correspond to a constrained system. To study the dynamics in an appropriate form, Dirac's method for constrained systems (Dirac, 1958, 1967) must be employed, and Dirac's analysis reveals that the constraints are second class. In order to obtain the transition probability density or the option price correctly, one must evaluate the propagator as a constrained Hamiltonian path-integral (Henneaux and Teitelboim, 1992), in a similar

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way to the high energy gauge theory models. In fact, for all stochastic volatility models, after integrating over momentum variables, one obtains an effective Euclidean Lagrangian path-integral over the volatility alone. The role of the second class constraints is determining the underlying asset price S completely in terms of volatility, so it plays no role in the path integral. In order to examine the effect of the constraints on the dynamics for both extreme limits, the probability density function is evaluated by using semi-classical arguments, in an analogous manner to that developed in Hagan et al. (2002), for the SABR model.

1. Introduction

The stochastic volatility models permit to generalize the Black–Scholes model to the non constant volatility case and are widely used to evaluate option prices in the financial world. A generic stochastic volatility model is characterized in continuous time by two coupled stochastic differential equations for the underlying asset price S and the volatility σ of the form [1,2]

$$dS = \mu(S, \sigma, t)dt + G(S, \sigma, t)dW_S(t) \tag{1}$$

$$d\sigma = r(S, \sigma, t)dt + q(S, \sigma, t)dW_\sigma(t) \tag{2}$$

where μ, G, r and q are arbitrary functions of S, σ and t that defines the stochastic volatility model completely. The two Brownian motions dW_S and dW_σ satisfy the product rule

$$dW_S\, dW_\sigma = \rho\, dt \tag{3}$$

in Itô's sense, and the parameter $-1 \leq \rho \leq 1$ is called the correlation parameter. Between the different plethora of models, the best known are the Heston model [3], the Ornstein–Uhlenbeck model [1], the CEV model [4,5], the SABR volatility model [6], the GARCH model [7], the 3/2 model, the Hull and White model [8] and the Chen model [9]. Other variations include incorporating a "jump diffusion" term, which gives rise to integro-differential equations [10] for the cost of the option. Some stochastic volatility models are even capable to capture some important statistical properties of real markets, called stylized facts, such as the autocorrelation and the leverage effect [11–14]. To account these stylized facts for the real financial data, empirical analysis implies that $|\rho|$ must be of the order of 0.5 [11,12]. Although the extreme case $\rho = \pm 1$ is not realized in the real world (from a statistical point of view), these values can be satisfied for "outliers" events in the sense of Ref. [15] and may be very important in a financial crisis scenario. Also, from a structural mathematical perspective, it is necessary to understand the behavior of the stochastic volatility models for the full range of its parameter values, in particular the case $\rho = \pm 1$.

In mathematical terms too, the stochastic system (1), (2) is equivalent to the Fokker–Planck equation for the transition probability density of the random variables S and σ, and the Heston stochastic volatility model, were explored in Ref. [16] through the solution of the Fokker–Planck equation. There, the fundamental quantities (such as the transition density itself) depend on ρ in such a way that these quantities are divergent for the extreme limit $\rho = \pm 1$. The same divergent behavior appears in Ref. [17], where the probability density of the SABR model was analyzed.

In an option pricing context, the propagator of the bi-dimensional Black–Scholes equation in terms of the path integrals was found in Ref. [18]. In this case, the propagator diverges again for the extremely correlated case $\rho = \pm 1$. The same divergence appears in Refs. [19,20], where propagators for the Black–Scholes equation for different stochastic volatility models, were constructed. Thus, one can ask: What is the reason for the divergent behavior of the probability density? What is the probability density function at $\rho = \pm 1$ itself? What is the propagator for the extremely correlated limit? In Ref. [21], the propagator for the Black–Scholes equation at the extreme limit $\rho = \pm 1$ was obtained for a particular class of stochastic volatility models. At the same time, the divergent behavior of the propagator was explained by the existence of constraints that appear only in the $\rho = \pm 1$ case.

This paper generalizes the results obtained in Ref. [21] for an arbitrary stochastic volatility model and shows that the divergent behavior of the transition probability density or the option price propagator is due to a universal property of the stochastic volatility models in the continuum: all of them are second class constrained systems at the extreme correlated limit.

This conclusion is achieved by considering the Fokker–Planck equation or the bi-dimensional Black–Scholes equation as a Euclidean quantum Schrödinger equation. Then, the analysis of the underlying classical mechanics of the quantum model, implies that stochastic volatility models at $\rho = \pm 1$ correspond to a constrained system. To study the dynamics in an appropriate form, Dirac's method for constrained systems [22,23] must be employed and Dirac's analysis reveals that the constraints are second class. In order to obtain the transition probability density or the option price correctly, one must evaluate the propagator as a constrained Hamiltonian path-integral [24], in a similar way to the high energy gauge theory models. In fact, for all stochastic volatility models, after integrating over momentum variables, one obtains an effective Euclidean Lagrangian path integral over the volatility alone. The role of the second class constraints is determining the underlying asset price S completely in terms of volatility, so it plays no role in the path integral. In order to examine the effect of the constraints on the dynamics of both extreme $\rho = \pm 1$ cases, the probability density function is evaluated by using semi-classical arguments, in an analogous way to that developed in Ref. [17], for the SABR models.

The paper is structured as follows: Sections 2 and 3 show that the Fokker–Planck equation for the transition probability density and the Black–Scholes equation for the option price, can be written as the same Euclidean quantum Schrödinger equation for a charged particle in external electric and magnetic fields. Section 4 is devoted to the analysis of the Euclidean framework both in classical and quantum terms in a generic way. In Section 5, the application of the Euclidean framework is done to the Fokker–Planck/Black–Scholes equations in order to determine its classical behavior and to establish how the second class constraints appear in the extreme correlated limit. Section 6 is devoted to the correct implementation of the propagator as a Hamiltonian constrained path integral. In Section 7, the semi-classical approximation is used to evaluate the transition probability density, with the object to determine the role of the constraints in semi-classical terms. Finally Section 8 contains the conclusions and further perspectives of this work.

2. Stochastic volatility models and the Fokker–Planck equation

In this section, the Fokker–Planck equation for a general stochastic volatility model will be presented and the interpretation of this equation as a Euclidean quantum Schrödinger equation, will permit to find the underlying associated classical dynamic of the quantum model.

The transition probability density $P = P(t', S', \sigma'|t, S, \sigma)$ for the stochastic differential equations (1) and (2) satisfies the Fokker–Planck equation [25]

$$\frac{\partial P}{\partial t} = \frac{1}{2}\frac{\partial^2 (G^2 P)}{\partial S^2} + \rho \frac{\partial^2 (GqP)}{\partial S \partial \sigma} + \frac{1}{2}\frac{\partial^2 (q^2 P)}{\partial \sigma^2} - \frac{\partial (\mu P)}{\partial S} - \frac{\partial (rP)}{\partial \sigma}. \tag{4}$$

It is convenient to separate the Fokker–Planck equation (4) into three types of terms:

(i) Kinetic: associated with second order differential operators,
(ii) Magnetic: related to first order differential operators,
(iii) Electric: generated by a multiplicative potential term
in the form

$$\frac{\partial P}{\partial t} = \frac{1}{2} G^2 \frac{\partial^2 P}{\partial S^2} + \rho G q \frac{\partial^2 P}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 P}{\partial \sigma^2} - A_S \frac{\partial P}{\partial S} - A_\sigma \frac{\partial P}{\partial \sigma} + \Phi P \tag{5}$$

where the magnetic potential terms are given by

$$A_S = A_S(S, \sigma, t) = \mu - \frac{\partial (G^2)}{\partial S} - \rho \frac{\partial (Gq)}{\partial \sigma} \tag{6}$$

$$A_\sigma = A_\sigma(S, \sigma, t) = r - \frac{\partial (q^2)}{\partial \sigma} - \rho \frac{\partial (Gq)}{\partial S} \tag{7}$$

and the electric potential is

$$\Phi = \Phi(S, \sigma, t) = -\frac{\partial (\mu)}{\partial S} - \frac{\partial (r)}{\partial \sigma} + \frac{1}{2}\frac{\partial^2 (G^2)}{\partial S^2} + \frac{1}{2}\frac{\partial^2 (q^2)}{\partial \sigma^2} + \rho \frac{\partial^2 (Gq)}{\partial S \partial \sigma}. \tag{8}$$

Eq. (5) can be interpreted as a Euclidean Schrödinger type equation, for the "wave function" probability density P,

$$\frac{\partial P}{\partial t} = \hat{H} P \tag{9}$$

with Hamiltonian operator

$$\hat{H} = \frac{1}{2} G^2 \frac{\partial^2}{\partial S^2} + \rho G q \frac{\partial^2}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2}{\partial \sigma^2} - A_S \frac{\partial}{\partial S} - A_\sigma \frac{\partial}{\partial \sigma} + \Phi \tag{10}$$

which corresponds to the Hamiltonian of a Euclidean charged quantum particle in the presence of external magnetic and electric fields. The propagator $K(S, \sigma, t|S', \sigma', t')$ for Eq. (9) can be written in terms of path integrals in a way which is analogous to the quantum mechanical description. In fact, the propagator in this instance, is just the transition probability density function P [25]

$$P(S, \sigma, t|S', \sigma', t') = K(S, \sigma, t|S', \sigma', t') = \langle S\sigma | e^{\hat{H}(t-t')} | S'\sigma' \rangle. \tag{11}$$

3. Stochastic volatility models and option pricing

The stochastic volatility models can be applied to option pricing. In this case, the stochastic system (1) and (2), together with the assumption of no-arbitrage and by using a self financing portfolio constructed from the underlying asset and options [1,2], imply the following generalized Black–Scholes equation for the option price $P(S, \sigma, \tau)$

$$\frac{\partial P}{\partial \tau} + \frac{1}{2}G^2 \frac{\partial^2 P}{\partial S^2} + \rho G q \frac{\partial^2 P}{\partial S \partial \sigma} + \frac{1}{2}q^2 \frac{\partial^2 P}{\partial \sigma^2} + r\left(S\frac{\partial P}{\partial S} - P\right) + (p - \lambda q)\frac{\partial P}{\partial \sigma} = 0 \tag{12}$$

where the function $\lambda = \lambda(S, \sigma, \tau)$ is called the "market price of risk". As mentioned in Ref. [2], one can use a risk neutral drift μ function in (1) in such a way that all dependences in λ disappear in (12).

If T is the maturity of the option, the above equation must be integrated with the final condition

$$P(S, \sigma, T) = \mathcal{C}(S, \sigma) \tag{13}$$

where $\mathcal{C}$ is called the contract function. If one defines the backward time $t = T - \tau$ the modified Black–Scholes equation (12) can be written as

$$\frac{\partial P}{\partial t} = \frac{1}{2}G^2 \frac{\partial^2 P}{\partial S^2} + \rho G q \frac{\partial^2 P}{\partial S \partial \sigma} + \frac{1}{2}q^2 \frac{\partial^2 P}{\partial \sigma^2} + rS\frac{\partial P}{\partial S} + (p - \lambda q)\frac{\partial P}{\partial \sigma} - rP \tag{14}$$

with $P = P(S, \sigma, t)$ and (14) must be integrated now with the initial condition $P(S, \sigma, 0) = \mathcal{C}(S, \sigma)$. The above equation can be written in the same form (9), with $\hat{H}$ the same Hamiltonian (10), but now the magnetic and electric potentials are given by

$$A_S = A_S(S, \sigma, t) = -rS \tag{15}$$

$$A_\sigma = A_\sigma(S, \sigma, t) = -(p - \lambda q) \tag{16}$$

$$\Phi = \Phi(S, \sigma, t) = -r. \tag{17}$$

Thus, the bi-dimensional Black–Scholes equation for the stochastic volatility models can be again interpreted as a Euclidean Schrödinger type equation, for the "wave option price" P. In this case, the option price can be obtained as

$$P(S, \sigma, t) = \int K(S, \sigma, t|S', \sigma', 0)\, \mathcal{C}(S', \sigma')\, \mathrm{d}S'\mathrm{d}\sigma' \tag{18}$$

where the propagator is

$$K(S, \sigma, t|S', \sigma', 0) = \langle S\sigma|\mathrm{e}^{\hat{H}t}|S'\sigma'\rangle. \tag{19}$$

In this way, both the Fokker–Planck equation for the transition probability and the Black–Scholes equation for the option pricing for an arbitrary stochastic volatility model, satisfy the same structural Euclidean Schrödinger equation (9). Because the analysis of the quantum model and its associated classical mechanics will be done directly in Euclidean time, it is necessary to make some considerations about the Euclidean formalism, which is the objective of the following section.

4. Euclidean framework

In this section, the analysis of the Euclidean path integral and Euclidean classical dynamics associated with Eq. (9) is done in a generic way.

4.1. Euclidean path integral

Consider the Euclidean Schrödinger type equation

$$\frac{\partial|\Psi(t)\rangle}{\partial t} = \hat{H}|\Psi(t)\rangle \tag{20}$$

where $\hat{H}$ is a time independent Hamiltonian. The solution of this equation, for an initial condition $|\Psi(t = 0)\rangle = |\Phi_0\rangle$ is (see [19] for details)

$$|\Psi(t)\rangle = \mathrm{e}^{\hat{H}t}|\Phi_0\rangle. \tag{21}$$

The wave function can be written as

$$\Psi(x, t) = \int K(x, t|x', 0)\Phi_0(x')\mathrm{d}x' \tag{22}$$

where the propagator is

$$K(x, t|x', 0) = \langle x|\mathrm{e}^{\hat{H}t}|x'\rangle. \tag{23}$$

By dividing the time t into N time steps of length $\epsilon = t/N$, the propagator can be written as

$$K = \langle x|\mathrm{e}^{\hat{H}\epsilon}\mathrm{e}^{\hat{H}\epsilon}\ldots\mathrm{e}^{\hat{H}\epsilon}|x'\rangle; \tag{24}$$

by inserting the identity operators

$$\hat{I} = \int |x\rangle\langle x| \mathrm{d}x \tag{25}$$

$$\hat{I} = \int |p_x\rangle\langle p_x| \frac{\mathrm{d}p_x}{2\pi} \tag{26}$$

the propagator is

$$K = \prod_{i=1}^{N-1} \int \mathrm{d}x_i \prod_{j=1}^{N} \int \frac{\mathrm{d}p_j}{2\pi} \langle x|\mathrm{e}^{\hat{H}\epsilon}|p_N\rangle\langle p_N|x_{N-1}\rangle \cdots \langle x_2|\mathrm{e}^{\hat{H}\epsilon}|p_2\rangle\langle p_2|x_1\rangle\langle x_1|\mathrm{e}^{\hat{H}\epsilon}|p_1\rangle\langle p_1|x'\rangle \tag{27}$$

but using the fact that

$$\langle p|x\rangle = \mathrm{e}^{-ipx} \tag{28}$$

$$\langle x|p\rangle = \mathrm{e}^{ipx} \tag{29}$$

$$\langle x|\mathrm{e}^{\hat{H}\epsilon}|p\rangle = \mathrm{e}^{H(x,p)\epsilon}\langle x|p\rangle \tag{30}$$

where $H(x, p)$ is the classical Hamiltonian associated to the Hamiltonian operator $\hat{H}$ through the relation

$$\langle x|\hat{H}|p\rangle = H(x, p)\langle x|p\rangle \tag{31}$$

the propagator is

$$K = \prod_{i=1}^{N-1} \int \mathrm{d}x_i \prod_{j=1}^{N} \int \frac{\mathrm{d}p_j}{2\pi} \mathrm{e}^{H(x_N,p_N)\epsilon+ip_N(x_N-x_{N-1})} \ldots \mathrm{e}^{H(x_2,p_2)\epsilon+ip_2(x_2-x_1)} \mathrm{e}^{H(x_1,p_1)\epsilon+ip_1(x_1-x_0)} \tag{32}$$

where $x_N = x$ and $x_0 = x'$. Finally

$$K = \prod_{i=1}^{N-1} \int \mathrm{d}x_i \prod_{j=1}^{N} \int \frac{\mathrm{d}p_j}{2\pi} \mathrm{e}^{\sum_{k=1}^{N}[ip_k \frac{(x_k-x_{k-1})}{\epsilon}+H(x_k,p_k)]\epsilon} \tag{33}$$

or in the limit $\epsilon \to 0$

$$K = \int \mathcal{D}x \int \mathcal{D}p_x \, \mathrm{e}^{A[x,p_x]} \tag{34}$$

where the Euclidean classical action A is

$$A[x, p_x] = \int L(x, \dot{x}, t)\mathrm{d}t \tag{35}$$

and the Euclidean Lagrangian L is defined by

$$L(x, \dot{x}, t) = ip_x\dot{x} + H(x, p_x, t) \tag{36}$$

and $\dot{x}$ means the time derivative with respect to t.

For a time dependent Hamiltonian, Eq. (34) is still valid, for details see for example [26].

4.2. Euclidean classical Hamiltonian equations

The classical Euclidean Hamiltonian equations can be deduced by imposing that the classical trajectories optimize the classical action:

$$\delta S = S[x + \delta x, p_x + \delta p_x] - S[x, p_x] = 0 \tag{37}$$

but

$$S[x + \delta x, p_x + \delta p_x] = \int [i(p_x + \delta p_x)(\dot{x} + \delta\dot{x}) + H(x + \delta x, p_x + \delta p_x)] \, \mathrm{d}t. \tag{38}$$

By expanding the Hamiltonian and assuming that the extremes of trajectories are fixed, one gets

$$S[x + \delta x, p_x + \delta p_x] = S[x, p_x] + \int \left(i\dot{x} + \frac{\partial H(x, p_x)}{\partial p_x}\right) \delta p_x \, \mathrm{d}t + \int \left(-i\dot{p}_x + \frac{\partial H(x, p_x)}{\partial x}\right) \delta x \, \mathrm{d}t + \cdots \tag{39}$$

hence

$$\delta S = \int \left(i\dot{x} + \frac{\partial H(x, p_x)}{\partial p_x}\right) \delta p_x \, \mathrm{d}t + \int \left(-i\dot{p}_x + \frac{\partial H(x, p_x)}{\partial x}\right) \delta x \, \mathrm{d}t + \cdots. \tag{40}$$

Thus, the classical Hamiltonian equations in Euclidean time are

$$-i\dot{x} = \frac{\partial H(x, p_x)}{\partial p_x}, \tag{41}$$

$$-i\dot{p}_x = -\frac{\partial H(x, p_x)}{\partial x}. \tag{42}$$

In n dimensions, the Poisson brackets may be defined by

$$\{A(x,p), B(x,p)\} \equiv \sum_i \left(\frac{\partial A(x,p)}{\partial x^i}\frac{\partial B(x,p)}{\partial p_i} - \frac{\partial B(x,p)}{\partial x^i}\frac{\partial A(x,p)}{\partial p_i} \right), \tag{43}$$

to rewrite Hamilton's equations in the form

$$-i\dot{x}^j = \{x^j, H\}, \tag{44}$$

$$-i\dot{p}_k = \{p_k, H\}. \tag{45}$$

5. Euclidean classical dynamics for the Fokker–Planck/Black–Scholes equations

In this section, the classical dynamic associated to the Fokker–Planck/Black–Scholes equations is studied. For the rest of the paper it is assumed that both functions G, q in Eqs. (1) and (2) satisfy $G(S, \sigma, t) \neq 0$ and $q(S, \sigma, t) \neq 0$ in order to obtain a non-trivial coupled stochastic system of equations for the underlying price S and the volatility σ.

5.1. Euclidean classical Hamiltonian dynamics

Use the fact that

$$\langle p_S\, p_\sigma | S\, \sigma \rangle = e^{-ip_S S - ip_\sigma \sigma} \tag{46}$$

$$\langle S\, \sigma | p_S\, p_\sigma \rangle = e^{ip_S S + ip_\sigma \sigma} \tag{47}$$

$$\langle S\, \sigma | \hat{H} | p_S\, p_\sigma \rangle = H(S, \sigma, p_S, p_\sigma) e^{ip_S S + ip_\sigma \sigma} \tag{48}$$

where the classical Hamiltonian $H(S, \sigma, p_S, p_\sigma)$ associated to the quantum Hamiltonian $\hat{H}$ given by (10) is

$$H(S, \sigma, p_S, p_\sigma) = -\frac{1}{2}G^2 p_S^2 - \rho G q p_S p_\sigma - \frac{1}{2} q^2 p_\sigma^2 - iA_S p_S - iA_\sigma p_\sigma + \Phi. \tag{49}$$

The corresponding Hamiltonian equations of motions are

$$-i\dot{S} = \frac{\partial H}{\partial p_S} \qquad -i\dot{\sigma} = \frac{\partial H}{\partial p_\sigma} \tag{50}$$

$$i\dot{p}_S = \frac{\partial H}{\partial S} \qquad i\dot{p}_\sigma = \frac{\partial H}{\partial \sigma} \tag{51}$$

and in this case the first two Hamiltonian equations (50) for S and σ give the system

$$\begin{pmatrix} i\dot{S} - iA_S \\ i\dot{\sigma} - iA_\sigma \end{pmatrix} = \mathbf{M} \begin{pmatrix} p_S \\ p_\sigma \end{pmatrix} = \begin{pmatrix} G^2 & \rho G q \\ \rho G q & q^2 \end{pmatrix} \begin{pmatrix} p_S \\ p_\sigma \end{pmatrix} \tag{52}$$

where the determinant of $\mathbf{M}$ is

$$\det(\mathbf{M}) = G^2 q^2 (1 - \rho^2). \tag{53}$$

When $\rho \neq \pm 1$ one has that $\det(\mathbf{M}) \neq 0$ and the momentum variables p_S and p_σ can be written in terms of velocities $\dot{S}$ and $\dot{\sigma}$ from (52). In this case the Hamiltonian system (50), (51) is regular and no constraints appear.

But when $\rho = \pm 1$, $\det(\mathbf{M}) = 0$ and the equations of the system (52) are not independent. In fact for $\rho = \pm 1$ the system (52) is explicitly

$$i\dot{S} - iA_S = G^2 p_S \pm G q p_\sigma \tag{54}$$

$$i\dot{\sigma} - iA_\sigma = \pm G q p_S + q^2 p_\sigma. \tag{55}$$

Multiplying (54) by q and (55) by G one obtains

$$(i\dot{S} - iA_S) q = G^2 p_S q \pm G q^2 p_\sigma \tag{56}$$

$$(i\dot{\sigma} - iA_\sigma) G = \pm G^2 q p_S + G q^2 p_\sigma. \tag{57}$$

Now, for $\rho = +1$ one can subtract (56) and (57) and for $\rho = -1$ one can add (56) and (57) to give the constraint

$$(i\dot{S} - iA_S) \mp (i\dot{\sigma} - iA_\sigma)\frac{G}{q} = 0. \tag{58}$$

The classical Hamiltonian for $\rho = \pm 1$ is

$$H^{\pm} = -\frac{1}{2}(Gp_S \pm qp_\sigma)^2 - iA_S p_S - iA_\sigma p_\sigma + \Phi. \tag{59}$$

Note that the magnetic and electric potentials (6)–(8) for the Fokker–Planck equation depend on $\rho = \pm 1$ explicitly, whereas for the Black–Scholes equation, those same potentials (15)–(17) are $\rho = \pm 1$ independent.

5.2. Euclidean classical singular Lagrangian dynamics

For $\rho = \pm 1$ the Euclidean Lagrangian $L^{\pm}$ associated to the Euclidean classical Hamiltonian $H^{\pm}$ becomes

$$L^{\pm} = ip_S\dot{S} + ip_\sigma\dot{\sigma} + H^{\pm} \tag{60}$$

that is

$$L^{\pm} = ip_S\dot{S} + ip_\sigma\dot{\sigma} - \frac{1}{2}(Gp_S \pm qp_\sigma)^2 - iA_S p_S - iA_\sigma p_\sigma + \Phi \tag{61}$$

which can be written as

$$L^{\pm} = (i\dot{S} - iA_S)p_S + (i\dot{\sigma} - iA_\sigma)p_\sigma - \frac{1}{2}(Gp_S \pm qp_\sigma)^2 + \Phi. \tag{62}$$

From Eq. (54) one obtains

$$Gp_S \pm qp_\sigma = \frac{(i\dot{S} - iA_S)}{G} \tag{63}$$

and

$$p_S = \frac{(i\dot{S} - iA_S)}{G^2} \mp \frac{qp_\sigma}{G}. \tag{64}$$

Replacing (63) and (64), respectively, in the third and first terms of (62) gives

$$L^{\pm} = \frac{(i\dot{S} - iA_S)^2}{G^2} \mp (i\dot{S} - iA_S)\frac{qp_\sigma}{G} + (i\dot{\sigma} - iA_\sigma)p_\sigma - \frac{1}{2}\frac{(i\dot{S} - iA_S)^2}{G^2} + \Phi \tag{65}$$

or

$$L^{\pm} = \frac{1}{2}\frac{(i\dot{S} - iA_S)^2}{G^2} + \left[(i\dot{\sigma} - iA_\sigma) \mp (i\dot{S} - iA_S)\frac{q}{G}\right]p_\sigma + \Phi. \tag{66}$$

Using now the constraint equation (58) in (66), one arrives finally to

$$L^{\pm} = -\frac{1}{2}\frac{(\dot{S} - A_S)^2}{G^2} + \Phi. \tag{67}$$

In order that the dynamical evolution of (67) be equivalent to the Hamiltonian evolution, the last Lagrangian must be supplemented with the constraint (58). To describe properly the classical constrained Lagrangian dynamics, one must apply Dirac's method [22,23] to the Lagrangian in (67). In this way, one needs to construct the extended Lagrangian

$$\tilde{L}^{\pm} = -\frac{1}{2}\frac{(\dot{S} - A_S)^2}{G^2} + \Phi + i\lambda\left[(\dot{S} - A_S) \mp (\dot{\sigma} - A_\sigma)\frac{G}{q}\right]. \tag{68}$$

The canonical Euclidean momentum definitions

$$ip_\sigma = \frac{\partial \tilde{L}^{\pm}}{\partial \dot{\sigma}} \tag{69}$$

$$ip_\lambda = \frac{\partial \tilde{L}^{\pm}}{\partial \dot{\lambda}} = 0 \tag{70}$$

give origin to the primary constraints

$$\phi_1^{\pm} = p_\sigma \pm \lambda\frac{G}{q} \simeq 0 \tag{71}$$

$$\phi_2 = p_\lambda \simeq 0. \tag{72}$$

Dirac's extended Hamiltonian is

$$H^{\pm}_{\rm ext} = -ip_S\dot{S} - ip_\sigma\dot{\sigma} - ip_\lambda\dot{\lambda} + \tilde{L}^{\pm} + \mu_1\phi_1^{\pm} + \mu_2\phi_2 \tag{73}$$

which written in terms of the momenta turns out to be

$$H^{\pm}_{\rm ext} = -\frac{1}{2}(Gp_S \pm qp_\sigma)^2 - iA_S p_S - iA_\sigma p_\sigma + \mu_1\left[p_\sigma \pm \lambda\frac{G}{q}\right] + \mu_2[p_\lambda]. \tag{74}$$

The (time evolution) consistency of the constraints implies

$$-i\dot{\phi}_1^{\pm} = \{\phi_1^{\pm}, H^{\pm}_{\rm ext}\} = 0 \tag{75}$$

$$-i\dot{\phi}_2 = \{\phi_2, H^{\pm}_{\rm ext}\} = 0 \tag{76}$$

which yield no new constraints if $\frac{G}{q} \neq 0$. Therefore, there are only two primary constraints. The Poisson bracket for the two primary constraints $\phi_1^{\pm}$ and ϕ_2 turns out to be

$$\{\phi_1^{\pm}, \phi_2\} = \pm\frac{G}{q} \tag{77}$$

so that Dirac's matrix is invertible when $\frac{G}{q} \neq 0$, a condition that is necessary to ensure the existence of the propagator, as discussed later. Then the two constraints $\phi_1^{\pm}$ and ϕ_2 are second class and can be eliminated by using Dirac brackets. Hence, one can set $\phi_1^{\pm}$ and ϕ_2 strongly equal to zero, that is,

$$\phi_1^{\pm} = p_\sigma \pm \frac{G}{q} = 0 \tag{78}$$

$$\phi_2 = p_\lambda = 0. \tag{79}$$

These last equations imply that the extended Hamiltonian can be written as

$$H^{\pm}_{\rm ext} = H^{\pm} \tag{80}$$

which coincides with the original expression (59). Hence, the Euclidean singular Lagrangian and Hamiltonian theories are equivalent.

6. The transition probability density as a path integral

The transition probability density "wave function" P is given by

$$P(S, \sigma, t) = \int K^{\pm}(S, \sigma, t|S', \sigma', 0)P_0(S', \sigma')\, dS'd\sigma' \tag{81}$$

where the propagator is

$$K^{\pm}(S, \sigma, t|S', \sigma', 0) = \langle S\sigma|e^{\hat{H}^{\pm}t}|S'\sigma'\rangle \tag{82}$$

and $\hat{H}^{\pm}$ is the Fokker–Planck/Black–Scholes Hamiltonian operator evaluated at $\rho = \pm 1$.

For the probability density P, the initial condition is just Dirac's delta [25]

$$P_0(S', \sigma') = \delta(S' - S'', \sigma' - \sigma'') \tag{83}$$

so the transition probability is the propagator itself

$$P(S, \sigma, t) = K(S, \sigma, t|S', \sigma', 0). \tag{84}$$

The Euclidean path integral for the propagator of systems with second class constraints is given in phase space by [24]

$$K^{\pm} = \int \mathcal{D}S\mathcal{D}\sigma\mathcal{D}\lambda\frac{\mathcal{D}p_S}{2\pi}\frac{\mathcal{D}p_\sigma}{2\pi}\frac{\mathcal{D}p_\lambda}{2\pi}\sqrt{\det(\mathbf{C})}\delta(\phi_1^{\pm})\delta(\phi_2)\exp(A^{\pm}[S, \sigma, \lambda, p_S, p_\sigma, p_\lambda]) \tag{85}$$

where $\mathbf{C}$ is Dirac's matrix constructed out of the second class constraints according to

$$\mathbf{C} = (C_{ij}) = \{\phi_i, \phi_j\} \tag{86}$$

and

$$A^{\pm}[S, \sigma, \lambda, p_S, p_\sigma, p_\lambda] = \int [ip_S\dot{S} + ip_\sigma\dot{\sigma} + ip_\lambda\dot{\lambda} + H^{\pm}(S, \sigma, \lambda, p_S, p_\sigma, p_\lambda)]\, dt \tag{87}$$

is the Euclidean action. In this case, Dirac's matrix is

$$\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} \{\phi_1^{\pm}, \phi_1^{\pm}\} & \{\phi_1^{\pm}, \phi_2\} \\ \{\phi_2, \phi_1^{\pm}\} & \{\phi_2, \phi_2\} \end{pmatrix} = \begin{pmatrix} 0 & \pm\frac{G}{q} \\ \mp\frac{G}{q} & 0 \end{pmatrix}$$

therefore,

$$\det(\mathbf{C}) = \frac{G^2}{q^2}. \tag{88}$$

In this way, the propagator is given by

$$K^{\pm} = \int \mathcal{D}S\mathcal{D}\sigma\mathcal{D}\lambda \frac{\mathcal{D}p_S}{2\pi}\frac{\mathcal{D}p_\sigma}{2\pi}\frac{\mathcal{D}p_\lambda}{2\pi}\frac{G}{q}\delta\left(p_\sigma \pm \lambda\frac{G}{q}\right)\delta(p_\lambda)\exp(A^{\pm}[S,\sigma,\lambda,p_S,p_\sigma,p_\lambda]) \tag{89}$$

where the action $A^{\pm}$ is

$$A^{\pm}[S,\sigma,\lambda,p_S,p_\sigma,p_\lambda] = \int\left[ip_S\dot{S} + ip_\sigma\dot{\sigma} + ip_\lambda\dot{\lambda} - \frac{1}{2}(Gp_S \pm qp_\sigma)^2 - iA_Sp_S - iA_\sigma p_\sigma + \Phi\right]\mathrm{d}t. \tag{90}$$

Integration on p_λ yields

$$K^{\pm} = \int \mathcal{D}S\mathcal{D}\sigma\mathcal{D}\lambda \frac{\mathcal{D}p_S}{2\pi}\frac{\mathcal{D}p_\sigma}{2\pi}\frac{1}{2\pi}\frac{G}{q}\delta\left(p_\sigma \pm \lambda\frac{G}{q}\right) \times \exp\left(\int\left[ip_S\dot{S} + ip_\sigma\dot{\sigma} - \frac{1}{2}(Gp_S \mp qp_\sigma)^2 - iA_Sp_S - iA_\sigma p_\sigma + \Phi\right]\mathrm{d}t\right) \tag{91}$$

whereas integration on p_σ gives rise to

$$K^{\pm} = \int \mathcal{D}S\mathcal{D}\sigma\mathcal{D}\lambda \frac{\mathcal{D}p_S}{2\pi}\frac{1}{(2\pi)^2}\frac{G}{q}\exp\left(\int\left[ip_S\dot{S} \mp i\lambda\frac{G}{q}\dot{\sigma} - \frac{1}{2}(Gp_S - G\lambda)^2 - iA_Sp_S \pm i\lambda\frac{G}{q}A_\sigma + \Phi\right]\mathrm{d}t\right). \tag{92}$$

After integrating over p_S [27] one gets

$$K^{\pm} = \int \mathcal{D}S\mathcal{D}\sigma\mathcal{D}\lambda \frac{1}{(2\pi)^3}\frac{G}{q}\sqrt{\frac{2\pi}{G^2\,\mathrm{d}t}}\exp\left(\int\left[-\frac{(\dot{S}-A_S)^2}{2G^2} + i\lambda\left[(\dot{S}-A_S) \mp \frac{G}{q}(\dot{\sigma}-A_\sigma)\right] + \Phi\right]\mathrm{d}t\right). \tag{93}$$

Integration on λ yields

$$K^{\pm} = \int \mathcal{D}S\mathcal{D}\sigma \frac{1}{(2\pi)^3}\frac{1}{q}\sqrt{\frac{2\pi}{\mathrm{d}t}}\delta\left[(\dot{S}-A_S) \mp \frac{G}{q}(\dot{\sigma}-A_\sigma)\right]\exp\left(\int\left[-\frac{(\dot{S}-A_S)^2}{2G^2} + \Phi\right]\mathrm{d}t\right). \tag{94}$$

Finally, after integrating on S one arrives to the principal result of this paper, which is stated as a theorem:

Theorem 1. *The propagator for the Fokker–Planck/Black–Scholes Hamiltonian (10) associated to an arbitrary stochastic volatility model (1), (2) when $\rho = \pm 1$, is given by the path integral*

$$K^{\pm} = \int \mathcal{D}\sigma \frac{1}{(2\pi)^3}\frac{1}{q}\sqrt{\frac{2\pi}{\mathrm{d}t}}\exp\left(\int\left[-\frac{(\dot{\sigma}-A_\sigma)^2}{2q^2} + \Phi\right]\mathrm{d}t\right). \tag{95}$$

In the last expression, the functions $q = q(S,\sigma,t)$, $\Phi = \Phi(S,\sigma,t)$ and $A_\sigma = A_\sigma(S,\sigma,t)$ must be evaluated in the asset price trajectory $S(t) = S_\sigma(t)$, where $S_\sigma(t)$ is the solution to the inhomogeneous differential equation

$$\dot{S} = A_S(S,\sigma,t) \pm \frac{G(S,\sigma,t)}{q(S,\sigma,t)}(\dot{\sigma} - A_\sigma(S,\sigma,t)) \tag{96}$$

for each trajectory $\sigma(t)$ of the path integral.

One interesting special case of the above theorem occurs when the q function depends on σ only, which gives the following corollary.

Theorem 2. *When the function q depends only on σ, that is, $q(S,\sigma,t) = q(\sigma)$, the transformation*

$$u(t) = \int \frac{\dot{\sigma}}{q(\sigma)}\mathrm{d}t \tag{97}$$

implies that the path integral given above can be written as

$$K^{\pm} = \int \mathcal{D}u \frac{1}{(2\pi)^3}\sqrt{\frac{2\pi}{\mathrm{d}t}} \,\exp\left(\int \left[-\frac{\left(\dot{u}-\bar{A}_\sigma\right)^2}{2} + \bar{\Phi}\right] \mathrm{d}t\right) \tag{98}$$

with

$$\bar{G}(S,u,t) = G(S,\sigma(u),t) \tag{99}$$

$$\bar{A}_S(S,u,t) = A_S(S,\sigma(u),t) \tag{100}$$

$$\bar{A}_\sigma(S,u,t) = \frac{A_\sigma(S,\sigma(u),t)}{q(\sigma(u))} \tag{101}$$

$$\bar{\Phi}(S,u,t) = \Phi(S,\sigma(u),t) \tag{102}$$

where $\sigma(u)$ is obtained by inverting σ as a function of u from Eq. (97), and $S = S_u(t)$ is the solution of the differential equation

$$\dot{S} = \bar{A}_S(S,u,t) \pm \bar{G}(S,u,t)(\dot{u} - \bar{A}_\sigma(S,u,t)) \tag{103}$$

for each trajectory $u(t)$ in the path integral (98). Note that in this case, the second class constraint measure $\frac{1}{q}$ disappears from the path integral.

7. Semi-classical approximation

In order to explore the role of the constraints in the dynamic, the semi-classical method will be used to find an approximation to the transition density probability K. For this, the solutions of the classical motion equations are needed. To obtain the classical equations of motion, in general one must search for the extreme of action given in the exponent of the path integral (95)

$$A = \int_{t_0}^{t} \left[-\frac{(\dot{\sigma}-A_\sigma)^2}{2q^2} + \Phi\right] \mathrm{d}\tau \tag{104}$$

subject to the constraint equation (96), that is, one must consider the extended action

$$\tilde{A} = \int_{t_0}^{t} \left[-\frac{(\dot{\sigma}-A_\sigma)^2}{2q^2} + \Phi\right] + \lambda(\tau)\left[\dot{S} - A_S(S,\sigma,\tau) \mp \frac{G(S,\sigma,\tau)}{q(S,\sigma,\tau)}(\dot{\sigma} - A_\sigma(S,\sigma,\tau))\right] \mathrm{d}\tau \tag{105}$$

where the dots denote the derivative with respect to the τ variable, and $\lambda(\tau)$ is a Lagrange multiplier.

Note that this optimization problem is analogous to an optimal control problem, where one optimizes a certain functional of the form

$$A = \int_{t_0}^{t} F(x(\tau), v(\tau), \tau)\mathrm{d}\tau \tag{106}$$

subject to the equation

$$\dot{x}(\tau) = f(x(\tau), v(\tau), \tau). \tag{107}$$

Here x is called a state variable (defined by a differential equation), whereas v is called a control variable (defined by an algebraic equation). To find the equations of motion of this system, one considers the extended action

$$\tilde{A} = \int_{t_0}^{t} F(x(\tau), v(\tau), \tau) + \lambda(\tau)[\dot{x}(\tau) - f(x(\tau), v(\tau), \tau)]\mathrm{d}\tau \tag{108}$$

and search for an extreme of this extended action with respect to x, v and λ. This optimization process implies that the Lagrangian multiplier $\lambda(\tau)$ satisfies the transversality condition

$$\lambda(t) = 0. \tag{109}$$

For more details about the optimal control theory problem, see for example [28].

In our case, the extended action (105) has the same structure of the optimal control problem action (108), but without the control variable. Only state variables S and σ appear in this case, plus the Lagrangian multiplier λ (that satisfies the transversality condition (109)). The Euler–Lagrange equations associated to the extended action (105) give a three dimensional system of differential coupled equations for the asset price S, the volatility σ and the Lagrange multiplier λ. These equations for the general case are very complicated and are not necessary to write them. More interesting is to analyze the case $q = q(\sigma)$. Here, one considers the action that appears in the exponent of the path integral (98)

$$A = \int_{t_0}^{t} \left[-\frac{\left(\dot{u}-\bar{A}_\sigma\right)^2}{2} + \bar{\Phi}\right] \mathrm{d}\tau \tag{110}$$

subject to the constraint (103) and where the extended action is

$$\tilde{A} = \int_{t_0}^{t} \left[-\frac{\left(\dot{u} - \bar{A}_\sigma\right)^2}{2} + \bar{\Phi} \right] + \lambda(\tau)[\dot{S} - \bar{A}_S(S, u, \tau) \mp \bar{G}(S, u, \tau)(\dot{u} - \bar{A}_\sigma(S, u, \tau))]\, \mathrm{d}\tau. \tag{111}$$

One important special case must be mentioned here, when the functions $\bar{G}$, $\bar{A}_S$ are separable in terms of S and u and $\bar{A}_\sigma$ and $\bar{\Phi}$ are independent of S

$$\bar{G}(S, u, t) = f(S)\bar{G}(u) \tag{112}$$

$$\bar{A}_S(S, u, t) = f(S)\bar{A}_S(u) \tag{113}$$

$$\bar{A}_\sigma(S, u, t) = \bar{A}_\sigma(u) \tag{114}$$

$$\bar{\Phi}(S, u, t) = \bar{\Phi}(u). \tag{115}$$

In this situation, the constraint equation (103) becomes

$$\dot{S} = f(S)\bar{A}_S(u) \pm f(S)\bar{G}(u)(\dot{u} - \bar{A}_\sigma(u)) \tag{116}$$

and by taking the transformation $x(\tau) = \int \frac{\dot{S}}{f(S)} \mathrm{d}\tau$ the last equation changes to

$$\dot{x} = \bar{A}_S(u) \pm \bar{G}(u)(\dot{u} - \bar{A}_\sigma(u)). \tag{117}$$

In this way, one considers the optimal problem for x, u and λ

$$\tilde{A} = \int_{t_0}^{t} \left[-\frac{\left(\dot{u} - \bar{A}_\sigma\right)^2}{2} + \bar{\Phi} \right] + \lambda(\tau)[\dot{x} - \bar{A}_S(u) \mp S\bar{G}(u)(\dot{u} - \bar{A}_\sigma(u))]\, \mathrm{d}\tau. \tag{118}$$

The Euler–Lagrange equations give the system

$$\dot{x} = \bar{A}_S(u) \pm \bar{G}(u)(\dot{u} - \bar{A}_\sigma(u)) \tag{119}$$

$$\dot{\lambda} = 0 \tag{120}$$

$$(\dot{u} - \bar{A}_\sigma(u))\frac{\partial \bar{A}_\sigma(u)}{\partial u} + \frac{\partial \bar{\Phi}(u)}{\partial u} + \frac{\mathrm{d}}{\mathrm{d}\tau}[(\dot{u} - \bar{A}_\sigma(u)) \pm \lambda\bar{G}(u)]. \tag{121}$$

The transversality condition for the Lagrange multiplier implies that $\lambda(\tau) = 0$ for $t_0 \le \tau \le t$, and the u dynamic becomes independent of the x dynamic and it is given by the following Newton equation for the u variable

$$\ddot{u} = -\frac{\partial V(u)}{\partial u} \tag{122}$$

where the effective potential is defined by

$$V(u) = \bar{\Phi}(u) - \frac{1}{2}[\bar{A}_\sigma(u)]^2. \tag{123}$$

The constraint equation (116) can be integrated to give the S dynamic by

$$\int \frac{\mathrm{d}S}{f(S)} = \int \bar{A}_S(u(\tau)) \pm \bar{G}(u(\tau)) \left(\dot{u}(\tau) - \bar{A}_\sigma(u(\tau))\right) \mathrm{d}\tau \tag{124}$$

where $u(\tau)$ are the solutions of the Newton equations (122).

In the following section, these results will be applied to study a special volatility model known as the SABR model.

7.1. Example: the SABR model with $C = 1$

The SABR model is defined by the system of stochastic differential equations

$$\begin{aligned} \mathrm{d}S &= \sigma C(S)\mathrm{d}W_S \\ \mathrm{d}\sigma &= \alpha\sigma \mathrm{d}W_\sigma. \end{aligned} \tag{125}$$

For the special case, $C(S) = 1$ the functions μ, G, r, q are given by

$$\begin{aligned} \mu &= 0 \\ G &= \sigma \\ r &= 0 \\ q &= \alpha\sigma. \end{aligned} \tag{126}$$

The magnetic and electric potentials associated to the Fokker–Planck equation for this model are

$$\begin{aligned} A_S &= \mp 2\alpha\sigma \\ A_\sigma &= \alpha^2\sigma \\ \Phi &= \alpha^2. \end{aligned} \tag{127}$$

Because q depends on σ only one can apply Theorem 2, and Eq. (97) gives $\sigma = e^{\alpha u}$. In this way

$$\begin{aligned} \bar{G} &= e^{\alpha u} \\ \bar{A}_S &= \mp 2\alpha e^{\alpha u} \\ \bar{A}_\sigma &= -2\alpha \\ \bar{\Phi} &= \alpha^2. \end{aligned} \tag{128}$$

The propagator is given by

$$K^{\pm} = \int \mathcal{D}u \frac{1}{(2\pi)^3}\sqrt{\frac{2\pi}{d\tau}}\ \exp\left(\int \left[-\frac{(\dot{u}+2\alpha)^2}{2} + \alpha^2\right] d\tau\right). \tag{129}$$

The extended Lagrangian in this case is

$$\tilde{L}^{\pm} = -\frac{(\dot{u}+2\alpha)^2}{2} + \alpha^2 + \lambda(\tau)[\dot{S} \pm 2\alpha e^{\alpha u} \mp e^{\alpha u}(\dot{u}+2\alpha)]. \tag{130}$$

The Euler–Lagrange equations for λ, u and S are in this case respectively

$$\dot{\lambda} = 0 \tag{131}$$

$$\ddot{u} \pm \left(\dot{\lambda} e^{\alpha u} + \lambda\alpha\dot{u}e^{\alpha u} + \lambda\,[2\alpha^2 e^{\alpha u} - \alpha e^{\alpha u}(\dot{u}+2\alpha)]\right) = 0 \tag{132}$$

$$\dot{S} = \mp 2\alpha e^{\alpha u} \pm e^{\alpha u}(\dot{u}+2\alpha). \tag{133}$$

The transversality condition $\lambda(t) = 0$ implies that the solution for λ in (131) is $\lambda(\tau) = 0$ for $t_0 \le \tau \le t$, so the equation for u reduces to

$$\ddot{u} = 0 \tag{134}$$

with a solution, for initial and final conditions $u(t_0) = u_0$, $u(t) = u$, given by

$$u(\tau) = \frac{u-u_0}{t-t_0}\tau + \frac{u_0 t - u t_0}{t-t_0}. \tag{135}$$

The solution of the constraint equation for S with the initial condition $S(t_0) = S_0$ is

$$S(\tau) = \pm\frac{1}{\alpha}\left(e^{\alpha\left(\frac{u-u_0}{t-t_0}\tau + \frac{u_0 t - u t_0}{t-t_0}\right)} \pm S_0\alpha - e^{\alpha u_0}\right). \tag{136}$$

The action (110) evaluated on the classical solutions is

$$A = -\frac{1}{2}\left(\frac{u-u_0}{t-t_0} + 2\alpha\right)^2 (t-t_0) + \alpha^2(t-t_0) \tag{137}$$

and the semi-classical solution (up to second order terms) for the propagator is [27]

$$K^{\pm} = \frac{1}{\sqrt{2\pi(t-t_0)}} \exp\left(-\frac{1}{2}\left(\frac{u-u_0}{t-t_0} + 2\alpha\right)^2 (t-t_0) + \alpha^2(t-t_0)\right) \tag{138}$$

which is the exact solution. In fact, if one considers the Lagrangian

$$L_u = -\frac{(\dot{u}+2\alpha)^2}{2} + \alpha^2 \tag{139}$$

on the exponent of (129), as the Lagrangian of a one-dimensional charged particle in constant magnetic and electric potentials, the classical Hamiltonian is then

$$H = -ip_u\dot{u} + L \tag{140}$$

that is

$$H = -\frac{p_u^2}{2} + 2i\alpha p_u + \alpha^2; \tag{141}$$

by quantizing this classical Hamiltonian through the relations

$$u \to \hat{u} = u \tag{142}$$

$$p_u \to \hat{p}_u = -i\frac{\partial}{\partial u} \tag{143}$$

one arrives to the quantum Hamiltonian

$$\hat{H} = \frac{1}{2}\frac{\partial^2}{\partial u^2} + 2\alpha\frac{\partial}{\partial u} + \alpha^2. \tag{144}$$

The corresponding Schrödinger equation is

$$\frac{\partial P(u,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 P(u,t)}{\partial u^2} + 2\alpha\frac{\partial P(u,t)}{\partial u} + \alpha^2 P(u,t). \tag{145}$$

The propagator for this equation can be obtained by a Fourier transform

$$P(u,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(k,t)\mathrm{e}^{iku}\mathrm{d}k \tag{146}$$

which gives

$$\frac{\mathrm{d}P(k,t)}{\mathrm{d}t} = \left(-\frac{1}{2}k^2 + 2ik\alpha + \alpha^2\right)P(k,t) \tag{147}$$

with solution

$$P(k,t) = \mathrm{e}^{\left(-\frac{1}{2}k^2+2ik\alpha+\alpha^2\right)t}; \tag{148}$$

by replacing it in (146), one gets

$$P(u,t) = \frac{1}{2\pi}\mathrm{e}^{\alpha^2 t}\mathrm{e}^{-\frac{(u+2\alpha t)^2}{2t}}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{1}{2}\left(\sqrt{t}k - i(\frac{u+2\alpha t}{\sqrt{t}})\right)^2}\mathrm{d}k \tag{149}$$

and after integrating over k, the following results hold

$$P(u,t) = \frac{1}{\sqrt{2\pi t}}\mathrm{e}^{\alpha^2 t}\mathrm{e}^{-\frac{(u+2\alpha t)^2}{2t}}. \tag{150}$$

Now, by shifting the u and t variables in $u-u_0$ and $t-t_0$ respectively one arrive finally to the propagator K for the Schrödinger equation (145)

$$K(u,t|u_0,t_0) = \frac{1}{\sqrt{2\pi(t-t_0)}}\mathrm{e}^{\alpha^2(t-t_0)}\mathrm{e}^{-\frac{((u-u_0)+2\alpha(t-t_0))^2}{2(t-t_0)}} \tag{151}$$

which is the same propagator obtained by the semi-classical approximation (138). Hence, the transition probability density of the SABR model with $C = 1$ is equal to the Euclidean propagator associated to a charged one-dimensional particle placed in a region with constant magnetic and electric potentials.

8. Conclusions and further research

Thus, stochastic volatility models in continuous time for the extreme correlated case, are second class constrained Hamiltonian systems. This is a universal property, that is, valid for arbitrary μ, G, r and q functions in the stochastic system (1), (2). Because the emergence of the constraint for $\rho = \pm 1$, its stochastic dynamic is completely different of the $-1 < \rho < 1$ case. Thus, one cannot take the limit $\rho \to \pm 1$ from the $-1 < \rho < 1$ world, procedure that, in fact, gives divergent quantities. The correct answer comes from the Hamiltonian Dirac's analysis of constrained systems. Its application for the Euclidean classical and quantum behaviors, permits determine the propagator of the Fokker–Planck/Black–Scholes equations in terms of a constrained path integral. Once the momenta integration of the Hamiltonian path integral is performed, one gets an unconstrained effective Lagrangian path integral. In all cases, the price of the underlying asset price $S(t)$ is completely determined by one of the second class constraints in terms of volatility $\sigma(t)$ and plays no active role in the path integral.

It is necessary to mention here, that the analysis of the classical equations of motion that underlie the quantum Euclidean financial systems, can be an important and effective tool, because it reveals some facts that the quantum analysis can overlook.

As an application, we study the SABR model with $C = 1$. In this case, the transition probability density at $\rho = \pm 1$ can be computed exactly using semi-classical arguments or in a direct form, and it is identical to the propagator of a charged Euclidean particle in the presence of constant electromagnetic potentials.

In forthcoming articles, we will explore in detail the semi-classical behavior of several stochastic volatility models at the extreme correlated limit, and compare it with the exact path integral (95) computed in a closed form or numerically, and use it to obtain the transition probability density or the price of a call, for each case.

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