PHYSICAL REVIEW D **89,** 084050 (2014)

Asymptotically locally AdS and flat black holes in Horndeski theory

Andres Anabalon,[1,2,*] Adolfo Cisterna,[3,†] and Julio Oliva[4,‡]

[1]*Departamento de Ciencias, Facultad de Artes Liberales y Facultad de Ingeniería y Ciencias, Universidad Adolfo Ibáñez, Viña del Mar, Chile*

[2]*Laboratoire de Physique, Université de Lyon, UMR 5672, CNRS, École Normale Supérieure de Lyon, 46 allé d'Italie, F-69364 Lyon Cedex 07, France*

[3]*Instituto de Física, Pontificia Universidad de Católica de Valparaíso, Avenida Universidad 330, Curauma, Valparaíso, Chile*

[4]*Instituto de Ciencias Físicas y Matemáticas, Universidad Austral de Chile, Valdivia, Chile*

(Received 16 December 2013; published 10 April 2014)

In this paper we construct asymptotically locally AdS and flat black holes in the presence of a scalar field whose kinetic term is constructed out from a linear combination of the metric and the Einstein tensor. The field equations as well as the energy-momentum tensor are second order in the metric and the field, therefore the theory belongs to the ones defined by Horndeski. We show that in the presence of a cosmological term in the action, it is possible to have a real scalar field in the region outside the event horizon. The solutions are characterized by a single integration constant, the scalar field vanishes at the horizon and it contributes to the effective cosmological constant at infinity. We extend these results to the topological case. The solution is disconnected from the maximally symmetric AdS background, however, within this family there exists a gravitational soliton which is everywhere regular. This soliton is therefore used as a background to define a finite Euclidean action and to obtain the thermodynamics of the black holes. For a certain region in the space of parameters, the thermodynamic analysis reveals a critical temperature at which a Hawking-Page phase transition between the black hole and the soliton occurs. We extend the solution to arbitrary dimensions greater than 4 and show that the presence of a cosmological term in the action allows one to consider the case in which the standard kinetic term for the scalar it is not present. In such a scenario, the solution reduces to an asymptotically flat black hole.

DOI: 10.1103/PhysRevD.89.084050 PACS numbers: 04.70.Bw, 04.50.Kd, 04.50.Gh, 04.70.Dy

I. INTRODUCTION

In many scenarios, the gravitational dynamics is successfully described by Einstein's theory of general relativity (GR), which has a large observational support [1]. However there is also a strong empirical motivation, mainly associated with astronomical observations, that alternative theories of gravity could be useful in order to describe phenomena such as dark matter and dark energy, or the process of inflation suffered by the early Universe. Many of these theories are based in the use scalar fields. The inclusion of scalar fields in physics has a vital role. Scalar fields for example appear in the Brans-Dicke theory as an attempt to incorporate Mach's principle in a gravitational theory [2], in inflationary theories, and as a possible candidate of dark matter. On the other hand in particle physics, the Higgs boson is a fundamental piece of the standard model, and its recent discovery gives a firm ground to the realization of scalar fields in nature. Moreover, scalar fields are also ubiquitous in string theory and Kaluza-Klein compactifications of higher dimensional theories.

The study of scalar-tensor theories have recently attracted considerable attention due to the development of Galileon theories and their applications [3]. In particular, these ideas led the rediscovery of the most general scalar-tensor theory which has second order field equations and second order energy-momentum tensor; a problem that was solved by Horndeski [4] in the early 1970s. In a curved four-dimensional background, the most general Lagrangian that can be constructed with these properties is given by

$$L = \beta_1 \delta^{abdc}_{efhi} R^{ef}_{ab} R^{hi}_{cd} + \beta_2 \delta^{acd}_{def} \nabla_a \phi \nabla^d \phi R^{ef}_{cd} + \beta_3 \delta^{ab}_{cd} R^{cd}_{ab} + \Xi + C \epsilon^{abdc} R^p{}_{qab} R^q{}_{pcd}, \tag{1}$$

where C is a constant, β_i are arbitrary functions of the scalar ϕ and Ξ is an arbitrary function of the scalar field and its squared gradient, i.e. $\Xi = \Xi(\nabla_a \phi \nabla^a \phi, \phi)$. The first term is a nonminimal coupling between the scalar and the four-dimensional Gauss-Bonnet density, the second includes a nonminimal coupling between the standard kinetic term and the Einstein tensor, the third one is a nonminimal coupling between the field and the Ricci scalar, while the fourth term since it is defined by an arbitrary function might include an additive term which may act as a cosmological term in the action. The last term is just the Pointryagin term

*andres.anabalon@uai.cl
†adolfo.cisterna.r@mail.pucv.cl
‡julio.oliva@uach.cl

 © 2014 American Physical Society

which is a boundary term and does not contribute to the field equations.[1] From here we see that it is possible to have scalar field Lagrangians whose kinetic term has nonminimal derivative couplings with the curvature.

Let us consider kinetic terms H which are quadratic in the derivatives of the field in arbitrary dimension D. Requiring second order energy-momentum tensor as well as linear and second order equation for the field, restricts H to be a linear combination of the following basic terms:

$$H^{(n)} = E^{(n)}_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi, \tag{2}$$

where $E^{(n)}_{\mu\nu}$ is the nth order Lovelock tensors[2]

$$E^{(n)\,\nu}{}_{\mu} = \delta^{\nu\alpha_1\ldots\alpha_{2n}}_{\mu\beta_1\ldots\beta_{2n}} R^{\beta_1\beta_2}{}_{\alpha_1\alpha_2}\ldots R^{\beta_{2n-1}\beta_{2n}}{}_{\alpha_{2n-1}\alpha_{2n}}. \tag{3}$$

The standard kinetic term is recovered with $n = 0$. Since $E^{(1)}_{\mu\nu}$ is proportional to the Einstein tensor, the first nonstandard term in (2) already includes a nonminimal kinetic coupling of the scalar and the curvature.

In this paper we will consider the following action principle:

$$I[g_{\mu\nu},\phi] = \int\sqrt{-g}d^Dx\left[\kappa(R-2\Lambda) - \frac{1}{2}(\alpha g_{\mu\nu} - \eta G_{\mu\nu})\nabla^\mu\phi\nabla^\nu\phi\right], \tag{4}$$

where we have taken the Einstein-Hilbert action with a cosmological term in the gravity sector, while the matter sector is given by a real scalar field with a nonminimal kinetic coupling (here $\kappa := \frac{1}{16\pi G}$). The possible values of the dimensionful parameters α and η will be determined below according with the positivity of the energy density of the matter field.

Numerical solutions in the case when an electromagnetic field is present were found in [7], where phase transitions to charged black holes with complex anisotropic scalar hair were explored. The first exact black hole solution to this system, in the case of a vanishing cosmological term Λ, was found by Rinaldi in [8]. There, the scalar field becomes imaginary in the domain of outer communication, and the weak energy condition (WEC) is violated outside the horizon.

Here we extend the results of [8] and show that the inclusion of a cosmological term in the action makes possible finding a black hole with a real scalar field outside the horizon. By a suitable regularization of the action, we explore the thermal properties of the spherically symmetric black holes. We show that for a region in the space of parameters, there is a phase transition as in general relativity with a negative cosmological constant We also extend the solution to the topological case in arbitrary dimension $D \geq 4$ and show that the cosmological term allows one to obtain a nontrivial solution when $\alpha = 0$. In this case we obtain an asymptotically flat black hole.

The field equations for the metric and the scalar field are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{\alpha}{2\kappa}T^{(1)}_{\mu\nu} + \frac{\eta}{2\kappa}T^{(2)}_{\mu\nu}, \tag{5}$$

$$\nabla_\mu[(\alpha g^{\mu\nu} - \eta G^{\mu\nu})\nabla_\nu\phi] = 0, \tag{6}$$

where[3]

$$T^{(1)}_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\lambda\phi\nabla^\lambda\phi,$$

$$\begin{aligned} T^{(2)}_{\mu\nu} =& \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi R - 2\nabla_\lambda\phi\nabla_{(\mu}\phi R^\lambda_{\nu)} - \nabla^\lambda\phi\nabla^\rho\phi R_{\mu\lambda\nu\rho} \\ &- (\nabla_\mu\nabla^\lambda\phi)(\nabla_\nu\nabla_\lambda\phi) + (\nabla_\mu\nabla_\nu\phi)\Box\phi + \frac{1}{2}G_{\mu\nu}(\nabla\phi)^2 \\ &- g_{\mu\nu}\left[-\frac{1}{2}(\nabla^\lambda\nabla^\rho\phi)(\nabla_\lambda\nabla_\rho\phi) + \frac{1}{2}(\Box\phi)^2 - \nabla_\lambda\phi\nabla_\rho\phi R^{\lambda\rho}\right]. \end{aligned}$$

We will consider a gauge fixed version of the most general, cohomogeneity one, static spacetime (which is not a direct product):

$$ds^2 = -F(r)dt^2 + G(r)dr^2 + r^2 d\Sigma_K^2, \tag{7}$$

where $d\Sigma_K$ is the line element of a closed, $(D-2)$-dimensional Euclidean space of a constant curvature $K = 0, \pm 1$. For $K = 1$, the space Σ_K is locally a sphere, for $K = 0$ it is locally flat and for $K = -1$ it locally reduces to a hyperbolic space. Hereafter we will consider a static and isotropic scalar field, i.e. $\phi = \phi(r)$.

The outline of the paper is as follows: in Sec. II the four-dimensional solution is given for arbitrary K. In order to constraint the couplings, the weak energy condition is imposed on the scalar field. In Sec. III, the geometry of the spherically symmetric solution is described in detail as well as its thermal properties. We compute the temperature, the mass and the entropy following the Hawking-Page approach, and show that the first law is satisfied. We prove that further restrictions on the integration constant come from requiring the black hole to have positive entropy, and show that for a certain region in the space of parameters there is a Hawking-Page phase transition. In Sec. IV, the solution in arbitrary dimension D is given. Finally in Sec. V, the solution in the special case $\alpha = 0$ is analyzed, which gives rise to an asymptotically locally flat black hole.

[1] An interesting application of this Lagrangian in the cosmological setup has been given in [5] where it can be seen that a particular case of Horndeski action provides a novel self-tuning mechanism.

[2] Lovelock tensors are the most general rank-two symmetric tensors which are divergency free and contain up to second order derivatives of the metric [6].

[3] We use a normalized symmetrization $A_{(\mu\nu)} := \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$.

In this paper we use the signature $(-,+,+,+)$, Greek and Latin indices stand for indices in the coordinate basis.

II. FOUR-DIMENSIONAL SOLUTION

For the static ansatz considered here, one has that the equation for the field (6) admits a first integral, which implies the relation

$$r\frac{F'(r)}{F(r)}=\left[K+\frac{\alpha}{\eta}r^2-\frac{C_0}{\eta}\frac{G(r)}{\psi(r)\sqrt{F(r)G(r)}}\right]G(r)-1, \quad (8)$$

where C_0 is an integration constant, $\psi(r) := \phi'(r)$, and $(')$ stands for derivation with respect to r. As in Ref. [8], in order to obtain an exact solution we (arbitrarily) choose $C_0 = 0$. Therefore, Eq. (8) in addition to the tt and rr components of (5), provides a consistent system which for $K = \pm 1$ and $\eta\Lambda \neq \alpha$, has the following solution:

$$F(r) = \frac{r^2}{l^2}+\frac{K}{\alpha}\sqrt{\alpha\eta K}\left(\frac{\alpha+\Lambda\eta}{\alpha-\Lambda\eta}\right)^2\frac{\arctan\left(\frac{\sqrt{\alpha\eta K}}{\eta K}r\right)}{r} -\frac{\mu}{r}+\frac{3\alpha+\Lambda\eta}{\alpha-\eta\Lambda}K, \quad (9)$$

$$G(r) = \frac{\alpha^2((\alpha-\eta\Lambda)r^2+2\eta K)^2}{(\alpha-\eta\Lambda)^2(\alpha r^2+\eta K)^2F(r)}, \quad (10)$$

$$\psi^2(r) = -\frac{2r^2\kappa\alpha^2(\alpha+\eta\Lambda)((\alpha-\eta\Lambda)r^2+2\eta K)^2}{\eta(\alpha-\eta\Lambda)^2(\alpha r^2+\eta K)^3F(r)}, \quad (11)$$

where μ is an integration constant. Here we have defined the effective (A)dS radius l by

$$l^{-2} := \frac{\alpha}{3\eta}. \quad (12)$$

In the case of a locally flat transverse section ($K = 0$) the solution reduces to a topological Schwarzschild solution with locally flat horizons [9]:

$$F(r) = \frac{r^2}{l^2}-\frac{\mu}{r}=\frac{1}{G(r)}, \quad (13)$$

$$\psi(r)^2 = -\frac{2\kappa(\alpha+\eta\Lambda)}{\alpha\eta}\frac{1}{F(r)}, \quad (14)$$

and can be obtained from the one with arbitrary K by taking the formal limit $K \to 0$. Note that in both cases, the solution obtained by replacing $\mu \to -\mu$ is the same as the former if we replace as well $r \to -r$.

As mentioned above, from Eq. (11) one can see explicitly that the sign of $\psi^2(r)$ is determined by the sign of the combination $\frac{(\alpha+\eta\Lambda)}{\alpha\eta}$, and therefore regardless the sign of η/α, the inclusion of a cosmological term in the action allows for the existence of asymptotically (A)dS black holes with a real scalar field in the outer domain of communication.

It can be seen that this solution is asymptotically locally dS or AdS for $\alpha/\eta < 0$ or $\alpha/\eta > 0$, respectively, since when $r \to \infty$ the components of the Riemann tensor go to

$$R^{\mu\nu}{}_{\lambda\sigma} \underset{r\to\infty}{\sim} -\frac{\alpha}{3\eta}\delta^{\mu\nu}_{\lambda\sigma} =: -\frac{1}{l^2}\delta^{\mu\nu}_{\lambda\sigma}, \quad (15)$$

justifying our previous definition of the effective (A)dS radius (12). The asymptotic expansion ($r \to \infty$) of the metric functions is

$$-g_{tt} \underset{r\to\infty}{=} \frac{r^2}{l^2}+\frac{3\alpha+\eta\Lambda}{\alpha-\eta\Lambda}K + \left(\frac{K}{2\alpha}\sqrt{\alpha\eta K}\left(\frac{\alpha+\eta\Lambda}{\alpha-\eta\Lambda}\right)^2\pi\sigma-\mu\right)\frac{1}{r}+O(r^{-2}), \quad (16)$$

$$g^{rr} \underset{r\to\infty}{=} \frac{r^2}{l^2}+\frac{7\alpha+\eta\Lambda}{3(\alpha-\eta\Lambda)}K + \left(\frac{K}{2\alpha}\sqrt{\alpha\eta K}\left(\frac{\alpha+\eta\Lambda}{\alpha-\eta\Lambda}\right)^2\pi\sigma-\mu\right)\frac{1}{r}+O(r^{-2}), \quad (17)$$

where σ is the sign of the combination ηK. For $\mu \neq 0$, there is a curvature singularity at $r = 0$ since for example the Ricci scalar diverges as

$$R \underset{r\to 0}{=} \frac{3(\alpha+\Lambda\eta)(\alpha-\Lambda\eta)^2}{4\eta K\alpha^2}\frac{\mu}{r}+O(1). \quad (18)$$

If the combination $(\alpha+\eta\Lambda)$ vanishes, one can see that the scalar field reduces to a constant and the metric reduces to the topological Schwarzschild-AdS solution of general relativity.

For $(\alpha+\eta\Lambda) \neq 0$, the metric is disconnected from the maximally symmetric AdS vacua, i.e. that in this case it is not possible to set μ to some value such that the metric reduces to AdS. Nevertheless for $\mu = 0$ it is possible to show that all the components of the Riemann tensor $R^{ab}{}_{cd}$ are finite when $r \to 0$. All the algebraic curvature invariants $\mathcal{I}$, i.e. those constructed out from contractions of the Riemann tensor without involving covariant derivatives, can be written as linear combinations of products of the components of the Riemann tensor with the index structure $R^{ab}{}_{cd}$ without involving metric factor. Therefore, the finiteness of the components of the Riemann tensor with this index structure ensures the finiteness of all the algebraic curvature invariants. For $\mu = 0$ the scalar field is finite at the origin, therefore invariants of the form $\phi^p \times \mathcal{I}$ will be finite for arbitrary p. This has important consequences, because it suggests considering the spacetime with $\mu = 0$ as a background to define a regularized Euclidean action. As described below, the solution with $\mu = 0$ actually defines a gravitational soliton.

If $\rho(r)$ is the energy density, then the total energy $\mathcal{E}$ is given by

$$\mathcal{E} = V(\Sigma)\int r^2\sqrt{G(r)}dr\rho(r), \tag{19}$$

where $V(\Sigma)$ stands for the volume of Σ. Therefore

$$T_{00} = \rho(r) := F(r)^{-1}T_{tt}, \tag{20}$$

where $T_{00} = e_0^\mu e_0^\nu T_{tt}$ and e_μ^a is a local frame. Now, the tt component of the energy momentum tensor reads

$$T_{tt} = -\frac{\kappa^2(\alpha+\Lambda\eta)}{\eta} \times F(r)\left[\frac{4K\eta^2(\alpha-\Lambda\eta)^2(\alpha r^2+\eta K)}{\alpha^2((\alpha-\Lambda\eta)r^2+2\eta K)^3}F(r)+1\right]. \tag{21}$$

As expected for a matter action that is quadratic in the derivatives of the field, the positivity of the energy density is in close relation with the reality of the scalar itself, which is determined by the sign of the combination $\frac{(\alpha+\eta\Lambda)}{\alpha\eta}$ as can be seen from Eq. (11). In the following section we analyze in detail the case $K = 1$ and compute the thermal properties of these black holes.

III. SPHERICALLY SYMMETRIC CASE

In the spherically symmetric case, the metric functions and the square of the derivative of the scalar field take the form

$$F(r) = \frac{r^2}{l^2} + \frac{1}{\alpha}\sqrt{\alpha\eta}\left(\frac{\alpha+\Lambda\eta}{\alpha-\Lambda\eta}\right)^2\frac{\arctan\left(\frac{\sqrt{\alpha\eta}}{\eta}r\right)}{r} - \frac{\mu}{r} + \frac{3\alpha+\Lambda\eta}{\alpha-\Lambda\eta}, \tag{22}$$

$$G(r) = \frac{\alpha^2((\alpha-\eta\Lambda)r^2+2\eta)^2}{(\alpha-\eta\Lambda)^2(\alpha r^2+\eta)^2F(r)}, \tag{23}$$

$$\psi^2(r) = -\frac{2r^2\kappa\alpha^2(\alpha+\eta\Lambda)((\alpha-\eta\Lambda)r^2+2\eta)^2}{\eta(\alpha-\eta\Lambda)^2(\alpha r^2+\eta)^3F(r)}. \tag{24}$$

Since the lapse function has a term including the combination $\sqrt{\alpha\eta}$, we can see that α and η need to have the same sign; $l^{-2} := \frac{\alpha}{3\eta}$ will be positive definite and therefore the spacetime is asymptotically locally AdS. Without losing generality, we consider α and η positive, since the solution with both α and η negative is equivalent to the former by changing $\mu \to -\mu$.

The reality of the field in the asymptotic region implies $(\alpha+\eta\Lambda) < 0$, which in our case imposes $\Lambda < -\frac{\alpha}{\eta} < 0$. Under these conditions one can see that the solution describes a black hole with a nondegenerate horizon for $\mu > 0$, which is located at $r = r_+$. Since the horizon is nondegenerate, $F(r_+) = 0$ while $F'(r_+) \neq 0$, therefore close to the horizon, we have

$$\psi(r) \underset{r\to r_+}{\sim} \frac{\zeta}{2\sqrt{r-r_+}} \Rightarrow \phi(r) \underset{r\to r_+}{\sim} \zeta\sqrt{r-r_+},$$

for some constant ζ, which implies that the scalar field vanishes at the horizon and it is nonanalytic there.

If $(\alpha+\eta\Lambda) \neq 0$, it is important to realize that it is not possible to switch off the scalar field and therefore our solution is not continuously connected with a globally maximally symmetric background. Nevertheless it is possible to show, as it occurs in the $\Lambda = 0$ case [8], that within this family, the only regular spacetime is obtained by setting $\mu = 0$. In this case the metric describes an asymptotically AdS gravitational soliton. Near $r = 0$, after a proper rescaling of the time coordinate, the soliton metric reduces to

$$ds^2_{\text{soliton}} \underset{r\to 0}{\sim} -\left(1-\frac{\Lambda}{3}r^2+O(r^4)\right)dt^2 + \left(1-\frac{3\alpha+2\Lambda\eta}{3\eta}r^2+O(r^4)\right)dr^2 + r^2d\Omega^2, \tag{25}$$

therefore one can see explicitly that it has a regular origin.

The thermal version of such a spacetime can be considered as the background metric to obtain a regularized Euclidean action along the lines of Hawking-Page [10], and to obtain therefore the thermodynamics of the black holes. This will be done in the next section.

A. Thermodynamics of the spherically symmetric black holes

The regularized Euclidean action is defined by

$$I_{\text{reg}} = I_E[g_{\mu\nu},\phi] - I_E[g^{(0)}_{\mu\nu},\phi^{(0)}], \tag{26}$$

where I_E and $g^{(0)}_{\mu\nu}$ and $\phi^{(0)}$ are the metric and the scalar field respectively for the gravitational soliton, which are obtained by setting $\mu = 0$ in the black hole solution defined by the functions (22) to (24). Removing the conical singularity at the horizon of the Euclidean black hole solution requires the period of the Euclidean time to be fixed as

$$\beta = 4\pi\sqrt{\frac{G'}{F'}}\bigg|_{r=r_+} = \frac{4\pi\eta(\alpha-\eta\Lambda)r_+}{\alpha(2\eta+(\alpha-\Lambda\eta)r_+^2)}. \tag{27}$$

In order for Eq. (26) to define a regularized action we need to consider a thermal gravitational soliton. If we denote the period of the Euclidean time for the soliton by β_0, we need to impose the redshifted temperatures to match:

$$\beta^2F(r=r_c,\mu) = \beta_0^2F(r=r_c,\mu=0), \tag{28}$$

and then take the limit $r_c \to \infty$ in the regularized action. Since in order to fulfill the weak energy condition Λ must be negative, we can define

$$l_0 := \sqrt{-\frac{3}{\Lambda}}. \tag{29}$$

The divergences in the regularized action cancel and further defining $x_+ := \sqrt{\frac{\alpha}{\eta}} r_+$, the regularized action reduces to

$$I_{\rm reg} = \frac{8\pi^2\kappa}{9}\frac{l^2 x_+}{l_0^2(2l_0^2+(l^2+l_0^2)x_+^2)}[3(l^2-l_0^2)^2\arctan(x_+) + (l^2-2l_0^2)(l^2+l_0^2)x_+^3 + 3(l_0^4-l^4+2l^2l_0^2)x_+]. \tag{30}$$

In terms of l_0 the WEC imposes the following constraint:

$$l^2 > l_0^2, \tag{31}$$

this is, the AdS length of the asymptotic region (l) has to be larger than the AdS length defined by the cosmological term in the action (l_0) given by (29).

As follows from the convention we are using defined in [10], in the canonical ensemble we relate the Euclidean action to the free energy by $I_{\rm reg} = \beta F$, therefore

$$M = \frac{\partial I_{\rm reg}}{\partial \beta} \quad \text{and} \quad S = \beta\frac{\partial I_{\rm reg}}{\partial \beta} - I_{\rm reg}. \tag{32}$$

Using our definition of the AdS radius l as well as the definition of l_0 given in (29), the temperature is given by

$$T = \frac{\sqrt{3}x_+}{4\pi l} + \frac{\sqrt{3}l_0^2}{2\pi l(l_0^2+l^2)x_+}, \tag{33}$$

while the mass reads

$$\begin{aligned} M = {} & \frac{2}{3^{3/2}}\frac{\kappa\pi l}{l_0^2(1+x_+^2)(l^2+l_0^2)((l^2+l_0^2)x_+^2-2l_0^2)} \\ & \times [3(1+x_+^2)(l^2-l_0^2)^2((l^2+l_0^2)x_+^2-2l_0^2)\arctan(x_+) \\ & - 2(l^2-2l_0^2)(l^2+l_0^2)^2x_+^7 \\ & - 2(l^2+5l_0^2)(l^2-2l_0^2)(l^2+l_0^2)x_+^5 \\ & + (l_0^6+7l^4l_0^2-13l^2l_0^4-3l^6)x_+^3 \\ & + 6l_0^2(l^2-3l_0^2)(l^2+l_0^2)x_+] \end{aligned}$$

and the entropy reduces to

$$S = \frac{8\pi^2 l^2\kappa x_+^2}{3l_0^2}\left[\frac{(l^2+l_0^2)(l^2-2l_0^2)x_+^4 + l_0^2(l^2-l_0^2)x_+^2 + 2l_0^4}{(1+x_+^2)(2l_0^2-(l_0^2+l^2)x_+^2)}\right]. \tag{34}$$

One can further check that with these expressions the first law of black hole thermodynamics,

$$dM = TdS, \tag{35}$$

is fulfilled.

For any positive x_+, there is a horizon and therefore from Eq. (34) one sees that requiring S to be positive might induce some restrictions on the couplings and x_+. Let us define the constant

$$\xi := \frac{l^2}{l_0^2} - 1, \tag{36}$$

which must be strictly positive if we want to have a nontrivial real scalar field outside of the horizon. The region in the plane ξ vs x_+ in which the entropy is positive is depicted in Fig. 1. We see that for $\xi > 1$ there is an upper bound on the radii of the black holes with positive entropy. For $\xi < 1$, there is a gap on the possible radii of the black holes with positive entropy.

As it occurs for Schwarzschild-AdS in vacuum, the expression for the temperature given in (33) has a minimum at

$$x_+ = x_0 := \frac{2l_0}{\sqrt{2l_0^2+l^2}}, \tag{37}$$

where the temperature takes its minimum value

$$T_0 := \frac{\sqrt{3}l_0}{\pi l\sqrt{2l_0^2+2l^2}}. \tag{38}$$

Therefore there are no black holes with $T < T_0$, while for a given temperature $T > T_0$ there are two possible black holes.

In the canonical ensemble the most probable configuration is the one with the lowest Helmholtz free energy F. Figure 2 shows the four possible behaviors of the free energy in terms of the temperature divided by the minimum temperature (T/T_0) for the large (continuous line) and small (dashed line) black holes.

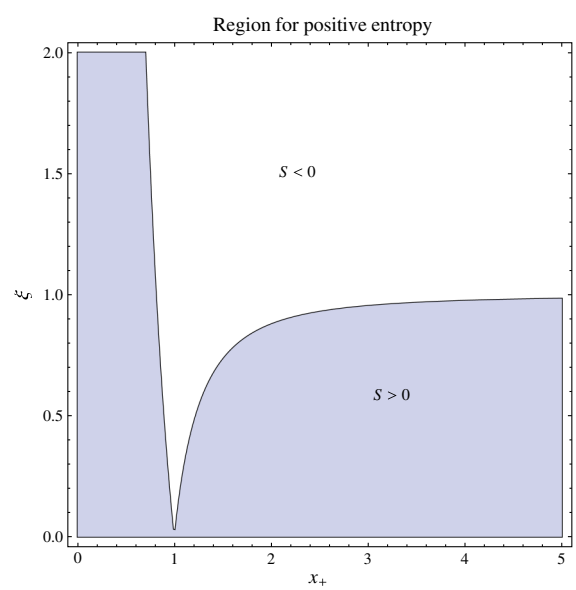

FIG. 1 (color online). The grey region corresponds to the region with positive entropy in the plane $\xi := \frac{l^2}{l_0^2} - 1$ vs x_+, while the white region stands for negative entropy. For $\xi > 1$, requiring $S > 0$ implies an upper bound on the black holes radii while for $0 < \xi < 1$ there is a gap on the possible radii of black holes with positive entropy.

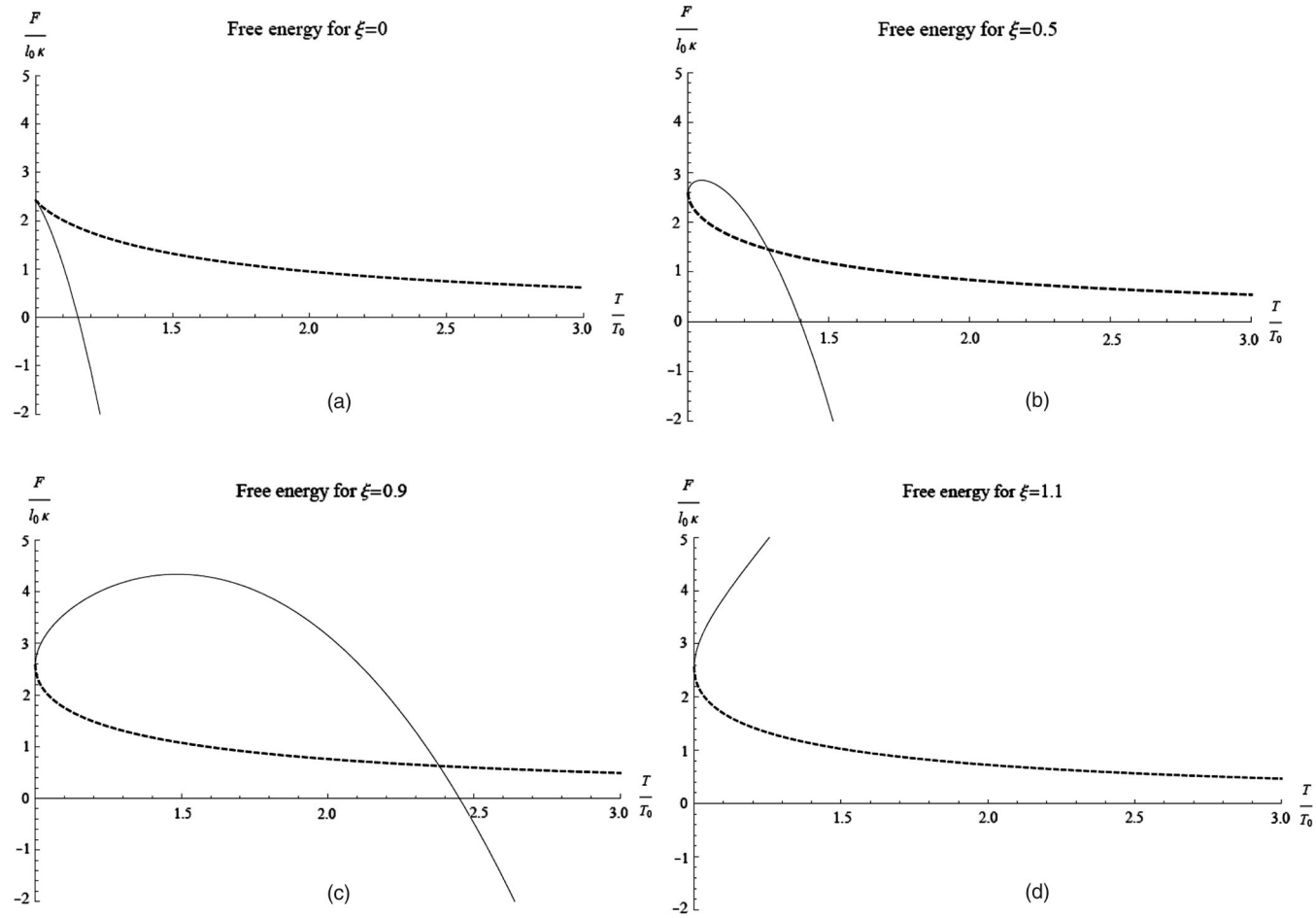

FIG. 2. The free energy for large (continuous) and small (dashed) black holes in terms of $\frac{T}{T_0}$, for $\xi = 0$ (a), $\xi = 0.5$ (b), $\xi = 0.9$ (c) and $\xi = 1.1$ (d).

For $\xi = 0$, the metric reduces to Schwarzschild-AdS; therefore the situation is like the one described by Hawking and Page in [10]. Small black holes always have positive free energy, while for large black holes there is a critical temperature above which the free energy becomes negative and the thermal AdS (the thermal soliton with $\xi = 0$) is less probable than the black hole. For $0 < \xi < 1$ Figs. 2(b) and 2(c) show that small black holes always have positive free energy and therefore they are less probable than the thermal soliton, nevertheless there is a range of temperatures for which the small black holes are more probable than large black holes. Above such temperature (which fulfills a transcendental equation in terms of ξ) there is a range in which the situation is like in GR in vacuum and therefore large black holes suffer a phase transition at some critical temperature above which the thermal soliton would tend to tunnel to black hole configurations. For $\xi \geq 1$, both the small and large black holes have positive free energy; therefore both are less probable than the corresponding thermal soliton, nevertheless large black holes are less probable than small black holes, again in opposition to what occurs in GR in vacuum.

IV. EXTENDING THE SOLUTION TO ARBITRARY DIMENSIONS D

If one considers the theory defined in the action principle (4) in arbitrary dimensions, one can see that the metric

$$ds^2 = -F(r)dt^2 + G(r)dr^2 + r^2 d\Sigma^2_{K,D-2} \tag{39}$$

defines a solution with the following metric functions:

$$F_D(r) = -\frac{\mu}{r^{D-3}} + \left[(D-2)K\eta - \frac{2\eta\Lambda}{(D-1)}r^2 + \frac{(\alpha+\eta\Lambda)^2}{(D-3)(D-2)(D+1)K\eta} r^4 {}_2F_1\left(1, \frac{D+1}{2}, \frac{D+3}{2}, -\frac{2\alpha}{(D-3)(D-2)K\eta}r^2\right)\right], \tag{40}$$

$$G_D(r) = \frac{A(r)}{B(r)}\frac{1}{F_D(r)}, \tag{41}$$

$$\psi_D^2(r) = -\frac{4\kappa(\Lambda\eta + \alpha)r^2}{\eta(2\alpha r^2 + \eta(D-2)(D-3)K)} G_D(r).$$

Here we have defined

$$A(r) := \Bigg[K^2\eta^3(D+3)(D-3)^2(D-2)^2((D-2)(D-3)K-2r^2\Lambda) + r^2(\alpha+\eta\Lambda)^2(D-2)(K\eta(D-2)(D-3)(D+3)\,{}_2F_1\left(1,\frac{D+1}{2},\frac{D+3}{2},-\frac{2\alpha r^2}{(D-3)(D-2)K\eta}\right) - 4\alpha\,{}_2F_1\left(2,\frac{D+3}{2},\frac{D+5}{2},-\frac{2\alpha r^2}{(D-3)(D-2)K\eta}\right)\Bigg)\Bigg],$$

$$B(r) := (D+3)\eta K^2(D-2)(D-3)^2((D-3)(D-2)K\eta+2\alpha r^2).$$

The constant μ is the only integration constant of the solution and ${}_2F_1$ is the hypergeometric function. In order to obtain this solution we had set to zero the integration constant appearing in the first integral of the equation of the field as in four dimensions. Let us define the dimensionless coordinate ρ by

$$\rho := \sqrt{\frac{2\alpha}{(D-3)(D-2)K\eta}}\,r. \tag{42}$$

The hypergeometric function appearing in Eq. (40) can be rewritten in terms of elementary functions in the following manner. For even dimensions $D = 2n$, we have

$${}_2F_1\left(1, n+\frac{1}{2}, n+\frac{3}{2}, -\rho^2\right) = (-1)^n(2n+1)\rho^{-2n-2}\left[\rho\arctan\rho + \sum_{j=1}^{n}\frac{(-1)^j}{2j-1}\rho^{2j}\right], \tag{43}$$

while for odd dimensions $D = 2n+1$, we use

$${}_2F_1(1, n+1, n+2, -\rho^2) = (-1)^n(n+1)\rho^{-2n-2}\left[\ln(\rho^2+1) + \sum_{j=1}^{n}\frac{(-1)^j}{j}\rho^{2j}\right]. \tag{44}$$

As in four dimensions, it can be seen that these metrics describe asymptotically locally AdS black holes with real scalar fields in the domain of outer communication.

V. ASYMPTOTICALLY FLAT BLACK HOLES SUPPORTED BY THE EINSTEIN-KINETIC COUPLING

Here we will show that the inclusion of a cosmological term in the action allows finding a new asymptotically locally flat black hole. In this case we will consider that the matter sector is given only by the kinetic term of the scalar field which is constructed with the Einstein tensor, namely we shall consider the action (4) with $\alpha = 0$. Thus, the action reduces to

$$I[g_{\mu\nu},\phi] = \int\sqrt{-g}d^4x\left[\kappa(R-2\Lambda) + \frac{\eta}{2}G_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi\right]. \tag{45}$$

In the same manner as before, the equation for the field allows a first integral which brings in an integration constant. If we set to zero such integration constant, then we find that for $K \neq 0$, the following metric defines a solution of the system:

$$ds^2 := -H(r)dt^2 + \frac{15(\Lambda r^2-2K)^2}{K}\frac{dr^2}{H(r)} + r^2 d\Sigma^2_{K,2}, \tag{46}$$

provided

$$H(r) := (60K^2 - 20\Lambda K r^2 + 3\Lambda^2 r^4) - \frac{\mu}{r}, \tag{47}$$

and the derivative of the scalar field is given by

$$\psi_0^2(r) := -\frac{30\kappa\Lambda r^2(\Lambda r^2-2K)^2}{\eta K^2 H(r)}. \tag{48}$$

The following comments are in order:

(i) As in the previous cases, at a possible nondegenerate horizon $r = r_+$ the lapse function $H(r_+)$ vanishes, and therefore the scalar field vanishes but it is nonanalytic at that point.

(ii) For vanishing Λ, the scalar field itself vanishes and the solution reduces to the topological Schwarzschild solution in flat space, which represents a black hole only in the spherically symmetric case ($K = 1$).

(iii) This solution is asymptotically locally flat since

$$R^{\mu\nu}{}_{\lambda\rho} \underset{r\to\infty}{\sim} 0. \tag{49}$$

(iv) In order to have a real scalar field in the region where $H(r)$ is positive, i.e. outside a possible event horizon, we need to impose $\Lambda/\eta < 0$.

(v) For nonvanishing μ, the metric has a singularity at the origin.

(vi) When K and Λ have the same sign, there is a curvature singularity at a finite radius $r = r_s := \sqrt{2K/\Lambda}$, for any value of μ.

(vii) For $\mu = 0$, the point $r = 0$ is a symmetric center and therefore the range of the radial coordinate is $r \geq 0$. If even more in this case $K/\Lambda < 0$ the metric is regular everywhere and therefore describes an asymptotically locally flat gravitational soliton.

(viii) When $K/\Lambda < 0$ the singularity at $r = r_s$ disappears and the singularity at the origin is surrounded by an event horizon provided $\mu > 0$. In this case the solution represents an asymptotically locally flat black hole.

(ix) For $K/\Lambda > 0$ the singularity at $r = r_s > 0$ is surrounded by an event horizon provided $\mu > \mu_c$ for a certain critical value of μ. In this case as well, the spacetime represents an asymptotically locally flat black hole.

The case $K = 0$ (and $\alpha = 0$) integrates in a different manner. As before, in order to have a vanishing integration constant from the first integral of the equation of the field, in addition to the field equations we need to impose $G^{rr} = 0$. The field equations therefore imply that Λ has to vanish and the metric and the derivative of the scalar field $[\psi_0(r) = \phi'(r)]$ are given by

$$ds^2 = -\frac{C}{r}dt^2 + \frac{dr^2}{G(r)} + r^2(dx^2 + dy^2), \tag{50}$$

$$\psi(r)^2 = -\frac{4\kappa}{\eta}G(r) + J\frac{G(r)^{3/2}}{\sqrt{r}}, \tag{51}$$

where C and J are integration constants and $G(r)$ is an arbitrary function. This is a degenerate case in which the system is underdetermined since one of the metric functions is completely arbitrary. Note that the function $G(r)$ cannot be absorbed by a diffeomorphism.

VI. DISCUSSION

We have found a new family of asymptotically AdS and locally flat black holes supported by scalar field in arbitrary dimensions. The theory we considered is a particular case of Horndeski theory and therefore the field equations and the energy-momentum tensor are of second order. For the ansatz considered, the equation for the field allows for a first integration giving rise to an integration constant. Following the steps of Rinaldi's work [8] we were able to obtain an exact solution imposing such integration constant to vanish. This imposes an extra constraint in the geometry that turns out to open a new family of nontrivial solutions. The inclusion of a cosmological term in the action allowed us to find a solution with a real scalar field outside the horizon, and allowed us to find asymptotically locally flat black hole solutions as well. It is important to note that the cosmological constant at infinity is not given by the cosmological Λ term in the action but in terms of the couplings α and η that appear in the kinetic term of the field [see (12)].

The solutions are not continuously connected with the maximally symmetric AdS or flat backgrounds since the scalar field cannot be turned off by setting the single integration constant to some value. Nevertheless, since our family of metrics contains a further integration constant, it is possible to show that within such a family there is a unique regular spacetime. Such spacetime is a gravitational soliton and we use it in the four dimensional, spherically symmetric case to defined a regularized Euclidean action and explore the thermodynamics of the black hole solution. A similar situation occurs with the AdS soliton, which can be considered as the background for some topological AdS black holes, as well as in gravity in $2 + 1$ with scalar fields, where the gravitational solitons are the right backgrounds to consider to give a microscopic description of the black hole entropies [11–13].

The thermodynamics of the asymptotically AdS black holes depends strongly on the ratio between the AdS length of the asymptotic region l which is determined by the couplings of the scalar and the "bared" AdS length l_0 constructed in terms of the cosmological term in the action Λ. We have shown that in a certain region of the space of parameters, the thermal soliton will tend to tunnel to large black holes. In opposition to what occurs in general relativity, there is also a range of temperatures for which small black holes are more probable than large black holes.

Even though it is probably not possible to give an exact, closed form for the general static black hole solution of Horndeski theory, there are some particular subsets of theories that might offer the possibility of finding exact solutions. For example, in dimensions higher than four, since the Lovelock tensors are nontrivial, it would be interesting to look for black holes with a scalar field in which the kinetic couplings are constructed with such Lovelock tensors. In some of those theories, the kinetic term will be constructed with the field equations of a gravity theory that has an extra symmetry [14], and this might help in the integration of the field equations. Other theories that belong to the Horndeski family are theories with invariance under local Weyl rescaling. In Ref. [15] it was shown that it is possible to construct theories that generalize the usual conformally coupled scalar field, including couplings of the scalar with Euler densities of higher degree [e.g. $\phi^4(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})$]. This was done by introducing a four-rank tensor which contains the curvature and derivatives of the field and transforms covariantly under local Weyl rescaling.[4] In

[4]In four dimensions a different approach for constructing theories with nonminimal coupling has been recently given in [16], allowing two scalar fields as well.

[15] it was shown that restricting to Lovelock-like combinations in such four-rank tensor ensures the field equations and energy-momentum tensor to be of second order and therefore those combinations are a subclass of Horndeski's theories, which in this case have an extra local symmetry that might be useful in the integration of the field equations.

ACKNOWLEDGMENTS

We thank Fabrizio Canfora, Francisco Correa, Alex Giacomini, Gaston Giribet, Nicolas Grandi, Mokhtar Hassaine and Sourya Ray for many useful conversations. We would like to thank as well the referee of this manuscript for a thorough revision and useful comments. Research of A. A. is supported in part by the FONDECYT Grant No. 11121187 and by the CNRS project "Solutions exactes en présence de champ scalaire". A. C. thanks the support of Beca CONICYT de Doctorado Nacional 2010. J. O. is supported in part by FONDECYT Grant No. 1141073. A. C. and J. O. appreciate the hospitality of the International Center of Theoretical Physics (ICTP) where part of this work was done.

Note added.—Recently, the article [17] appeared on arXiv which has some overlap with our four-dimensional solution.

[1] T. Damour, *Gravitation and Experiment*, Progress in Mathematical Physics (Birkhäuser Basel, Basel, 2007), Vol 52, pp. 1–49.
[2] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).
[3] A. Nicolis, R. Rattazzi, and E. Trincherini, Phys. Rev. D **79**, 064036 (2009).
[4] G. W. Horndeski, Int. J. Theor. Phys. **10**, 363 (1974).
[5] C. Charmousis, E. J. Copeland, A. Padilla, and P. M. Saffin, Phys. Rev. Lett. **108**, 051101 (2012).
[6] D. Lovelock, J. Math. Phys. (N.Y.) **12**, 498 (1971).
[7] T. Kolyvaris, G. Koutsoumbas, E. Papantonopoulos, and G. Siopsis, Classical Quantum Gravity **29**, 205011 (2012).
[8] M. Rinaldi, Phys. Rev. D **86**, 084048 (2012).
[9] J. P. S. Lemos, Phys. Lett. B **353**, 46 (1995).
[10] S. W. Hawking and D. N. Page, Commun. Math. Phys. **87**, 577 (1983).
[11] F. Correa, C. Martinez, and R. Troncoso, J. High Energy Phys. 01 (2011) 034.
[12] F. Correa, C. Martinez, and R. Troncoso, J. High Energy Phys. 02 (2012) 136.
[13] F. Correa, A. Faundez, and C. Martinez, Phys. Rev. D **87**, 027502 (2013).
[14] J. Zanelli, Classical Quantum Gravity **29**, 133001 (2012).
[15] J. Oliva and S. Ray, Classical Quantum Gravity **29**, 205008 (2012).
[16] A. Padilla, D. Stefanyszyn, and M. Tsoukalas, Phys. Rev. D **89**, 065009 (2014).
[17] E. Babichev and C. Charmousis, arXiv:1312.3204.