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Abstract

A Lagrangian formalism is used to study the motion of a spinning massive particle in Friedmann–Robertson–Walker and Gödel spacetimes, as well as in a general Schwarzschild-like spacetime and in static spherically symmetric conformally flat spacetimes. Exact solutions for the motion of the particle and general exact expressions for the momenta and velocities are displayed for different cases. In particular, the solution for the motion in spherically symmetric metrics is presented in the equatorial plane. The exact solutions are found using constants of motion of the particle, namely its mass, its spin, its angular momentum, and a fourth constant, which is its energy when the metric is time-independent, and a different constant otherwise. These constants are associated to Killing vectors. In the case of the motion on the Friedmann–Robertson–Walker metric, a new constant of motion is found. This is the fourth constant which generalizes previously known results obtained for spinless particles. In the case of general Schwarzschild-like spacetimes, our results allow for the exploration of the case of the Reissner–Nordstrom–(Anti)de Sitter metric. Finally, for the case of the conformally flat spacetimes, the solution is explicitly evaluated for different metric tensors associated to a universe filled with static perfect fluids and electromagnetic radiation. For some combination

of the values of the constants of motion the particle trajectories may exhibit spacelike velocity vectors in portions of the trajectories.

1. Introduction

Relativistic spinless test particles follow geodesics according to the equivalence principle. On the other hand, it is known that spinning massive test particles (tops) follow non geodesic paths when moving on gravitational fields [1–5]. The pioneering works of Mathisson [4] and Papapetrou [5] showed that the equations of motion for tops are non geodesic, deriving them as limiting cases of rotating fluids moving in gravitational fields. On the contrary, massless spinning particles (such as photons) do follow null geodesics as showed by Mashhoon [6] who used the Mathisson–Papapetrou formalism for his derivation. Thus, one can argue that the equivalence principle (interpreted as stating that test particles in a gravitational field follow geodesics) is valid only for spinless point test particles. Extended particles are, in general, subject to tidal forces and follow, therefore, non geodesic paths.

The rigorous derivation for the non geodesic equations of motion obeyed by tops moving on a gravitational background can be obtained using a Lagrangian formalism. The first derivation was obtained by Hojman [1, 2] (using a flat spacetime formalism developed by Hanson and Regge [7]). In this Lagrangian formulation, the velocity u^μ and the canonical momentum P^μ vectors are, in general, not parallel. For the motion of tops in the electromagnetic and/or gravitational fields, the square of the mass $m^2 (\equiv P^\mu P_\mu > 0)$ is conserved implying that the momentum vector remains timelike along the motion. However, the velocity vector may become spacelike [1, 7–9]. It is worth mentioning that the spin (the other Casimir function of the Poincaré group) $J^2 (\equiv (1/2) S^{\mu\nu} S_{\mu\nu})$ is also conserved for a top moving on any curved background. A proper treatment of the lack of parallelism between velocity and momentum is best achieved with a Lagrangian formulation of the motion of tops, because otherwise the canonical momentum cannot be appropriately defined. Besides, while the Mathisson–Papapetrou formulation gives rise to third-order equations of motion, the Lagrangian approach gives rise to second-order ones [1, 10]. It is important to stress that the treatment presented here is a pole–dipole model of a particle.

Recently, the interest for the motion of tops in curved spacetimes has staged a comeback [11–17]. In particular, in a previous work, the motion of a top on a Schwarzschild background was studied in detail [14]. It was found that the equations of motion can be solved exactly, and that the spin of the particle modifies the motion significantly as compared to the spinless particle geodesic motion. Furthermore, this formalism has been used to show that photons must be massless [15].

In the same spirit, here we exhibit exact solutions for the equations describing the motion of a top in cosmological spacetimes, as well as in general static spherically symmetric spacetimes. The cosmological models studied are the Friedmann–Robertson–Walker (FRW) and the Gödel spacetimes. The motion of a massive spinning particle is exactly solved for motion in the equatorial plane on the FRW metric. It is important to stress that, in this case, we find a new conserved quantity which cannot be expressed in terms of the Killing vectors of the metric. This constant is a generalization of a well-known constant for spinless particles. For Gödel

spacetimes we solve completely the motion of the spinning particle in the plane $z = 0$. On the other hand, we study a general Schwarzschild-like spacetime, showing that it is possible to find exact solutions of the motion generalizing the results of [14]. This is shown for the concrete case of the Reissner–Nordstrom–(Anti)de Sitter metric. Finally, a static spherically symmetric conformally flat spacetime is studied. To show the dependence of the exact solution for the orbits of the top on the conformal factor of the metric, we explore the conformally flat spacetime of a universe composed by radiation and a static perfect fluid with a cosmological equation of state. We exhibit the exact solution for the cases of a universe filled with uniform electromagnetic radiation only, a matter-dominated universe, a radiation-dominated universe, and a universe filled with dark energy.

In general, the motion of the top in every metric is described in terms of the constants of motion of the particle: its mass, its spin, its angular momentum, and its energy. The latter are constructed from the symmetries (Killing vectors) of the metric tensor. Also, for all the models, general exact expressions for the momenta and velocities are shown.

In addition to the general exact solutions for the motion of the spinning massive particle, an interesting consequence of the top's spin is highlighted. We show that, in these metrics, massive tops described by this theory may reach spacelike velocities in portion of the trajectories. This can be achieved if their constants of motion satisfy certain relations. This kind of behavior is extensive to almost all the metric studied in this work, and it seems to be a robust effect of the motion of a spinning massive particle. Similar results were obtained for a trajectory of a top on a Schwarzschild spacetime [14]. This remarkable outcome is, nevertheless, not uncommon. Theoretical results involving superluminal propagation of massive spinning particles and fields in interaction with electromagnetic or gravitational fields have been previously reported in the literature by Velo and Zwanziger [8], Hanson and Regge [7], Hojman [1] and Hojman and Regge [9]. On the other hand, although some experiments have reported hints of superluminal group velocity in optical fibers [18], there is no solid experimental evidence of superluminal neutrino propagation [19–21][6].

The paper is organized as follows: in section 2 we introduce the Lagrangian theory for the motion of the top. In section 3, we exhibit the exact solution to the equations of motion for the FRW metric. In section 4 we show the exact result for the top's motion in a Gödel spacetime. Later we study the Schwarzschild-like metric in section 5, in conjunction with the Reissner–Nordstrom–(Anti)de Sitter metric. In section 6, we study the motion of a top in a spherical symmetric conformally flat spacetime with the cases of a universe filled with a static perfect fluid and radiation. Finally, in section 7, conclusions are presented.

2. Lagrangian theory for tops in gravitational fields

The theory of a spinning massive particle in a curved spacetime was developed in [1, 2] and reviewed in [14]. In this section, we present a brief summary of the motion of a top on a gravitational field. For a detailed description, we refer the reader to the previously mentioned articles.

Let us denote the position of the relativistic (spherical) top by a four vector x^μ, while its orientation is defined by an orthonormal tetrad $e_{(\alpha)}{}^\mu$. A gravitational field is described as usual in terms of the metric field $g_{\mu\nu}$ [1, 2]. The tetrad vectors satisfy $g_{\mu\nu} e_{(\alpha)}{}^\mu e_{(\beta)}{}^\nu \equiv \eta_{(\alpha\beta)}$, with $\eta_{(\alpha\beta)} \equiv \text{diag}(+1, -1, -1, -1) = \eta^{(\alpha\beta)}$, and have, therefore, six independent components. The velocity vector u^μ is defined in terms of an arbitrary parameter λ by

$$u^\mu \equiv \frac{dx^\mu}{d\lambda}. \tag{1}$$

The antisymmetric angular velocity tensor $\sigma^{\mu\nu}$ is

$$\sigma^{\mu\nu} \equiv \eta^{(\alpha\beta)} e_{(\alpha)}{}^\mu \frac{De_{(\beta)}{}^\nu}{D\lambda} = -\sigma^{\nu\mu}, \tag{2}$$

where the covariant derivative $De_{(\beta)}{}^\nu/D\lambda$ is defined in terms of the Christoffel symbols $\Gamma^\nu{}_{\rho\tau}$, as usual, by

$$\frac{De_{(\beta)}{}^\nu}{D\lambda} \equiv \frac{de_{(\beta)}{}^\nu}{d\lambda} + \Gamma^\nu{}_{\rho\tau} e_{(\beta)}{}^\rho u^\tau. \tag{3}$$

The general covariance is achieved unambiguously at the level of the Lagrangian formulation [1] due to the fact that only first derivatives of the dynamical variables are used in its construction. If no Lagrangian theory for a system of special relativistic equations of motion is known, the introduction of gravitational interactions cannot be unambiguously implemented.

A possible Lagrangian $L = L(a_1, a_2, a_3, a_4)$ is constructed as an arbitrary function of four invariants $a_1 \equiv u^\mu u_\mu$, $a_2 \equiv \sigma^{\mu\nu}\sigma_{\mu\nu} = -\text{tr}(\sigma^2)$, $a_3 \equiv u_\alpha \sigma^{\alpha\beta}\sigma_{\beta\gamma}u^\gamma$ and $a_4 \equiv \det(\sigma)$,

$$L(a_1, a_2, a_3, a_4) = (a_1)^{1/2} \mathcal{L}(a_2/a_1, a_3/(a_1)^2, a_4/(a_1)^2), \tag{4}$$

such that the action $S = \int L \, d\lambda$, be λ-reparametrization invariant (the speed of light c is set equal to 1). $\mathcal{L}$ is an arbitrary function of three variables. Note that, unlike the spinless case, it is not necessary that a_1 be positive to have a real Lagrangian (see [14] for an extended discussion about this issue).

The conjugated momentum vector P_μ and antisymmetric spin tensor $S_{\mu\nu}$ are defined by

$$P_\mu \equiv \frac{\partial L}{\partial u^\mu}, \quad S_{\mu\nu} \equiv \frac{\partial L}{\partial \sigma^{\mu\nu}} = -S_{\nu\mu}. \tag{5}$$

As usual, the equations of motion are obtained by considering the variation of the action S with respect to (ten) independent variations δx^μ and (the covariant generalization of) $\delta\theta^{\mu\nu} \equiv \eta^{(\alpha\beta)} e_{(\alpha)}{}^\mu \delta e_{(\beta)}{}^\nu = -\delta\theta^{\nu\mu}$. The final equations of motion turn out to be non geodesic [1, 2]

$$\frac{DP^\mu}{D\lambda} = -\frac{1}{2} R^\mu{}_{\nu\alpha\beta} u^\nu S^{\alpha\beta}, \tag{6}$$

and

$$\frac{DS^{\mu\nu}}{D\lambda} = S^{\mu\lambda}\sigma_\lambda{}^\nu - \sigma^{\mu\lambda}S_\lambda{}^\nu = P^\mu u^\nu - u^\mu P^\nu. \tag{7}$$

These results hold for arbitrary $\mathcal{L}$. The dynamical variables P^μ and $S^{\mu\nu}$ are interpreted as the ten generators of the Poincaré group.

In order to restrict the spin tensor to generate rotations only, the Tulczyjew constraint [22] is considered [1, 7]

$$S^{\mu\nu}P_\nu = 0. \tag{8}$$

However, we would like to stress that, for a suitably chosen $\mathcal{L}$, this constraint is a consequence of the theory, i.e., the Tulczyjew constraint is derived from this Lagrangian formalism (for details, see [14]). This is one of the strengths of this Lagrangian theory.

It can also be proved [14] that both the top mass m and its spin J are conserved quantities

$$m^2 \equiv P^\mu P_\mu, \tag{9}$$

$$J^2 \equiv \tfrac{1}{2} S^{\mu\nu}S_{\mu\nu}. \tag{10}$$

Finally, a conserved quantity C_ξ given by

$$C_\xi \equiv P^\mu \xi_\mu - \tfrac{1}{2} S^{\mu\nu} \xi_{\mu;\nu}, \tag{11}$$

can be associated to any Killing vector ξ_μ of the metric

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0. \tag{12}$$

As the general theory is established, we proceed now in the following sections to find the exact solutions for the motion of a top in cosmological spacetimes using the FRW and the Gödel metrics. We also study the motion in general static spherically symmetric spacetimes with Schwarzschild-like and conformally flat metrics.

3. Exact solution for cosmological Friedmann–Robertson–Walker spacetimes

Let us consider as a first case the FRW metric, which is given by the following line element in spherical coordinates

$$ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu = c^2 \, dt^2 - a(t)^2 g(r) \, dr^2 - a(t)^2 r^2 (d\theta^2 + \sin^2\theta \, d\phi^2), \tag{13}$$

where r is the radial distance, θ and ϕ are the polar and azimuthal angles respectively, and $g(r) = 1/(1 - kr^2)$. Here $a(t) \equiv a$ is the time-dependent scale factor of the universe, whereas k assumes three possible values $k = -1, 0, 1$, denoting a universe with negative spatial curvature, spatially flat or with positive spatial curvature respectively. From now on, we explicitly display the speed of light c in our calculations.

Before solving the equations of motion for the top, it is useful to exhibit the three Killing vectors of the FRW metric explicitly

$$\begin{aligned}
\xi_\mu^0 &= (0, 0, 0, -a^2 r^2 \sin^2\theta), \\
\xi_\mu^1 &= (0, 0, a^2 r^2 \sin\phi, a^2 r^2 \sin\theta \cos\theta \cos\phi), \\
\xi_\mu^2 &= (0, 0, -a^2 r^2 \cos\phi, a^2 r^2 \sin\theta \cos\theta \sin\phi).
\end{aligned} \tag{14}$$

Using these Killing vectors, we can solve the equations of motion (6) and (7) in general. In what follows, we study the motion in the equatorial plane $\theta = \pi/2$, which is defined to be orthogonal to the conserved angular momentum vector. This choice simplifies the analysis because, in this case, $S^{r\theta} = S^{t\theta} = S^{\phi\theta} = 0$, and also $\dot\theta = 0 = P^\theta = \dot{P}^\theta$. Thus, the top remains in the plane orthogonal to the total angular momentum if it was initially there.

We will make use of constants of motion to find the top's trajectory in a FRW spacetime. The mass m^2 and spin J^2 are always conserved in any spacetime metric. The formalism provides three constants of motion, which can be found using the three (angular momentum) Killing vectors. Two of the three components of the angular momentum vector have been used to define the equatorial plane. The third component j is the angular momentum component perpendicular to the equatorial plane. The problem can be completely solved if we find a fourth constant of motion. In this case, the energy of the top motion is not conserved because the metric is time-dependent. Therefore, to find the last constant we have to integrate the equations of motion.

From the constants of motion (9) and (10) we find

$$m^2 c^2 = c^2 (P^t)^2 - a^2 [g (P^r)^2 + r^2 (P^\phi)^2], \tag{15}$$

$$J^2 = a^2 g (a^2 r^2 (S^{r\phi})^2 - c^2 (S^{tr})^2) - a^2 c^2 r^2 (S^{t\phi})^2, \tag{16}$$

where we can identify J with the top's spin. Making use of the Killing vectors, we find another constant which corresponds to the conserved angular momentum component orthogonal to the orbital plane

$$j = -ar[r\dot{a}S^{t\phi} + a(rP^{\phi} + S^{r\phi})], \tag{17}$$

where $\dot{a} \equiv \mathrm{d}a/\mathrm{d}t$.

To find the fourth constant, we first display the Tulczyjew contraints (8)

$$a^2 r^2 P^{\phi} S^{t\phi} + a^2 g P^r S^{tr} = 0, \tag{18}$$

$$c^2 P^t S^{tr} + a^2 r^2 P^{\phi} S^{r\phi} = 0, \tag{19}$$

$$a^2 g P^r S^{r\phi} - c^2 P^t S^{t\phi} = 0. \tag{20}$$

The explicit form of the equations of motion can be obtained form (6) and (7). The equations of motion for the momentum (6) are

$$\frac{ag\dot{r}\ddot{a}S^{tr}}{c^2} + \frac{ar^2\dot{\phi}\ddot{a}S^{t\phi}}{c^2} + \frac{ag\dot{r}\dot{a}P^r}{c^2} + \frac{ar^2\dot{\phi}\dot{a}P^{\phi}}{c^2} + \dot{P}^t = 0, \tag{21}$$

$$\frac{\ddot{a}S^{tr}}{a} + \frac{r^2\dot{\phi}\dot{a}^2 S^{r\phi}}{c^2} + \frac{\dot{a}P^r}{a} + \frac{\dot{r}\dot{a}P^t}{a} + \frac{\dot{r}g'P^r}{2g} + \frac{r\dot{\phi}g'S^{r\phi}}{2g^2} - \frac{r\dot{\phi}P^{\phi}}{g} + \dot{P}^r = 0, \tag{22}$$

$$\frac{\ddot{a}S^{t\phi}}{a} - \frac{g\dot{r}\dot{a}^2 S^{r\phi}}{c^2} + \frac{\dot{\phi}\dot{a}P^t}{a} + \frac{\dot{a}P^{\phi}}{a} - \frac{\dot{r}g'S^{r\phi}}{2gr} + \frac{\dot{\phi}P^r}{r} + \frac{\dot{r}P^{\phi}}{r} + \dot{P}^{\phi} = 0, \tag{23}$$

where the symbol $'$ denotes derivatives with respect to r. On the other hand, the equations of motion for the spin (7) are

$$-\frac{ar^2\dot{\phi}\dot{a}S^{r\phi}}{c^2} + \frac{\dot{a}S^{tr}}{a} + \frac{\dot{r}g'S^{tr}}{2g} - \frac{r\dot{\phi}S^{t\phi}}{g} + P^r - \dot{r}P^t + \dot{S}^{tr} = 0, \tag{24}$$

$$\frac{ag\dot{r}\dot{a}S^{r\phi}}{c^2} + \frac{\dot{a}S^{t\phi}}{a} - \dot{\phi}P^t + P^{\phi} + \frac{\dot{\phi}S^{tr}}{r} + \frac{\dot{r}S^{t\phi}}{r} + \dot{S}^{t\phi} = 0, \tag{25}$$

$$\frac{\dot{r}\dot{a}S^{t\phi}}{a} + \frac{2\dot{a}S^{r\phi}}{a} - \frac{\dot{\phi}\dot{a}S^{tr}}{a} + \frac{\dot{r}g'S^{r\phi}}{2g} - \dot{\phi}P^r + \dot{r}P^{\phi} + \frac{\dot{r}S^{r\phi}}{r} + \dot{S}^{r\phi} = 0. \tag{26}$$

Following with our analysis we have to use the constraints (18) and (19), as well as the equations for the constants of motion (15) and (16), to get the relation

$$S^{tr} = \kappa P^{\phi}, \tag{27}$$

with the function κ defined as

$$\kappa \equiv \pm\frac{Jr}{c^2\sqrt{gm}}.$$

Thus, solving for the spin components in terms of the momentum components, and using both constraints as well as equation (27), we get

$$S^{t\phi} = -\frac{g\kappa P^r}{r^2}, \tag{28}$$

$$S^{r\phi} = -\frac{c^2\kappa P^t}{a^2 r^2}, \tag{29}$$

which their time derivatives are readily calculated as

$$\dot{S}^{tr} = \kappa\left(\frac{(2-g)\dot{r}P^{\phi}}{r} + \dot{P}^{\phi}\right), \tag{30}$$

$$\dot{S}^{t\phi} = -\frac{g\kappa((g-2)\dot{r}P^r + r\dot{P}^r)}{r^3}, \tag{31}$$

$$\dot{S}^{r\phi} = \frac{c^2\kappa(P^t(2r\dot{a} + ag\dot{r}) - ar\dot{P}^t)}{a^3 r^3}, \tag{32}$$

where we have used the relation $g' = 2(g-1)g/r$. Replacing the spin components and their derivatives in the previous set (21)–(26) we can find a set of equations for the momentum components

$$\kappa\left(\frac{\dot\phi \dot a P^t}{a} + \frac{\dot a P^\phi}{a} + \frac{\dot\phi P^r}{r} + \frac{\dot r P^\phi}{r} + \dot P^\phi\right) + P^r - \dot r P^t = 0, \tag{33}$$

$$\kappa\left(-\frac{g\dot a P^r}{ar^2} - \frac{g\dot r \dot a P^t}{ar^2} - \frac{g^2 \dot r P^r}{r^3} + \frac{g\dot r P^r}{r^3} - \frac{g\dot P^r}{r^2} + \frac{\dot\phi P^\phi}{r}\right) - \dot\phi P^t + P^\phi = 0, \tag{34}$$

$$\kappa\left(-\frac{c^2 \dot P^t}{a^2 r^2} - \frac{g\dot r \dot a P^r}{ar^2} - \frac{\dot\phi \dot a P^\phi}{a}\right) - \dot\phi P^r + \dot r P^\phi = 0, \tag{35}$$

$$\kappa\left(\frac{ag\dot r\ddot a P^\phi}{c^2} - \frac{ag\dot\phi\ddot a P^r}{c^2}\right) + \frac{ag\dot r\dot a P^r}{c^2} + \frac{ar^2\dot\phi\dot a P^\phi}{c^2} + \dot P^t = 0, \tag{36}$$

$$\frac{\dot a P^r}{a} + \frac{\dot r\dot a P^t}{a} + \kappa\left(\frac{c^2\dot\phi P^t}{a^2 g r^2} - \frac{c^2\dot\phi P^t}{a^2 r^2} + \frac{\ddot a P^\phi}{a} - \frac{\dot\phi\dot a^2 P^t}{a^2}\right) + \frac{g\dot r P^r}{r} - \frac{r\dot\phi P^\phi}{g} - \frac{\dot r P^r}{r} + \dot P^r = 0, \tag{37}$$

$$\frac{\dot\phi\dot a P^t}{a} + \frac{\dot a P^\phi}{a} + \kappa\left(\frac{c^2 g\dot r P^t}{a^2 r^4} - \frac{c^2\dot r P^t}{a^2 r^4} - \frac{g\ddot a P^r}{ar^2} + \frac{g\dot r\dot a^2 P^t}{a^2 r^2}\right) + \frac{\dot\phi P^r}{r} + \frac{\dot r P^\phi}{r} + \dot P^\phi = 0. \tag{38}$$

The process of finding the new constant of motion consists first on substracting equation (38) multiplied by κ from equation (33) to obtain

$$P^r\left(\frac{g\kappa^2\ddot a}{ar^2} + 1\right) + \dot r P^t\left(-\frac{c^2 g\kappa^2}{a^2 r^4} + \frac{c^2\kappa^2}{a^2 r^4} - \frac{g\kappa^2\dot a^2}{a^2 r^2} - 1\right) = 0. \tag{39}$$

In the same fashion we can add equation (34) to equation (37) multiplied by $\kappa g/r^2$ to get

$$P^\phi\left(\frac{g\kappa^2\ddot a}{ar^2} + 1\right) + \dot\phi P^t\left(-\frac{c^2 g\kappa^2}{a^2 r^4} + \frac{c^2\kappa^2}{a^2 r^4} - \frac{g\kappa^2\dot a^2}{a^2 r^2} - 1\right) = 0. \tag{40}$$

These two above equation can be solved for P^r and P^ϕ in terms of P^t as

$$P^r = \alpha\dot r P^t, \quad P^\phi = \alpha\dot\phi P^t, \tag{41}$$

where the function α depends on time only

$$\alpha \equiv \frac{a^2 c^4 m^2 + J^2\dot a^2 + c^2 J^2 k}{a^2 c^4 m^2 + a J^2\ddot a}.$$

Note that $\alpha = 1$ if $J = 0$. Now, to get P^t use equations (41) in (35) to find

$$\frac{a\alpha c^2\dot P^t P^t}{\dot a} = (P^t)^2(-a^2\alpha^2 g\dot r^2 - a^2\alpha^2 r^2\dot\phi^2). \tag{42}$$

At the same time, using (41) in the constant of motion (9) we get

$$c^2 m^2 = (P^t)^2(-a^2\alpha^2 g\dot r^2 - a^2\alpha^2 r^2\dot\phi^2 + c^2). \tag{43}$$

Subtracting (43) from (42) we finally find

$$\frac{\dot a((P^t)^2 - m^2)}{a\alpha} + \dot P^t P^t = 0, \tag{44}$$

which has the solution

$$P^t = \sqrt{\frac{2\sigma}{a^2 c^4 m^2 + J^2\dot a^2 + c^2 J^2 k} + m^2}, \tag{45}$$

where σ is a new (integration) constant of motion. This constant is one of the important results of this work and deserves more attention. The new constant can be written in terms of P^t as

$$\sigma = \frac{m^2 c^4}{2} \left[a^2 \left(1 + \frac{J^2 H^2}{m^2 c^4} \right) + \frac{J^2 k}{m^2 c^2} \right] [(P^t)^2 - m^2] = \frac{\beta}{2} [(P^t)^2 - m^2], \quad (46)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter and

$$\beta \equiv a^2 m^2 c^4 + J^2 \dot{a}^2 + c^2 J^2 k.$$

The constant (46) can be written as $\sigma = K_{\mu\nu} P^\mu P^\nu$, associated to the tensor

$$K_{\mu\nu} = \frac{\beta}{2} \left(\mathcal{U}_\mu \mathcal{U}_\nu - \frac{1}{c^2} g_{\mu\nu} \right), \quad (47)$$

where $\mathcal{U}_\mu = (1, 0, 0, 0)$. The constant σ has a well-known analogue when $J = 0$ for spinless particles [23]. It is important to emphasize that $K_{\mu\nu}/m^2$ is a Killing tensor for $J = 0$ only. When $J \neq 0$ we have not been able to write the new constant of motion in terms of a Killing tensor. Therefore, we have found a generalization for that constant in the case of a massive spinning particle $J \neq 0$ in FRW spacetimes. As far as we know, this is the first time that the constant (46) has been found for the motion of massive spinning particles on a FRW metric. The new extra terms (proportional to J^2) are interesting. Note how the first of these two terms combines to introduce a correction of the spin that takes in account the expansion of the universe through the Hubble parameter. Similarly, the second term of these two terms tell us how the curvature of the space affects the value of this constant. As we can see, this constant is richer in information than the spinless case counterpart.

Finally, making use of (46) to complete determine P^t, we can obtain the other momenta (41). Also, we can calculate the line element (13) as

$$\frac{ds^2}{c^2 dt^2} = 1 - \frac{1}{\alpha^2} - \frac{m^2}{\alpha^2 (P^t)^2} = 1 - \frac{a^2 \sigma \dot{\beta}^2}{2\beta^2 \dot{a}^2 (2\sigma + \beta m^2)}, \quad (48)$$

which can be written in terms of the constants of motion

$$\frac{ds^2}{c^2 dt^2} = 1 - \frac{2\sigma (J^2 \dot{H} + J^2 H^2 + c^4 m^2)^2}{(J^2 H^2 + c^4 m^2)^2 (a^2 m^2 (J^2 H^2 + c^4 m^2) + 2\sigma)}. \quad (49)$$

The line element (49) for a top in FRW metric is no longer positive definite. It can either vanish or be negative depending on the values of the different constants involved in its expression. Therefore, the top may follow lightlike or spacelike trajectories when $ds^2 = 0$ or $ds^2 < 0$, for appropriate values of the constants. This results are possible only if $\dot{H} \neq 0$.

This feature of the line element on portion of the trajectories is a characteristic of dynamics of massive spinning particles. It has been previously reported for the top's motion in Schwarzschild spacetimes [14].

As a final exercise for the FRW metric, we can calculate $\dot{r}$ and $\dot{\phi}$ using the equations (17), (28), (29) and (43), to obtain

$$\dot{r} = \frac{\dot{\beta}(\sqrt{g}\kappa\dot{a}(jr - c^2\kappa P^t) \pm ar\Upsilon)}{2\beta\sqrt{g}r\dot{a}P^t(a^2 r^2 + g\kappa^2 \dot{a}^2)}, \quad \dot{\phi} = \frac{(\alpha g\kappa r\dot{r}a\dot{a} + c^2\kappa)P^t - jr}{a^2 \alpha r^3 P^t}, \quad (50)$$

where

$$\Upsilon = \sqrt{-r^2(a^2 c^2 m^2 r^2 + c^2 g\kappa^2 m^2 \dot{a}^2 + j^2) + c^2(P^t)^2(a^2 r^4 + g\kappa^2 r^2 \dot{a}^2 - c^2\kappa^2) + 2c^2 j\kappa rP^t}.$$

4. Exact solution for cosmological Gödel spacetimes

Consider a universe described by the Gödel metric which is given by [24]

$$g^{\mu\nu} = \begin{pmatrix} c^2 & 0 & c\,e^{xw_0} & 0 \\ 0 & -1 & 0 & 0 \\ c\,e^{xw_0} & 0 & \frac{1}{2}\,e^{2xw_0} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{51}$$

in rectangular coordinates, where w_0 is a constant related to the angular velocity of the rotating universe. This metric has five Killing vectors

$$\xi_\mu^0 = (c^2, 0, c\,e^{w_0 x}, 0),$$

$$\xi_\mu^1 = (0, 0, 0, -1),$$

$$\xi_\mu^2 = \left(c\,e^{w_0 x}, 0, \frac{1}{2}\,e^{2w_0 x}, 0\right),$$

$$\xi_\mu^3 = \left(-cw_0 y\,e^{w_0 x}, -1, -\frac{1}{2}w_0 y\,e^{2w_0 x}, 0\right),$$

$$\xi_\mu^4 = \left(-\frac{1}{2}cw_0 y^2\,e^{w_0 x} - \frac{c\,e^{-w_0 x}}{w_0}, -y, -\frac{1}{4}w_0 y^2\,e^{2w_0 x} - \frac{3}{2w_0}, 0\right). \tag{52}$$

It becomes apparent that there are solutions describing trajectories in the plane $z = 0$. Therefore, we work in this plane with $P^z = 0$ and $\dot{P}^z = 0$ (this also implies that every z-component of the spin as well as their time derivatives vanish).

We follow the same procedure than in the previous section. First we find the constants of motion (9) and (10) as

$$m^2 c^2 = c^2 (P^t)^2 + 2c\,e^{w_0 x} P^t P^y + \frac{1}{2}\,e^{2w_0 x}(P^y)^2 - (P^x)^2, \tag{53}$$

$$J^2 = -c^2 (S^{tx})^2 - \frac{1}{2}c^2\,e^{2w_0 x}(S^{ty})^2 + 2c\,e^{w_0 x} S^{tx} S^{xy} - \frac{1}{2}\,e^{2w_0 x}(S^{xy})^2, \tag{54}$$

while using the Killing vectors we can calculate the following non-vanishing constants of motion

$$E = \frac{1}{2}c(2cP^t + e^{w_0 x}(2P^y + w_0 S^{xy})), \tag{55}$$

$$C_2 = \frac{1}{2}\,e^{w_0 x}(2cP^t + w_0(e^{w_0 x} S^{xy} - cS^{tx}) + e^{w_0 x} P^y), \tag{56}$$

$$C_3 = \frac{1}{2}(-w_0\,e^{w_0 x}(2cyP^t - cS^{ty} + y\,e^{w_0 x} P^y) + w_0^2 y(-e^{w_0 x})(e^{w_0 x} S^{xy} - cS^{tx}) - 2P^x), \tag{57}$$

$$C_4 = -\frac{1}{2}cw_0 y^2\,e^{w_0 x} P^t - \frac{c\,e^{-w_0 x} P^t}{w_0} + \frac{1}{4}cw_0^2 y^2\,e^{w_0 x} S^{tx} - \frac{1}{2}c\,e^{-w_0 x} S^{tx} + \frac{1}{2}cw_0 y\,e^{w_0 x} S^{ty}$$

$$- \frac{1}{4}w_0 y^2\,e^{2w_0 x} + P^y - \frac{3P^y}{2w_0} - yP^x - \frac{1}{4}w_0^2 y^2\,e^{2w_0 x} S^{xy} + \frac{S^{xy}}{2}. \tag{58}$$

On the other hand, the two independent Tulczyjew constraint equations (8) read

$$P^x S^{tx} = c\,e^{w_0 x} P^t S^{ty} + \frac{1}{2}\,e^{2w_0 x} P^y S^{ty}, \tag{59}$$

$$P^x S^{xy} = c^2 P^t S^{ty} + c\,e^{w_0 x} P^y S^{ty}. \tag{60}$$

The above constants of motion must be used along with the equations of motion for the top in the Gödel spacetimes. The (non identically vanishing) momentum equations of motion (6) are

$$w_0(2\dot{x}(2cP^t - cw_0 S^{tx} + e^{w_0 x} P^y + 2w_0 e^{w_0 x} S^{xy}) + (2c + \dot{y} e^{w_0 x})(cw_0 e^{w_0 x} S^{ty} + 2P^x))$$
$$+ 4c\dot{P}^t = 0, \tag{61}$$

$$w_0(\dot{y} e^{w_0 x}(2cP^t + w_0(3 e^{w_0 x} S^{xy} - 2cS^{tx}) + 2 e^{w_0 x} P^y) + 2c(w_0(e^{w_0 x} S^{xy} - cS^{tx}) + e^{w_0 x} P^y))$$
$$+ 4\dot{P}^x = 0, \tag{62}$$

$$2 e^{w_0 x} \dot{P}^y - w_0(\dot{x}(2cP^t + w_0 e^{w_0 x} S^{xy}) + c(w_0 e^{w_0 x} S^{ty}(c + \dot{y} e^{w_0 x}) + 2P^x)) = 0, \tag{63}$$

whereas the equations for the spin (7) become

$$\frac{1}{2} cw_0 e^{w_0 x} S^{ty} - \frac{w_0 \dot{x} e^{w_0 x} S^{xy}}{2c} - \dot{x}P^t + P^x + w_0 \dot{x} S^{tx} + \dot{S}^{tx} + \frac{1}{2} w_0 \dot{y} e^{2w_0 x} S^{ty} = 0, \tag{64}$$

$$- cw_0 e^{-w_0 x} S^{tx} + \frac{w_0 \dot{y} e^{w_0 x} S^{xy}}{2c} - \dot{y}P^t + P^y + w_0 \dot{x} S^{ty} + \dot{S}^{ty} + w_0 S^{xy} = 0, \tag{65}$$

$$cw_0 \dot{x} e^{-w_0 x} S^{tx} + \frac{1}{2} cw_0 \dot{y} e^{w_0 x} S^{ty} - \dot{y}P^x + \dot{x}P^y + \dot{S}^{xy} = 0. \tag{66}$$

Now, using equations (53), (54), (59), and (60), we can find the first solution for the spin component

$$S^{ty} = e^{-w_0 x} \varrho P^x, \tag{67}$$

where we define

$$\varrho \equiv \frac{\sqrt{2} J}{c^2 m}. \tag{68}$$

With these results and definitions, we can replace (67) in (60), and use the constant (55) to obtain

$$S^{xy} = \frac{E\varrho e^{-w_0 x}}{2c\varrho w_0 + 4}. \tag{69}$$

Also, combining (59), (60) and (67) we get

$$\frac{2 e^{w_0 x} S^{xy} - 2cS^{tx}}{\varrho} = c e^{w_0 x} P^y. \tag{70}$$

We solve for the last component of the spin using equations (70), (69) and the constants C_3 and E

$$S^{tx} = \frac{\varrho e^{-w_0 x}(E\varrho w_0 e^{w_0 x} - C_3(c\varrho w_0 + 2))}{2(c^2 \varrho^2 w_0^2 - 4)}, \tag{71}$$

and replacing S^{tx} and S^{xy} in (70) we find

$$P^y = \frac{e^{-2w_0 x}(cC_3(c\varrho w_0 + 2) - 2E e^{w_0 x})}{c(c^2 \varrho^2 w_0^2 - 4)}. \tag{72}$$

To solve for P^t we use equation (60) along with the previous results to get

$$P^t = \frac{E - 2cC_3 e^{-w_0 x}}{2c^2(c\varrho w_0 - 2)}. \tag{73}$$

Lastly, for P^x we can use (53)

$$P^x = \pm \frac{e^{-w_0 x}}{2c(\lambda^2 - 4)} \Big[e^{2w_0 x}(E^2(\lambda^2 - 4\lambda - 4) - 4c^4(\lambda^2 - 4)^2 m^2)$$
$$- 2c^2 C_3^2(\lambda + 2)^2 + 8cC_3 E(\lambda + 2) e^{w_0 x} \Big]^{1/2}, \tag{74}$$

where we have defined the dimensionless constant $\lambda \equiv c\varrho w_0 = (\sqrt{2} J w_0)/(cm)$. Now, our objective is to solve for $\dot{y}$ and $\dot{x}$. Let us start by observing that performing a time derivative of equation (67) yields

$$\dot{S}^{ty} = \varrho\, e^{-w_0 x}\dot{P}^x - w_0\dot{x}S^{ty}. \tag{75}$$

Replacing $\dot{S}^{ty}$ in (65) and combining this result with (62) we get, after some algebra, an expression for $\dot{y}$. In the same way, to get $\dot{x}$, we use equation (63) along with the results for the spin and momentum components. Both results are

$$\dot{y} = \frac{2(\lambda-2)\dot{P}^x}{C_3 w_0}, \tag{76}$$

$$\dot{x} = -\frac{2c^2(\lambda-2)(\lambda+2)^2\, e^{w_0 x}P^x}{2cC_3(\lambda+2)^2 + E((\lambda-4)\lambda-4)\, e^{w_0 x}}. \tag{77}$$

We now solve for P^x. Using (54), (59), (60) and (53) we get

$$P^x = \pm\left(c\sqrt{\frac{2(S^{tx})^2}{c^2\varrho^2} - m^2 - (P^t)^2}\right). \tag{78}$$

Differentiate with respect to time and get

$$\dot{P}^x = \frac{c^2}{P^x}\left(\frac{2\dot{S}^{tx}S^{tx}}{c^2\varrho^2} - \dot{P}^t P^t\right), \tag{79}$$

while the time derivatives we need are

$$\dot{S}^{tx} = \frac{C_3\lambda\dot{x}\, e^{-w_0 x}}{2c(\lambda-2)}, \quad \dot{P}^t = \frac{C_3 w_0\dot{x}\, e^{-w_0 x}}{c(\lambda-2)}. \tag{80}$$

Thus, finally, using (80), (79) and (77) into (76) we can show that

$$\dot{y} = -\frac{2c(\lambda+2)\, e^{-w_0 x}(cC_3(\lambda+2) - 2E\, e^{w_0 x})}{2cC_3(\lambda+2)^2 + E(\lambda^2-4\lambda-4)\, e^{w_0 x}}. \tag{81}$$

In the same spirit than the previous section we can evaluate the line element of the trajectories of the top in Gödel spacetimes to inquire about the nature of its orbits. With all the previous results, and using the metric (51), we can find the line element of the spin particle

$$\frac{ds^2}{c^2\, dt^2} = \frac{\dot{y}^2\, e^{2w_0 x} - 2\dot{x}^2 + 4c\dot{y}\, e^{w_0 x}}{2c^2} + 1 = \frac{4c^2\, e^{2w_0 x}(4E^2\lambda^2/c^2 + c^2(\lambda-2)^2(\lambda+2)^4 m^2)}{(2cC_3(\lambda+2)^2 + E(\lambda^2-4\lambda-4)\, e^{w_0 x})^2}. \tag{82}$$

As it can be readily seen, the line element is always timelike. Thus, massive spinning particles moving in the plane $z = 0$ of Gödel spacetimes never follow lightlike or spacelike trajectories.

5. Exact solution for general Schwarzschild-like spacetimes

In the preceding sections we have obtained the exact solutions of the motion of tops in cosmological scenarios. We showed that it could be possible for the top to follow trajectories which may be (at least partially) described by lightlike or spacelike line elements. Now we will study the top's dynamics in a general spacetime which is the generalization of the results for the Schwarzschild metric studied in [14].

Consider a general Schwarzschild-like metric in spherical coordinates, given by the line element

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = g(r)\, dt^2 - \frac{c^2}{g(r)}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{83}$$

where now $g(r) \equiv g$ is a generic function with radial dependence only. Examples of this kind of metrics includes the Reissner–Nordstrom–(Anti)de Sitter case.

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As in previous sections, we start listing the Killing vectors of this metric

$$\xi_\mu^0 = (g, 0, 0, 0),$$
$$\xi_\mu^1 = (0, 0, 0, -r^2 \sin^2\theta),$$
$$\xi_\mu^2 = (0, 0, r^2 \sin\phi, r^2 \cos\theta \cos\phi \sin\theta),$$
$$\xi_\mu^3 = (0, 0, -r^2 \cos\phi, r^2 \cos\theta \sin\theta \sin\phi). \tag{84}$$

We can use these Killing vectors to find the constants of motion. In this case, as the metric is time-independent, we will be able to find the four constants of motion in a straightforward manner. To study the top's trajectories, in a way similar to the one used for the FRW metric, let us consider its motion in the plane $\theta = \pi/2$. First, we can find the constants of motion (9) and (10) as

$$m^2 c^2 = -\frac{c^2 (P^r)^2}{g} + g(P^t)^2 - r^2 (P^\phi)^2, \tag{85}$$

$$J^2 = -c^2 (S^{tr})^2 + \frac{c^2 r^2}{g} (S^{r\phi})^2 - gr^2 (S^{t\phi})^2. \tag{86}$$

And using the Killing vectors, we can find the constants of motion

$$E = gP^t - \frac{g' S^{tr}}{2}, \tag{87}$$

$$j = -r(rP^\phi + S^{r\phi}). \tag{88}$$

For the case of the general Schwarzschild-like metric, the constant E now corresponds to the energy of the top, whereas j is the conserved angular momentum orthogonal to the plane of the motion. On the other hand, the Tulczyjew contraints (8) for this case read

$$-\frac{P^r S^{tr} c^2}{g} - r^2 P^\phi S^{t\phi} = 0, \tag{89}$$

$$r^2 P^\phi S^{r\phi} + gP^t S^{tr} = 0, \tag{90}$$

$$\frac{c^2 P^r S^{r\phi}}{g} - gP^t S^{t\phi} = 0. \tag{91}$$

The momentum equations (6) for this metric turn out to be

$$\dot{P}^t + \frac{P^r g'}{2g} + \frac{\dot{r} P^t g'}{2g} - \frac{r\dot{\phi} S^{t\phi} g'}{2c^2} - \frac{\dot{r} S^{tr} g''}{2g} = 0, \tag{92}$$

$$\dot{P}^r + \frac{gP^t g'}{2c^2} - \frac{gr\dot{\phi} P^\phi}{c^2} - \frac{r\dot{\phi} S^{r\phi} g'}{2c^2} - \frac{gS^{tr} g''}{2c^2} - \frac{\dot{r} P^r g'}{2g} = 0, \tag{93}$$

$$\dot{P}^\phi + \frac{\dot{\phi} P^r}{r} + \frac{\dot{r} P^\phi}{r} + \frac{\dot{r} S^{r\phi} g'}{2gr} - \frac{gS^{t\phi} g'}{2c^2 r} = 0, \tag{94}$$

while the spin equations (7) are

$$\dot{S}^{tr} + P^r - \dot{r} P^t - \frac{gr\dot{\phi} S^{t\phi}}{c^2} = 0, \tag{95}$$

$$\dot{S}^{t\phi} - \dot{\phi} P^t + P^\phi + \frac{\dot{\phi} S^{tr}}{r} + \frac{\dot{r} S^{t\phi}}{r} + \frac{S^{r\phi} g'}{2g} + \frac{\dot{r} S^{t\phi} g'}{2g} = 0, \tag{96}$$

$$\dot{S}^{r\phi} - \dot{\phi} P^r + \dot{r} P^\phi + \frac{\dot{r} S^{r\phi}}{r} + \frac{gS^{t\phi} g'}{2c^2} - \frac{\dot{r} S^{r\phi} g'}{2g} = 0. \tag{97}$$

It is important to mention that the constants of motion (87) and (88) can be derived from the above set of equations. With this full set of equations (85)–(97) we can solve completely for the motion of the top. Using equations (89) and (90) we get

$$\frac{r^4(P^\phi)^2(S^{r\phi})^2}{g} - \frac{gr^4(P^\phi)^2(S^{t\phi})^2}{c^2} = g(P^t)^2(S^{tr})^2 - \frac{c^2(P^r)^2(S^{tr})^2}{g}. \tag{98}$$

Thereby, using (85) and (86) we can solve for S^{tr} finding that

$$S^{tr} = \pm \frac{Jr}{c^2 m} P^\phi. \tag{99}$$

New relations among the constants of motion can be found using the constants (87), (88) and equation (90). They produce the relation

$$ES^{tr} - jrP^\phi = r^3(P^\phi)^2 - \tfrac{1}{2}g'(S^{tr})^2, \tag{100}$$

which can be used to find the solution of P_ϕ. Using equation (99), after some algebra, we can find

$$P_\phi = \frac{-j \pm EJ/(mc^2)}{\eta - 1}, \tag{101}$$

where we introduce the notation

$$\eta \equiv \frac{J^2 g'}{2c^4 m^2 r},$$

in a similar fashion to the parameter defined in [14]. Now we are able to solve for P_t. Using equations (87), (99) and (101), we find that

$$P_t = \frac{E \mp jJg'/(2mc^2 r)}{1 - \eta}, \tag{102}$$

which lead us to easily solve for P^r by using (9)

$$P^r = \pm \frac{1}{c}\sqrt{P_t^2 - g\left(c^2 m^2 + \frac{P_\phi^2}{r^2}\right)}. \tag{103}$$

We would like to solve for $\dot{r}$ and $\dot{\phi}$. With this purpose in mind, we have to find first the components of the spin in terms of the momenta. By using constraints (89) and (90) along with equation (99) we can show that the other components of the spin may be expressed as

$$S^{t\phi} = \mp \frac{JP^r}{mgr}, \quad S^{r\phi} = \mp \frac{JgP^t}{mc^2 r}. \tag{104}$$

To find $\dot{r}$, we multiply equation (91) by $\pm Jr/(c^2 m)$ and then we subtract equation (95). With the help of the spin components (104) we get

$$\frac{g'P^r J^2}{2m^2 c^4 r} - \frac{\dot{r} g' P^t J^2}{2m^2 c^4 r} \pm (r\dot{P}^\phi + \dot{r}P^\phi)\frac{J}{c^2 m} - P^r + \dot{r}P^t - \dot{S}^{tr} = 0, \tag{105}$$

which, using (99), allow us to find the solution

$$\dot{r} = \frac{P^r}{P^t} = \frac{gP^r}{P_t}. \tag{106}$$

To solve for $\dot{\phi}$ we replace $\dot{r}$ from the previous equation in equation (95), as well as, we have to use (99) and equation (101) (to find $\dot{P}^\phi$). After some algebra we get

$$\dot{\phi} = \frac{gP_\phi(g' - \eta r g'')}{(\eta - 1)r^2 g' P_t}. \tag{107}$$

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When the general case is reduced to the Schwarzschild spacetime, where $g(r) = c^2(1 - 2r_0/r)$ and r_0 is half of the Schwarzschild radius, the dynamics of the top's motion described in [14] is recovered.

Now we can seek for the line element (83) for this metric. This is written in the plane $\theta = \pi/2$ as

$$\frac{\mathrm{d}s^2}{c^2\,\mathrm{d}t^2} = \frac{g}{c^2} - \frac{\dot{r}^2}{g} - \frac{r^2\dot{\phi}^2}{c^2} = \frac{m^2(1-\Lambda)}{(P^t)^2}, \tag{108}$$

where we have defined the parameter

$$\Lambda \equiv \frac{P_\phi^2}{c^2 m^2 r^2}\left[\frac{(\eta r g''/g' - 1)^2}{(\eta - 1)^2} - 1\right] = \frac{(-j \pm EJ/(c^2 m))^2}{c^2(\eta - 1)^2 m^2 r^2}\left[\frac{(\eta r g''/g' - 1)^2}{(\eta - 1)^2} - 1\right]. \tag{109}$$

As in the previous sections, according to the sign of $\mathrm{d}s^2$ the solution may describe timelike, lightlike or spacelike orbits, depending on the value of Λ. We can see that here, analogously to the case of the Schwarzschild metric [14], Λ can take different values depending on the constants of motion of the top, such as its mass, its energy E, its angular momentum j, and its spin. If $\Lambda < 1$, the top follows timelike trajectories. Instead if $\Lambda = 1$ or $\Lambda > 1$, the spinning particle follows lightlike or spacelike trajectories (at least partially).

5.1. Reissner–Nordstrom–(Anti)de Sitter metric

To evaluate the previous calculations in explicitly scenario, let us calculate Λ, from (109), for the Reissner–Nordstrom–(Anti)de Sitter. This metric is given by

$$g(r) = c^2\left(1 - \frac{2GM}{c^2 r} + \frac{\kappa GQ^2}{c^4 r^2} - \frac{\lambda r^2}{3}\right), \tag{110}$$

where G is the gravitational constant, κ is Coulomb's constant, Q is the charge of the black hole, M its mass and λ is the cosmological constant.

For this case, the parameter (109) becomes

$$\Lambda = \frac{27c^6 GJ^2 r^6 (3c^2 Mr - 4\kappa Q^2)(c^2 jm + EJ)^2(6c^6 m^2 r^4 + 2c^4 J^2 \lambda r^4 + 3c^2 GJ^2 Mr - 6\kappa GJ^2 Q^2)}{(3c^6 m^2 r^4 + c^4 J^2 \lambda r^4 - 3c^2 GJ^2 Mr + 3\kappa GJ^2 Q^2)^4}. \tag{111}$$

We have found that searching for parameter combinations that produce $\Lambda \geqslant 1$ is a very difficult task when selecting known values for different particles. For example, trying to make an electron accelerate to the speed of light is practically impossible on an extremely large range of central object masses (from Earth like to extreme black holes) under even the most extreme scenarios for values of E and j. Another interesting result is that the cosmological constant has no effect on the line element when there is no central object present ($M = Q = 0$). This can easily be seen from equation (111).

6. Exact solution for static spherically symmetric conformally flat spacetimes

As well as in the other metrics studied in this work, the motion of a top in a conformally flat spacetime with spherical symmetry may be solved exactly. The conformally spherical line element is

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu, \qquad g_{\mu\nu} = \Omega^2 \eta_{\mu\nu}, \tag{112}$$

where $\Omega \equiv \Omega(r)$ is the spherical symmetric conformal factor, and $\eta_{\mu\nu}$ as the flat spacetime metric written in spherical coordinates, such that $\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = c^2\,\mathrm{d}t^2 - \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$.

Because the metric is time-independent, we can find a general expression for the conserved energy using the general conserved quantity (11). We get the conserved energy E for the motion of the top

$$E = P_t - c^2 \Omega \Omega' S^{tr}, \tag{113}$$

associated to the Killing vector $\xi^1_\mu = (c^2\Omega^2, 0, 0, 0)$. Similarly, the three components of the conserved vectorial angular momentum of the top can be found in a similar fashion. Without loss of generality, we again restrict ourselves to motion in the equatorial plane $\theta = \pi/2$, as in the previous sections. The conserved angular momentum component orthogonal to the orbital plane is

$$j = -\Omega(\Omega + r\Omega')rS^{\phi r} - P_\phi, \tag{114}$$

which can be checked from (11) using the Killing vector $\xi^2_\mu = (0, 0, 0, -r^2\Omega^2)$.

On the other hand, the mass and spin conservation laws give rise to new relations. From (9) we find that

$$\frac{r^2}{c^2}P_t^2 - r^2P_r^2 - P_\phi^2 = r^2\Omega^2 m^2 c^2, \tag{115}$$

whereas from (10) we have

$$J^2 = \Omega^4 \frac{(S^{\phi r})^2}{P_t^2}\left(r^2 P_t^2 - c^2 P_\phi^2 - c^2 r^2 P_r^2\right), \tag{116}$$

where use has been made of the constraints (8)

$$S^{tr} = -\frac{S^{\phi r}P_\phi}{P_t}, \qquad S^{t\phi} = \frac{S^{\phi r}P_r}{P_t}. \tag{117}$$

We notice that equations (115) and (116) can be used to get the condition

$$S^{\phi r} = \pm\frac{JP_t}{mr\Omega^3 c^2}. \tag{118}$$

This condition allows us to rewrite the conserved energy and angular momentum in terms of momenta. Using (117) and (118), we can write the energy (113) and the angular momentum (114) as

$$E = P_t \pm \frac{J\Omega'}{rm\Omega^2}P_\phi, \tag{119}$$

$$j = \mp\frac{J(\Omega + r\Omega')}{\Omega^2 mc^2}P_t - P_\phi. \tag{120}$$

Notice that two solutions for the energy and the angular momentum have emerged now. Their origin are the two possible solutions for $S^{\phi r}$ of the condition (118) related to the fact that the spin vector may be parallel or antiparallel to the angular momentum vector.

The previous relations can be solved for P_t and P_ϕ. It is again convenient to define the dimensionless auxiliary function

$$\eta \equiv \eta(r) = \frac{\Omega'(\Omega + r\Omega')J^2}{r\Omega^4 m^2 c^2}, \tag{121}$$

in order to find the top's momentum from (119) and (120)

$$P_\phi = \frac{1}{1-\eta}\left(-j \mp \frac{EJ(\Omega + r\Omega')}{mc^2\Omega^2}\right), \tag{122}$$

$$P_t = \frac{1}{1-\eta}\left(E \pm \frac{jJ\Omega'}{mr\Omega^2}\right). \tag{123}$$

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Lastly, from the condition (9), we get

$$P_r = \pm \left[\frac{P_t^2}{c^2} - \frac{P_\phi^2}{r^2} - m^2 c^2 \Omega^2 \right]^{1/2}. \tag{124}$$

Once we find the momenta in terms of the conserved quantities, it is possible to solve the system for the velocities $\dot{r}$ and $\dot{\phi}$. With the help of (117), the equations of motion (7) for S^{tr} and $S^{t\phi}$ become

$$P^t \dot{r} - P^r = \frac{S^{\phi r} P_\phi}{P_t^2} \frac{DP_t}{D\lambda} - \frac{DS^{\phi r}}{D\lambda} \frac{P_\phi}{P_t} - \frac{S^{\phi r}}{P_t} \frac{DP_\phi}{D\lambda}, \tag{125}$$

$$P^t \dot{\phi} - P^\phi = -\frac{S^{\phi r} P_r}{P_t^2} \frac{DP_t}{D\lambda} + \frac{DS^{\phi r}}{D\lambda} \frac{P_r}{P_t} + \frac{S^{\phi r}}{P_t} \frac{DP_r}{D\lambda}, \tag{126}$$

where this time we have not written explicitly the covariant derivatives. Using (6), these equations may be solved for $\dot{r}$ and $\dot{\phi}$ in the equatorial plane to give

$$\dot{\phi} = \frac{\zeta \gamma c^2}{r^2} \left(\frac{P_\phi}{P_t} \right), \quad \dot{r} = \zeta c^2 \left(\frac{P_r}{P_t} \right), \quad \frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{\gamma}{r^2} \left(\frac{P_\phi}{P_r} \right), \tag{127}$$

where we have defined

$$\zeta = (\eta - 1) \left[\eta + 1 - \frac{J^2}{\Omega^3 m^2 c^2 r} (2\Omega' + r\Omega'') \right]^{-1}, \tag{128}$$

and

$$\gamma = (1 - \eta)^{-1} \left[1 + \frac{J^2}{\Omega^4 m^2 c^2} ((\Omega')^2 - \Omega\Omega'') \right]. \tag{129}$$

Thereby, the motion of the spinning massive particle has been solved exactly, remembering that J may be identified with the top's spin. It is worth noting that the preceding expressions coincide with the usual results for geodesic motion when the spin is neglected, $J = 0$, being $\eta = 0$, $P_\phi = -j$, $P_t = E$, and $P_r^2 = E^2/c^2 - j^2/r^2 - m^2 c^2 \Omega^2$. Thus, the velocities are reduced to $\dot{r} = -c^2 P_r/E$ and $\dot{\phi} = c^2 j/(r^2 E)$.

Again, another interesting aspect of the motion of the top is the evaluation of its interval (112). This becomes

$$\frac{\mathrm{d}s^2}{c^2 \, \mathrm{d}t^2} = \Omega^2 (1 - \zeta^2) + \frac{m^2 c^4 \zeta^2 \Omega^4}{(P_t)^2} (1 - \Lambda), \tag{130}$$

where we have introduced the parameter

$$\Lambda = \frac{(P_\phi)^2 (\gamma^2 - 1)}{\Omega^2 m^2 r^2 c^2}. \tag{131}$$

From (130) we realize that, even for massive particles, there could be some initial conditions such that $\mathrm{d}s^2 \leqslant 0$, at least in part of the top's trajectories. The contribution of Λ depends on the conformal factor and also depends strongly on the value of the mass, being important for small mass values. When the spin is neglected, then $\mathrm{d}s^2 = c^2 \, \mathrm{d}t^2 \Omega^4 (m^2 c^4/E^2) > 0$, and the top always travels in timelike trajectories.

To show this behavior with explicit conformal metric, we will study different scenarios where the conformal metric is relevant.

6.1. Conformally flat spacetime for static perfect fluid and radiation

As an example, we study the motion of a test top in a conformally spherically flat universe filled with a static perfect fluid and radiation. We will focus our attention in four different cases: radiation only, matter-dominated, radiation-dominated and inflation-dominated universes. The universe will be filled with a static fluid, with energy density ϵ and pressure p, and a electromagnetic radiation field $F_{\mu\nu}$ which has origin in a electrostatic potential, i.e., its only non-vanishing component is F_{0r}.

Assuming that the fluid pressure p is proportional to the fluid energy density ϵ, the Einstein equations for the conformal metric (112) can be solved exactly. A general conformal factor can be found [25]

$$\Omega(r) = Q_1 (r^{-2/(1+3\alpha)} - Q_2\beta)^{(1+3\alpha)/2}, \tag{132}$$

where Q_1 and Q_2 are constants, $\alpha = p/\epsilon$ is the constant ratio between pressure and the energy density, and $\beta = (1+3\alpha)^{(3+3\alpha)/(1+3\alpha)}$. Also, the energy density has the form [25]

$$\epsilon = \frac{3c^2\beta Q_2}{4\pi G Q_1^2(1+3\alpha)} r^{-\frac{4+6\alpha}{1+3\alpha}} (r^{-2/(1+3\alpha)} - Q_2\beta)^{-3-3\alpha}, \tag{133}$$

where G is the gravitational constant. Notice that we must require $Q_2 \geqslant 0$ in order to have a positive semidefinite energy density. The radiation field can be found to be

$$F^2 = \frac{-c^2 r^{\frac{-6-6\alpha}{1+3\alpha}}}{G Q_1^2(1+3\alpha)} \left(r^{\frac{-2}{1+3\alpha}} - Q_2\beta \right)^{-3-3\alpha} \left[2 + 6\alpha - 4(2+3\alpha)\beta Q_2 r^{\frac{2}{1+3\alpha}} \right], \tag{134}$$

where $F^2 \equiv F_{\mu\nu}F^{\mu\nu} = 2F_{0r}F^{0r} = -2(F_{0r})^2/(\Omega^4 c^2)$. Because $Q_2 \geqslant 0$, this solution could have an intrinsic singularity in the metric [25].

6.1.1. Uniform electromagnetic radiation universe.

If $Q_2 = 0$, the conformal factor will be simply

$$\Omega = \frac{Q}{r}. \tag{135}$$

This conformal metric describing this kind of universe is known as the Bertotti–Robinson solution [26–28]. It is important to notice from (133) that the energy density and the pressure vanish. Therefore, there is no fluid. On the other hand, from (134), we get that $F^2 = -2c^2/(GQ^2)$ is constant. Thus, this universe is filled only with a static uniform electromagnetic radiation field.

The motion of a top in this universe can be studied using the Bertotti–Robinson metric. From (121) we find that $\eta \equiv 0$. This fact simplifies the previous expressions, finding from (122), (123) and (129), that

$$P_\phi = -j, \quad P_t = E \pm \frac{jJ}{mQr}, \quad \gamma = 1 - \frac{J^2}{m^2c^2Q^2}, \tag{136}$$

respectively. Also, from (128) we get $\zeta = -1$. This implies that the nature of the interval ds^2 of the top particle (130) is controlled by Λ only. From (131) we find that

$$\Lambda = \frac{j^2 J^2}{m^6 c^6 Q^6} (J^2 - 2m^2 c^2 Q^2). \tag{137}$$

With these quantities we can evaluate the velocities (127), and thus, the motion of the spinning particle is completely described.

Lastly, notice that Λ can be positive, negative or null depending on the properties of the top and of the spacetime through Q. This determines the sign of the interval ds^2, which it is

$$ds^2 = \frac{dt^2}{r^4 P_t^2 m^4 Q^2}(2m^2 c^2 Q^2 j^2 J^2 + m^6 c^6 Q^6 - j^2 J^4), \tag{138}$$

where P_t is given in (136). If the previous expression for ds^2 is positive, the spinning massive particle follows timelike orbits. Otherwise, the particle can travel in lightlike or spacelike orbits. Interesting enough is that this condition depends only on the properties of the particle j, J and m, and not on the top's path. Thus, depending on the top's properties, the previous desciption is for the motion of bradyons, luxons or tachyons [29].

6.1.2. Matter-dominated universe. If we choose $\alpha = 0$, then we describe a matter-dominated universe [30]. In this case, the energy density is dominant over the pressure, and the fluid is composed by non-relativistic matter (cold dust, $p = 0$). For this case, the conformal factor will be given by

$$\Omega = Q_1 \left(\frac{1}{r^2} - Q_2\right)^{1/2}, \tag{139}$$

making the evaluation of the η function a straightforward calculation

$$\eta = \frac{J^2 Q_1^4 Q_2}{r^4 m^2 c^2 \Omega^6} > 0. \tag{140}$$

This allows us to write the other auxiliary functions in terms of η. We find that $\zeta = (\eta - 1)/(2\eta + 1)$ and $\gamma = [1 + 3\eta - \eta/(r^2 Q_2)]/(1 - \eta)$. In this case, the momenta (122) and (123) take the form

$$P_\phi = \frac{1}{1 - \eta}\left(-j \pm \frac{EJQ_1^2 Q_2}{mc^2 \Omega^3}\right), \quad P_t = \frac{1}{1 - \eta}\left(E \mp \frac{jJQ_1^2}{mr^4 \Omega^3}\right). \tag{141}$$

With these values, the evaluation of P_r or of F^2 could now be done using (124) and (134).

Along with this results, it is possible to obtain an expression for the interval (130) in this spacetime. As in the previous case, the spin introduces enough freedom to enrich the behavior of ds^2. It can be shown from (130) that

$$\frac{ds^2}{c^2 dt^2} = \frac{\Omega^2}{(\eta + 1 - 2\eta^2)^2 P_t^2}\left[3\eta(2 + \eta)(1 - \eta)^2 P_t^2 + m^2 c^4 \Omega^2 (1 - \eta)^4\right.$$
$$\left. - (1 - \eta)^2 P_\phi^2 \left(\frac{8\eta(1 + \eta)c^2}{r^2} + \left(\frac{J^2 Q_1^4}{\Omega^6 c\, m^2 r^7}\right)^2 - \frac{(2 + 6\eta)J^2 Q_1^4}{\Omega^6 m^2 r^8}\right)\right], \tag{142}$$

written in terms of η given in (140) and the momenta (141). If this expression for ds^2 is null, the motion will be lightlike, and it will be spacelike if the right-hand side of (142) is negative. Otherwise, the trajectory will be timelike. In addition to its dependence on the top's properties, the above expression for ds^2 also depends on r, implying that the particle can achieve velocities larger than speed of light in certain regions of its path. A rigorous evaluation of this condition requires the knowledge of the top's energy E, mass m and angular momentum j, and the constants Q_1 and Q_2 of the metric.

6.1.3. Radiation-dominated universe. The radiation-dominated phase of the universe (very early universe) can be studied for $\alpha = 1/3$ [30], implying an equation of state for ultra-relativistic matter. In this case, the conformal factor (132) becomes

$$\Omega = Q_1 \left(\frac{1}{r} - 4Q_2\right), \tag{143}$$

which produces now that

$$\eta = \frac{4J^2 Q_1^2 Q_2}{r^3 m^2 c^2 \Omega^4} > 0. \tag{144}$$

Then, the auxiliary functions become $\zeta = (\eta - 1)/(\eta + 1)$, and $\gamma = [1 + 2\eta - \eta/(4rQ_2)]/(1 - \eta)$, whereas the momenta are

$$P_\phi = \frac{1}{1-\eta}\left(-j \pm \frac{4EJQ_1 Q_2}{mc^2 \Omega^2}\right), \quad P_t = \frac{1}{1-\eta}\left(E \mp \frac{jJQ_1}{mr^3 \Omega^2}\right), \tag{145}$$

which can be used to find the velocities of the particle.

Again, using the previous solution an explicit expression for the top's interval could be found. We again obtain a possibility for a behavior different to a timelike interval. It can be shown from (130) that

$$\frac{ds^2}{c^2\, dt^2} = \frac{\Omega^2}{(1-\eta^2)^2 P_t^2}\left[4\eta(1-\eta)^2 P_t^2 + m^2 c^4 \Omega^2 (1-\eta)^4 \right.$$
$$\left. - (1-\eta)^2 P_\phi^2 \left(\frac{3\eta(2+\eta)c^2}{r^2} + \frac{J^4 Q_1^4}{\Omega^8 m^4 c^2 r^{10}} - \frac{(2+4\eta)J^2 Q_1^2}{\Omega^4 m^2 r^6} \right) \right], \tag{146}$$

in terms of η defined in (144) and momenta (145). Thus, ds^2 could define timelike, lightlike or spacelike motion for the spinning massive particle, if it is positive, null or negative, respectively. Anew, the lightlike or the spacelike behavior of the top's interval will depend on the distance and on its properties. For some appropriated values of the energy, mass and angular momentum of the massive particle, the spinning massive particle can be have different behaviors for the velocities in some part of the trajectory.

6.1.4. Inflation scenario. As a final example, we perform a theoretical exercise studying a universe filled with dark energy. In this case the universe will be in an inflationary state with a cosmological constant. A simplest case is to consider a fluid with the equation of state $\alpha = -1$ [30, 31]. Thus, the conformal factor acquires the form

$$\Omega = \frac{Q_1}{r - Q_2}, \tag{147}$$

while from (121) we find that $\eta = Q_2 J^2/(Q_1^2 r m^2 c^2)$. Remarkably, this metric produces huge simplifications. It implies that $\zeta = -1$. Also, we obtain that $\gamma = r(J^2 - Q_1^2 m^2 c^2)/(Q_2 J^2 - rQ_1^2 m^2 c^2)$, and the momenta are

$$P_\phi = \frac{1}{1-\eta}\left(-j \pm \frac{EJQ_2}{mc^2 Q_1}\right), \quad P_t = \frac{1}{1-\eta}\left(E \mp \frac{jJ}{mrQ_1}\right). \tag{148}$$

Finally, we get that the interval (130) of the top particle is controlled only by Λ (due to the fact that $\zeta^2 = 1$). For this case, we get

$$\frac{ds^2}{c^2\, dt^2} = \frac{m^2 c^2 \Omega^4}{\left(Q_1^2 r m^2 c^2 - Q_2 J^2\right)^4 P_t^2}\left[c^2\left(J^2 Q_2 - Q_1^2 r m^2 c^2\right)^4 \right.$$
$$\left. - J^2 (r - Q_2)^3 (jQ_1 mc^2 \mp EJQ_2)^2 \left(J^2(r + Q_2) - 2Q_1^2 r m^2 c^2\right) \right]. \tag{149}$$

The interval ds^2 could be null or negative, and the motion could be lightlike or spacelike. This condition also depends on the distance. So, a spinning massive particle moving in a universe with dark energy could reach velocities larger than speed of light due to its spin.

7. Conclusions

We have find exact solutions for the motion of a spinning massive particle in different spacetimes. First, we study two are cosmological models, the Friedmann–Robertson–Walker and Gödel spacetimes. Later we study a general Schwarzschild-like spacetime and finally the static spherically symmetric conformally flat spacetime.

The top's motion in each metric is exhibited in detail. For the case of the FRW spacetime, we found a new conserved quantity that allow us to solve completely the problem. For the general Schwarzschild-like spacetime the case of the Reissner–Nordstrom–(Anti)de Sitter metric is shown, whereas for the conformally flat spacetimes, we show different solutions for different universes filled with a perfect fluid and/or radiation.

In any case, the solutions are written in terms of the momenta and velocities of the spin particle. The spin strongly modifies the dynamics of the spinning particle (as compared to the spinless case), which modifies the momenta, the velocities, and, in particular, the nature of its line element for the orbital motion. Although most of the solutions produce timelike trajectories with $ds^2 > 0$, for various cases there are some specific relations between the energy, the spin and the angular momentum that can give rise to luminal ($ds^2 = 0$) or superluminal ($ds^2 < 0$) motion for massive particles. These are the cases of the motion for the spin particle in the FRW, general Schwarzschild-like and static conformally flat spacetimes. These results generalize those found for spinning massive particles in a Schwarzschild background [14]. The results presented here seem to indicate that the effects due to spin are robust.

Acknowledgments

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