

# Hot Fluids and Nonlinear Quantum Mechanics

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**Abstract** A hot relativistic fluid is viewed as a collection of quantum objects that represent interacting elementary particles. We present a conceptual framework for deriving nonlinear equations of motion obeyed by these hypothesized objects. A uniform phenomenological prescription, to affect the quantum transition from a corresponding classical system, is invoked to derive the nonlinear Schrödinger, Klein–Gordon, and Pauli–Schrödinger and Feynman–GellMaan equations. It is expected that the emergent hypothetical nonlinear quantum mechanics would advance, in a fundamental way, both the conceptual understanding and computational abilities, particularly, in the field of extremely high energy-density physics.

**Keywords** Nonlinear quantum mechanics · Fluids · Temperature · High energy density physics

## 1 Introduction

This paper is an attempt to construct a nonlinear quantum mechanics (NQM) that will describe an assembly of thermally interacting (exchanging energy) elementary particles. Using classical nomenclature, such an assembly may be called a fluid (it can have charge and other internal degrees of freedom). The object that emerges from this quantum formulation will be called a Fluidon.

The moniker of nonlinear quantum mechanics is, conventionally, applied to a rather vast body of literature examining whether “nonlinear quantum mechanics” is possible at a fundamental level. Nonlinear generalizations of the Schrodinger equation (in particular with

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an additional logarithmic nonlinear term) have been proposed, and critically examined by several authors. References [1–13, 15–17] form a representative, but by no means an exhaustive, set dealing with the strengths and weaknesses of the theory. The polemic discussion on the “fundamentality” of nonlinear quantum mechanics is still continuing, and to the best of our knowledge, there is no experimental evidence of departures from the predictions of the standard quantum mechanics. The essence of the for and against arguments, extracted from the introduction of Ref. [16], may be summarized as:

- a) The main idea is to replace the linear Hamiltonian operator generating the dynamics of the state vector with a state-dependent operator in such a way that the norm of the state vector remains constant. The properties of such nonlinear observables were subsequently studied in detail by Weinberg.
- b) Gisin and Polchinski [9, 10, 12] argued that the existence of dynamical nonlinearities in quantum mechanics might lead to undesirable physical features, such as the possibility of superluminal EPR communication.
- c) Main issue associated with dynamics of the so called Mielnik-Kibble-Weinberg (MKW) type of quantum mechanics is that the evolution of a density matrix depends in general on the specific choice of pure-state mixture underlying the initial density matrix.
- d) Thus, by observing the dynamics, one might be able to infer the particular choice of mixture associated with the initial density matrix (Haag, Bannier and Waniewski [13, 14]) which is contrary to the fundamental notion that the initial density matrix is a sufficient characterization of the initial state of the system, and that its composition is irrelevant.
- e) Another potentially problematic aspect of nonlinear quantum mechanics was suggested by Peres [15], who provided an example where the Von Neumann entropy decreases with time.
- f) Tight experimental bounds on deviations away from the linear evolution law have been found, while criticism of the MKW theory has been strengthened further by Mielnik [4].
- g) As a consequence of these studies, one might conclude that a consensus has emerged to the effect that nonlinear quantum mechanics of the MKW type must be ruled out on physical grounds.
- h) By proposing a constrained QM, Brody et al. [16] demonstrate that although the resulting equations of motion are nonlinear, the associated evolution is autonomous and hence independent of the choice of initial mixture- a possible viable step towards an acceptable generalization of the unitary evolution of quantum mechanics.

The work presented here, however, has no direct connection to the branch of investigation described above; we are not proposing a fundamental revision of the basic linear equations of QM. Instead, our goal is to hypothesize a “quantum” formalism that may be able to describe collective nonlinear modes of a hot interactive collection of fluid elements (Fluidons). The theory developed here is both heuristic, and hypothetical, and its usefulness will have to be eventually tested. The most likely arena will be the extremely high energy-density physics.

The Historical background relevant to the present work is rather extensive. Exploring the classical-quantal interplay to create, perhaps, a basis for a classical style visualization of quantum mechanics has attracted a variety of researchers ever since the dawn of quantum mechanics. Although, to the best of our knowledge, this quest has not yielded very fruitful

insights, the interest in this deep and fascinating subject has waxed and waned but it has never disappeared. In the literature, two main trends are easily discernible:

- 1) *Quantum to Classical*: The first attempts were focused in the construction of a fluid-like system from the standard equations of quantum mechanics, the Schrödinger, the Pauli–Schrödinger, the Klein–Gordon, the Feynman–Gellmann and the Dirac equations [18–41]. In the earlier years of quantum mechanics, this enterprise had a limited scope; the investigators were content to devise appropriate fluid-like variables obeying the “expected” fluid-like dynamics epitomized in the continuity and the momentum balance equations. Quantum mechanics entered the latter through a variety of “quantum forces” proportional to powers of  $\hbar$ ; the simplest easily recognizable quantum force came from the well-known Bohm potential. The fluidized system, of course, was equivalent to the original quantum one. In more recent years, however, an impressive regeneration of the fluidization project was fueled by attempts to investigate the collective macroscopic motions accessible to a fluid (plasma) whose elementary constituents follow the laws of quantum rather than classical dynamics [42]. Much progress has been made in the direction of building and exploring macroscopic theory of quantum plasmas [41, 43–51] as opposed to the earlier efforts that mostly consisted of casting quantum mechanics into a fluid-like mould. New plasma phenomena, originating in the quantum nature of the constituent particles, have come into light.
- 2) *Classical to Quantum*: The reverse path has also been heavily travelled- there exist many “derivations” of the elementary equations of quantum mechanics from classical kinetic (phase space) and fluid (configuration space) theories [52–54, 56]; the quantum equations follow when very specific choices of the distribution function (or the fluid variables) are made. In most of these investigations, barring a very recent paper [57], the sought after result was the standard linear equations of quantum mechanics, the Schrödinger [52, 53], the Pauli–Schrödinger [54], the Klein–Gordon and the Feynman–Gell-Mann version [55] of the Dirac equation [56].

This paper would fall in the second category. It is difficult to obtain a full expression of the state of a fluid with collective features like temperature (and an internal energy) in an elementary linear quantum equation. We will, therefore, seek a quantum description appropriated to a relativistically hot fluid. Not unexpectedly, this description turns out to define a nonlinear quantum mechanics; the nonlinearity is an expression of the internal energy of the fluid. The latter insures that the fluid is not just a collection of noninteracting “quantum” particles.

Our procedure is fully inspired by the very original formulation of quantum mechanics where, for example, the Schrödinger and the Klein–Gordon equations were derived by the respective identifications

$$E_{nonrel} = i\hbar\partial_t, \quad \mathbf{p} = \frac{\hbar}{i}\nabla \quad (1)$$

$$p^\mu = \frac{\hbar}{i}\partial^\mu \quad (2)$$

where  $p^\mu = [E, \mathbf{p}]$  is the momentum four-vector,  $E_{nonrel} = E - m$ . Throughout this paper, the speed of light  $c = 1$ . The Schrödinger equation can be viewed as a phenomenological equation obtained by the above mentioned identification.

Our objective is to seek a similar, hopefully unique, quantum prescription to develop a (phenomenological) quantum theory of a hot fluid. We will illustrate the general development by working with the simplest example of such a system - a perfect isotropic field.

### 2 Hot Relativistic Perfect Fluid

The energy momentum tensor for an isotropic relativistic (both thermally and kinematically) perfect fluid composed of particles with no internal degree of freedom may be written as

$$T^{\mu\nu} = \frac{n}{w} f p^\mu p^\nu + \Pi \eta^{\mu\nu}, \tag{3}$$

where  $p^\mu$  is the fluid kinematic momentum,  $\Pi$  is the pressure,  $f$  is the enthalpy,  $n$  is the invariant number density, and  $w$  is a constant with dimensions of mass to insure that  $T^{\mu\nu}$  has the dimensions of energy density. Although the energy momentum tensor may seem to represent a "river" of a perfect fluid, it is assumed that the "river" is actually "made" of individual particles. The energy momentum tensor of (3), for example, corresponds to an assembly of particles obeying a relativistic Maxwellian distribution. Following [58], we choose the Minkowsky signature tensor to be  $\eta^{\mu\nu} = [-1, 1, 1, 1]$ . Notice that  $T^{\mu\nu}$  in (3) is the same as the expression in standard textbooks because  $nfw = U + \Pi$ , where  $U = T^{00}$  is the proper energy density.

The invariant density  $n$  appears in the continuity equation obeyed by the fluid

$$\partial_\mu (n p^\mu) = 0. \tag{4}$$

For a fluid that has a relativistic temperature in addition to a relativistic directed motion, the bare kinematic momentum ceases to be the right dynamic variable in terms of which the equations of motion have a "simple structure"; it also ceases to be the appropriate variable in terms of which, via the minimal prescription  $p^\mu \rightarrow p^\mu + q A^\mu$ , one can correctly incorporate electromagnetism into the fluid dynamics ( $A^\mu$  is the vector potential). It was shown in a 2003 paper [59] that, for a relativistically hot fluid, the thermally modified "effective momentum"

$$p^\mu_{eff} \equiv P^\mu = f p^\mu, \tag{5}$$

is endowed with both of these desirable properties. The effective momentum, obviously, reflects the thermal enhancement of the rest mass by a factor  $f$  (that depends only on the thermal distribution of the particles). Using (5), one may write (3) and (4) as

$$T^{\mu\nu} = g P^\mu P^\nu + \Pi \eta^{\mu\nu}, \tag{6}$$

$$\partial_\mu (g P^\mu) = 0. \tag{7}$$

where  $g = n/(fw)$ .

#### 2.1 Quantization for the Effective Momentum

It is proposed that the transition to the equivalent "quantum mechanics of a Fluidon" is affected by the prescription

$$g P^\mu P^\nu \implies g \left( P^\mu P^\nu + \frac{\hbar}{2i} \partial^\mu \frac{\hbar}{2i} \partial^\nu \ln g \right). \tag{8}$$

This prescription looks both similar to, and distinct from the conventional prescription  $p^\mu = -i\hbar\partial^\mu$  invoked in particle quantum mechanics. This prescription, in essence, introduces

relativistic Bohm potential to affect the transition from a classical system to a quantum system. Comments on this prescription will be offered below after the required equations are obtained. The transformed energy-momentum tensor [via prescription (8)]

$$T_q^{\mu\nu} = g P^\mu P^\nu + \Pi \eta^{\mu\nu} - \frac{\hbar^2}{4} g \partial^\mu \partial^\nu \ln g, \tag{9}$$

has been labelled with the subscript  $q$  [to differentiate it from (6)]. We will first work out the quantum dynamics of a free Fluidon; the only forces that the Fluidon experiences are inertial and thermal. Then, the equation of motion

$$0 = \partial_\mu T_q^{\mu\nu}, \tag{10}$$

in conjunction with the continuity (7), yields ( $\zeta = \ln(n/f)$ )

$$0 = \frac{1}{g} \partial^\nu \Pi + P^\mu \partial_\mu P^\nu - \frac{\hbar^2}{4} \partial^\nu \left( \frac{1}{2} \partial_\mu \zeta \partial^\mu \zeta + \square \zeta \right) \tag{11}$$

Notice that the quantum prescription (8) was so chosen that the term proportional to  $\hbar$  appears as a perfect four-gradient in the equation of motion. The other two terms on the right hand side of (11) will also become perfect four gradients provided: 1) the effective momentum, itself, is the four gradient of a scale field,

$$P^\mu = \partial^\mu S, \tag{12}$$

that is, it is vorticity-free in relativistic space time [56], and 2) there exists an equation of state of the form

$$\Pi = \Pi(g) \tag{13}$$

When both these conditions are satisfied, the equation of motion may be integrated to yield

$$d = \bar{\Pi} + \frac{1}{2} (\partial_\mu S) (\partial^\mu S) - \frac{\hbar^2}{4} \left( \frac{1}{2} \partial_\mu \zeta \partial^\mu \zeta + \square \zeta \right) \tag{14}$$

where  $d$  is an integration constant and  $\bar{\Pi}(g)$  is determined by

$$\bar{\Pi} = \int d(\ln g) \frac{d\Pi}{dg}. \tag{15}$$

In terms of  $S$  and  $\zeta$ , the continuity equation reads

$$0 = \partial_\mu \partial^\mu S + \partial_\mu \zeta \partial^\mu S, \tag{16}$$

In two direct and straightforward algebraic manipulations, consisting of first introducing an intermediate variable,

$$\Omega = \frac{\zeta}{2} + \frac{i}{\hbar} S, \tag{17}$$

and then defining the wave function [34, 40]

$$\ln \Psi = \Omega \implies \Psi = \sqrt{\frac{n}{f}} e^{iS/\hbar}, \tag{18}$$

we derive from (14) and (16), the sought after quantum equation

$$\left[ \hbar^2 \partial_\mu \partial^\mu - \bar{\Pi} + d \right] \Psi = 0, \tag{19}$$

obeyed by the Fluidon. Interestingly, the probability density

$$\Psi^* \Psi = \frac{n}{f} \tag{20}$$

associated with the Fluidon is not the original number density of the fluid elements; it is modulated by the thermal factor  $f$ .

In order to analyze the content of (19), we must comment on the thermodynamic quantities- the pressure  $\Pi$ , the enthalpy factor  $f$ , and the equation of state- that have appeared in our derivation. In general, these quantities will depend on the nature of the particles (bosons or fermions or the Maxwell- Boltzmann approximation), and the energy regime (relativistic, non-relativistic). Although, one would deal with Fermions through the Pauli-Schrodinger and Feynman-GellMann equations (appearing later in the text), here we give illustrative examples based, primarily, on the Maxwell-Boltzmann approximation for which analytical results are available. For the relativistic Maxwell Boltzmann, the enthalpy factor turns out to be  $f = K_3(m/T)/K_2(m/T)$  where  $K_s$  are the MacDonald functions of order  $s$  (Hankel functions of the second kind); the fluid obeys the implicit and complicated equation of state  $\Pi \exp[-(m/T)f] = T^2 K_2(m/T)$ . It is useful to note that, in the non relativistic limit ( $m/T \gg 1$ ),  $f = 1 + 5T/(2m)$ , and the equation of state becomes the familiar  $\Pi/T^{5/2} = Const = \Pi/n^{5/3}$ . In the super relativistic limit,  $f$  approaches  $4T/m$  and the equation of state, again, takes the well-known ultra-relativistic form  $\Pi/T^4 = Const = \Pi/n^{4/3}$ . For the Maxwell-Boltzman systems, the pressure goes to zero as temperature goes to zero. For Fermions, however, there is a residual degeneracy pressure even in the limit of zero temperature. The reader is referred to [59, 64, 66] for details.

### 2.2 Interpretation-Conjectures

The hypothesized Fluidon quantum equation is likely to open exciting new channels for exploring many body physics via a relatively simple nonlinear equation. We will attempt to extricate some obvious features of the rich “content” of the quantum (19):

- 1) Let us first examine the zero temperature limit: For the spineless Klein-Gordon fluid, zero T implies no internal thermal energy, and no thermal pressure  $\Pi$  (hence no  $\bar{\Pi}$ ). In this limit, the corresponding enthalpy factor goes to unity  $f \rightarrow 1$  as shown in the last paragraph of the preceding section. The nonlinear quantum equation, then, reduces to the standard linear Klein-Gordon (KG) equation

$$[\hbar^2 \partial_\mu \partial^\mu - m^2]\Psi = 0, \tag{21}$$

when the constant  $d$  is chosen to be  $-m^2$ . The wave function  $\Psi$  also simplifies to

$$\Psi = e^\Omega = \sqrt{n} e^{iS/\hbar} \tag{22}$$

so that  $n = \Psi^* \Psi$  represents the conventional probability density. Thus, the quanta emerging from the application of the quantization prescription (8) to a fluid with no internal energy (and with a vorticity free kinematic momentum), obey the linear KG equation, exactly like a free elementary particle with the relativistic energy-momentum relation  $E^2 = p^2 + m^2$ . Thus a Klein-Gordon fluid at zero temperature and may be perceived as consisting of non-interacting free quanta. This identification/recognition defines the baseline for further discussion.

- 2) An exciting new vista opens up when we switch on the internal thermal energy; that will reinstate both  $\bar{\Pi}(g)$  and  $f$ . The first direct consequence is that  $g = (1/w)n/f =$

$\Psi^* \Psi/w$  implying  $\bar{\Pi}(g) \equiv \bar{\Pi}(\Psi^* \Psi)$ . The dynamics associated with the Fluidon (born out of a hot fluid) is, then, governed by the nonlinear KG equation

$$\left[ \hbar^2 \partial_\mu \partial^\mu - \bar{\Pi}(\Psi^* \Psi/w) + d \right] \Psi = 0, \tag{23}$$

where the nonlinear term is displayed as a function of the probability density  $\Psi^* \Psi$ . The exact expression for the nonlinear term will, of course, depend on the equation of state. *As remarked earlier, the wavefunction  $\Psi$  no longer measures the conventional probability density  $n$ , but a thermally weighted effective probability density.*

- 3) The transition from a cold to a hot fluid translates into a profound transition for the respective fluid quanta - the cold Fluidons are non-interacting and follow a linear KG equation, while the hot Fluidons are interactive and obey a KG equation with an added nonlinear term; the nonlinear term reflects the internal thermal energy. One could, perhaps, dare to venture that the proposed nonlinear quantum mechanics may constitute a proper “replacement” of the standard linear quantum mechanics for describing an “elementary” constituent of a system of thermally interacting particles.

### 2.3 The Nonlinear Term

To get a feel for the nonlinear term, let us take a specific equation of state

$$\Pi = ag^\Gamma \longrightarrow \bar{\Pi} = \frac{a\Gamma}{\Gamma - 1} g^{\Gamma-1} \tag{24}$$

This choice converts (23) into

$$\left[ \hbar^2 \partial_\mu \partial^\mu - \lambda(\Psi^* \Psi)^{\Gamma-1} + d \right] \Psi = 0, \tag{25}$$

where  $\lambda = (a/w^{\Gamma-1})\Gamma/(\Gamma - 1)$  is a fluid specific constant. For normal matter,  $\Gamma > 1$  implying that both  $\lambda$ , and the exponent of the nonlinear term, are positive definite. For exotic matter,  $\lambda$  need not be positive definite. Investigation of this nonlinear equation is likely to bring out very interesting physics. For instance, for  $\Gamma = 2$ , (25) corresponds to the highly investigated  $\Psi^4$  theories that have been invoked as models for spontaneous symmetry breaking (when the vacuum does not have the symmetries of the Lagrangian). The choice  $d = -m^2$ , leads to a nonlinear extension of the KG field, but if  $d = \mu^2 > 0$ ,  $\lambda > 0$ , the field develops a finite vacuum expectation value.

It is interesting to note that the Fluidon energy momentum relation is fundamentally different from that of a linear KG particle. It is explicitly seen for a particle with real mass ( $d = -m^2$ ), by substituting the energy- momentum eigenfunction

$$\Psi = \Psi_0 e^{i\mathcal{P}_\mu x^\mu / \hbar}, \tag{26}$$

where  $\mathcal{P}_\mu = [\mathcal{E}, \mathbf{P}]$  is the four momentum and  $\Psi_0$  is a constant amplitude, into (23). We find

$$\mathcal{E}^2 = \mathbf{P}^2 + m^2 + \lambda \Psi_0^{2(\Gamma-1)}; \tag{27}$$

the last amplitude dependent term is the expression of nonlinearity. This contribution could be viewed as an effective mass and could dominate the intrinsic rest mass for sufficiently high pressure.

In the context of preceding discussion, Fluidons could be viewed as “many-body” quanta that carry (through the nonlinearity in the equation) the extra information characteristic of an interacting many-body system like a hot fluid. The Fluidons, governed by the nonlinear KG (25), are thermally interacting spin-less bosons. In the next section we will talk about Fluidons that emerge when the hot fluid has additional internal structure.

### 2.4 Switching on the Electromagnetic Field

If the hot fluid is composed of electrically charged Fluidons, then the electromagnetic interaction can be readily worked in. The most obvious and straightforward procedure is to begin by appropriately change the equation of motion  $\partial_\mu T_q^{\mu\nu} = 0$  to

$$\partial_\mu T^{\mu\nu} = q \frac{n}{w} F^{\mu\nu} p_\mu = q g F^{\mu\nu} P_\mu \tag{28}$$

where  $q$  is the elementary charge,  $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$  is the Faraday tensor, and  $A^\mu = [\phi, \mathbf{A}]$  is the electromagnetic four potential. Following exactly the procedure laid out in Section 2.2 - but demanding that, now, the combination  $P^\mu + qA^\mu$  be a perfect gradient

$$P^\mu + qA^\mu = \partial^\mu S, \tag{29}$$

one ends up deriving the equation of motion of the KG Fluidon subjected to an electromagnetic field,

$$\left[ -(i\hbar\partial_\mu - qA_\mu)(i\hbar\partial^\mu - qA^\mu) - \lambda\bar{\Pi}(\Psi^*\Psi) + d \right] \Psi = 0, \tag{30}$$

where we have switched back to the general form of the nonlinear term  $\lambda\bar{\Pi}(\Psi^*\Psi)$ . Indeed, we would have gotten this result without any extra calculation - by simply invoking the minimal coupling prescription (pertinent to a hot relativistic fluid) that was worked out in Ref. [59],  $gp^\mu = P^\mu \rightarrow P^\mu + qA^\mu$ , precisely what transpired in going from (12) to (29). When this replacement is made, the effective fluid momentum  $P^\mu$  will, in general, have vorticity but it is the generalized momentum,  $P^\mu + qA^\mu$ , that is vorticity free [59].

### 2.5 Non Relativistic Limit- Landau Ginzburg Model

Here we follow the opposite of the standard practice of building the theory from the simpler non-relativistic to the more involved relativistic systems. The non relativistic version of the nonlinear KG equation follows by recalling that, in this limit, the rest mass is much greater than the kinetic and thermal energies ( $\mathcal{E} \simeq m$ ). Then

$$\mathcal{E}^2 - m^2 = (\mathcal{E} - m)(\mathcal{E} + m) \simeq 2m E_{nonrel} \tag{31}$$

where  $E_{nonrel}$  is the energy without the rest mass. Substituting (31) into (27), switching on the electromagnetic field, and affecting  $E_{nonrel} \rightarrow i\hbar\partial_t, \mathcal{P}_k \rightarrow -i\hbar\partial_k$  (latin indices go from 1-3 as greek indices go from 0-3), we obtain

$$i\hbar \frac{\partial\Psi}{\partial t} = \left[ -\frac{\hbar^2}{2\mu} \left( \partial^k - \frac{iq}{\hbar} A^k \right)^2 + q\phi + \lambda|\Psi^*\Psi|^{(\Gamma-1)} \right] \Psi, \tag{32}$$

a nonlinear Schrödinger equation satisfied by a non relativistic Fluidon interacting with an electromagnetic field. For  $\Gamma = 2$ , the Schrodinger Fluidon follows the same equation as the Super-electron of the Landau-Ginzburg model [60]. Notice that we could just as well derive (32) by directly working with the non relativistic limit of the energy momentum tensor.

For arbitrary  $\Gamma$ , and even for equations of state more complicated than the adiabatic law (24), the non relativistic Fluidon will always will be described by what may be called a generalized Nonlinear Schrödinger equation (NSE) where the nonlinear term can be an arbitrary function  $f(|\Psi^*\Psi|)$  [61]. The NSE has been one of most investigated equations and has yielded results of immense significance spanning classes of stable localized solutions (including solitons) [61]. Thus the investigative apparatus to study Fluidon dynamics is already in place.

Note that there are papers in literature where the opposite path, fluidization of the nonlinear Schrödinger equations [30–40], was explored.

### 3 Hot Fluids with an Internal Degree of Freedom

The single component KG Fluidon represents a many-body manifestation of a system of thermally interacting spin-less bosons. Being relativistic counterpart of the Schrödinger Fluidon, it may be invoked to model the many-body behavior of an assembly of relativistic electrons, but such a description cannot be fully adequate since the model lacks, *inter alia*, the intrinsic spin of the constituent particles.

In this section we will carry out fluid-quantization for a perfect fluid whose elementary constituents have intrinsic spin. Ideally, in analogy with the preceding sections on the nonlinear KG, one would expect to be able to derive a nonlinear Dirac equation for the spin half particles. Although the algebra for such an exercise is straightforward (though long and tedious), there are some conceptual problems that we have not yet been able to resolve. In fact, we have not been able to derive a nonlinear Dirac equation per se; what we have obtained, instead, is the nonlinear generalization of the so called Feynman-Gell-Mann (FG) equation [55]. Although we defer discussions on the ramifications of the FG to a forthcoming paper, we will summarize the derivation in [Appendix](#).

It is natural to expect that a phenomenological non relativistic equation for a spin-half Fluidon will turn out to be a nonlinear Pauli–Schrödinger equation.

#### 3.1 The Non-relativistic Spin-1/2 Fluidon - Nonlinear Pauli–Schrödinger (PS) Equation

In the non-relativistic limit, the vector part of classical energy-momentum conservation law,  $0 = \partial_\mu T^{\mu\nu}$ , reduces to

$$\partial_t T_{0k} + \partial_j T_{jk} = F_k, \tag{33}$$

where  $T_{0k} = mnv_k$  are the mixed, while  $T_{jk} = mnv_j v_k + p \delta_{jk}$  are the space components of the energy-momentum tensor (corresponding to a spinless particle). In (33),  $F_k$  represents all the external forces acting on the system, and  $p$  is the fluid pressure that obeys an equation of state  $p = p(n)$ . We have chosen to work with the velocity rather than the momentum variables for this section.

In order to introduce appropriate spin induced modifications to the energy momentum tensor, we have to draw from the pioneering work of Takabayasi [62], and the procedures worked out in [63]. The first part we borrow is the spin induced stress

$$\sigma_{jk} = (\hbar^2/4m)n\partial_j \Sigma_i \partial_k \Sigma_i, \tag{34}$$

where  $\Sigma_i = \varphi^\dagger \sigma_i \varphi$  is the spin field associated with the Fluidon,  $\sigma_i$  are the Pauli matrices, and  $\varphi$  is the normalized Fluidon spinor ( $\varphi^\dagger \varphi = 1, \Sigma_i \Sigma_i = 1$ ). This term can be viewed as phenomenological and is chosen to reproduce the linear PS equation when the fluid pressure is zero. One is, of course, seeking the equation that  $\varphi$  obeys.

Adding  $\sigma_{jk}$  to the classical stress and applying the non relativistic limit of the quantum prescription (8), we obtain, to the leading order, the appropriate quantum stress

$$\begin{aligned} T_{0k}^q &= mnv_k, \\ T_{jk}^q &= mnv_j v_k - \frac{\hbar^2}{4m} n \partial_j \partial_k \ln n + p \delta_{jk} + \sigma_{jk}, \end{aligned} \tag{35}$$

With this input, the use of the continuity equation

$$\partial_t n + \partial_j (n v_j) = 0, \tag{36}$$

and the identity  $(1/n)\partial_j(n\partial_j\ln n) = \partial_k(2\partial_j^2\sqrt{n}/\sqrt{n})$ , the quantum translation of (33) becomes

$$m\partial_t v_k + m v_j \partial_j v_k = F_k + \frac{\hbar^2}{2m} \partial_k \left( \frac{\partial_j^2 \sqrt{n}}{\sqrt{n}} \right) - \frac{\partial_j \sigma_{jk}}{n} - \frac{\partial_k p}{n}, \tag{37}$$

where  $F_k = eE_k + e\epsilon_{klm}v_l B_m + (2\mu/\hbar)\Sigma_l\partial_k B_l$  ( $\mu = e\hbar/2m$ ), is composed of the standard Lorentz force, and the additional force originating in the internal spin interacting with the magnetic field [62]. The electromagnetic fields may, as usual, be expressed in terms of the scalar ( $\phi$ ) and vector potentials ( $A_i$ ):  $E_k = -\partial_k\phi - \partial_t A_k$ , and  $B_i = \epsilon_{ijk}\partial_j A_k$ .

For the fluid equations to be complete, one needs an equation for the evolution of the spin field  $\Sigma$ . Fortunately, such an equation was derived in Ref. [62]

$$(\partial_t + v_j \partial_j)\Sigma_i = \frac{2\mu}{\hbar}\epsilon_{ijk}\Sigma_j B_k + \frac{1}{mn}\epsilon_{ijk}\Sigma_j \partial_l(n \partial_l \Sigma_k). \tag{38}$$

In order to arrive at the quantum Fluidon equation, that is equivalent to the fluid system described by (33)-(38), we first define the irrotational or vorticity free combination ( $S$  is a scalar to be later identified with action)

$$\partial_j S = m v_j + i\hbar\varphi^\dagger \partial_j \varphi + e A_i, \tag{39}$$

that converts the system of (37)-(38) into the equation  $\partial_k \chi = 0$  (see Ref. [63]), where

$$\chi = \partial_t S - i\hbar\varphi^\dagger \partial_t \varphi + \frac{1}{2}m v_i v_i - \frac{\hbar^2 \partial_j^2 \sqrt{n}}{2mn} + \frac{\hbar^2}{8m} \partial_i S_j \partial_i S_j + \frac{2\mu}{\hbar} S_j B_j + e\phi + \bar{\Pi}, \tag{40}$$

and where the function  $\bar{\Pi} = \bar{\Pi}(n)$ , just as in the KG case, is related to the pressure via

$$n \frac{\partial \bar{\Pi}}{\partial n} = \frac{\partial p}{\partial n}. \tag{41}$$

Choosing the solution  $\chi = 0$ , and literally repeating the KG procedure, (36), (38) and (40) are found to be equivalent to the non-linear PS equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \left( \partial_k - \frac{ie}{\hbar} A_k \right)^2 + \mu B_j \sigma_j + e\phi + \Pi(\Psi^\dagger \Psi) \right] \Psi, \tag{42}$$

for the wavefunction

$$\Psi = \sqrt{n} e^{iS/\hbar} \varphi, \tag{43}$$

where  $\varphi$  is a two component spinor; the norm of the wave function,  $\Psi^\dagger \Psi = n$ , is the Fluidon density. For the polytropic equation of state,  $p = n^\Gamma$ , the nonlinearity takes the form  $\Pi = \lambda n^\Gamma = \lambda(\Psi^\dagger \Psi)^\Gamma$  where  $\lambda = a\Gamma/(\Gamma - 1)$

Nonlinear Pauli-Schrodinger equations have been also studied in Refs. [48, 65]

#### 4 An Exact Solution of the Nonlinear Klein–Gordon (NKG) - Maxwell System

The primary aim of this paper was developing the conceptual foundations for deriving the equations of nonlinear quantum mechanics governing a variety of fluidon fields. This was mostly accomplished in Section 2 and Section 3, and in Appendix. Now we present an exact nonlinear solution that describes the propagation of a circularly polarized electromagnetic

wave in a fully relativistic hot spin less charged fluid. The physics of this system is contained in the NKG equation in the presence of an electromagnetic field [(30) with  $d = -m^2$ ]

$$[(i\hbar\partial_\mu - qA_\mu)(i\hbar\partial^\mu - qA^\mu) + \lambda\bar{\Pi}(\Psi^*\Psi) + m^2]\Psi = 0, \tag{44}$$

and the Maxwell equation

$$\partial_\mu\partial^\mu A^\nu = -4\pi J^\nu, \tag{45}$$

where  $J^\nu$  is the current associated with the KG fields satisfying (44). The four-vector potential of a circularly polarized electromagnetic wave (propagating along the  $z$  direction) has only transverse spatial components. It may be represented as ( $A_0 = 0 = A_z$ )

$$\mathbf{A} = A [\hat{e}_x \cos(kz - \omega t) - \hat{e}_y \sin(kz - \omega t)], \tag{46}$$

where  $\omega(k)$  is the frequency (wave number), and  $A$  is the constant amplitude of the wave. For this wave it is easy to note that  $\mathbf{A} \cdot \mathbf{A} = A^2$ ,  $\nabla \cdot \mathbf{A} = 0$ , and the current density to complete the Maxwell equation is

$$\mathbf{J} = -\frac{q^2(\Psi^*\Psi)}{m}\mathbf{A} = -\frac{q^2n}{mf}\mathbf{A} \tag{47}$$

where where the normalization  $\Psi^*\Psi = n/f$  (20) has been invoked.

With these simplifications, and the further assumption that the wave function has only time and no space dependence, i.e.,

$$\Psi = \sqrt{\frac{n}{f}} \exp\left(\frac{i\gamma}{\hbar}t\right), \tag{48}$$

the resulting NKG equation

$$\hbar^2 \frac{\partial^2 \Psi}{\partial t^2} + q^2 A_0^2 \Psi + \lambda \bar{\Pi}(\Psi^* \Psi) + m^2 \Psi = 0, \tag{49}$$

and the relevant part of the Maxwell equation

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = 4\pi \mathbf{J}, \tag{50}$$

are, simultaneously, solved provided the conditions ( $\omega_p = \sqrt{4\pi q^2 n/m}$  is the plasma frequency)

$$\gamma^2 = m^2 + q^2 A_0^2 + \lambda \bar{\Pi} \left(\frac{n}{f}\right), \tag{51}$$

$$\omega^2 - k^2 = \frac{\omega_p^2}{f}, \tag{52}$$

are satisfied. The former is the effective fluidon energy ( $E = \gamma mc^2$ ) appropriately modified by thermodynamics; the modifications enter the expression through the nonlinear term proportional to  $\lambda$ . The latter reflects a profound change in the dispersion relation (the standard one will be  $\omega^2 - k^2 = \omega_p^2$ ) brought about by the fact that for the fluidon the norm of the wave function stands for an effective density, that equals the standard density divided by the enthalpy factor  $f$ . Since for normal matter  $f > 1$  (for highly relativistic matter  $f \gg 1$ ) [64], the new dispersion relation implies a reduced effective plasma frequency revealing the possibility of a pressure induced transparency. A hot fluidon for the same rest mass density  $n$  will be more transparent to light than a cold system. For highly relativistic temperatures, the effect can be enormous.

### 5 Summary

A phenomenological quantum prescription, very similar to the standard quantum prescription ( $p^\mu = -i\hbar\partial^\mu$ ), applied to a hot relativistic fluid yields a thermal quantum object (fluidon) that obeys a nonlinear quantum equation - a nonlinear modification of the linear Klein–Gordon equation (that becomes a nonlinear Schrödinger equation in the non relativistic limit). The same quantum prescription results in a nonlinear Pauli-Schrödinger equation when the spin degree of freedom is appropriately included in a non-relativistic treatment. The derivation of a nonlinear Dirac equation to describe a Fluidon corresponding to a fluid composed of relativistic spin half particles is underway.

The nonlinear quantum mechanics, proposed in this paper, is likely to be a new and exciting tool for investigating extremely high energy density matter. Analysis of the dynamics in this extreme state of matter is made highly challenging by a deep interaction between relativistic ( kinematic and thermal) and quantum phenomena.

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### Appendix

To accommodate spin (an internal degree of freedom) in a relativistic theory, we must take the following essential steps:

- 1) Modify the classical energy momentum tensor to include a new, relativistically correct, internal (spin generated) stress term

$$\mathcal{T}^{\mu\nu} = g P^\mu P^\nu + \Pi\eta^{\mu\nu} + g \frac{\hbar^2}{8} \partial^\mu M_{\alpha\beta} \partial^\nu M^{*\alpha\beta}, \tag{53}$$

which will transform to the “quantum” energy momentum tensor

$$\mathcal{T}_q^{\mu\nu} = g \left( P^\mu P^\nu + \frac{\hbar}{2i} \partial^\mu \frac{\hbar}{2i} \partial^\nu \ln g \right) + \Pi\eta^{\mu\nu} + g \frac{\hbar^2}{8} \partial^\mu M_{\alpha\beta} \partial^\nu M^{*\alpha\beta} \tag{54}$$

when subjected to the prescription (8). The spin stress is expressed in terms of the fully antisymmetric tensor

$$M_{\alpha\beta} = \frac{\Psi^\dagger \sigma_{\alpha\beta} \Psi}{\Psi^\dagger \Psi}, \tag{55}$$

where  $\sigma_{\alpha\beta} = i\gamma_\alpha\gamma_\beta$ , and  $\gamma^\alpha$  are the Dirac matrices.

- 2) In addition to the contribution to the energy momentum tensor, we have to include the effect of spin-electromagnetic field interaction (through the associated magnetic moment). The complete equation of motion

$$\partial_\mu \mathcal{T}_q^{\mu\nu} = qg F^{\nu\mu} P_\mu + dg M_{\alpha\beta} \partial^\nu F^{\alpha\beta}, \tag{56}$$

then, expands to

$$\frac{1}{g} \partial^\nu \Pi + P^\mu \partial_\mu P^\nu - \frac{\hbar^2}{4} \partial^\nu \left( \frac{1}{2} \partial_\mu \zeta \partial^\mu \zeta + \square \zeta \right) = q F^{\nu\mu} P_\mu + \kappa M_{\alpha\beta} \partial^\nu F^{\alpha\beta} - \frac{\hbar^2}{8} \left[ \partial_{\mu\zeta} \partial^\mu M_{\alpha\beta} \partial^\nu M^{*\alpha\beta} + \partial_\mu (\partial^\mu M_{\alpha\beta} \partial^\nu M^{*\alpha\beta}) \right], \tag{57}$$

where the last three terms on the r.h.s are all spin dependent. Here the constant  $\kappa = -q\hbar/4c$ . In spite of the immensely exaggerated complication of (57) vis a vis (11), the passage to the spin Fluidon quantum equation is quite straightforward, though highly tedious. These are the main steps and the final results:

- a) Following the derivation of the KG Fluidon, we, first find the spin generalized momentum that is a perfect four gradient (vorticity free). Such a combination is found to be

$$P^\mu + qA^\mu - \varsigma \partial^\mu \omega = \partial^\mu S \tag{58}$$

where  $\varsigma$  and  $\omega$  are functions of spin to be explicitly displayed later. The four curl of the effective momentum  $P^\mu$  yields

$$\partial^\mu P^\nu - \partial^\nu P^\mu = -q F^{\mu\nu} - \hbar \Omega^{\mu\nu} \tag{59}$$

where

$$\Omega^{\mu\nu} = -\partial^\mu \varsigma \partial^\nu \omega + \partial^\nu \varsigma \partial^\mu \omega \tag{60}$$

is a kind of spin specific vorticity or curvature, which (with a weight  $\hbar$ ) accentuates the electromagnetic curvature  $F^{\mu\nu}$  (with a weight  $q$ ).

- b) then we introduce the following definitions: The wave function [56]

$$\Psi = \begin{pmatrix} \psi \\ -\psi \end{pmatrix} \tag{61}$$

is a four component spinor with

$$\psi = \sqrt{g} e^{iS/\hbar} \varphi \tag{62}$$

thus the relativistic nature of the spin is represented through a spinor field. Here  $S$  is the eikonal defined by (58),  $g = n/f$ , as in the KG case, is the thermally modified number density, and

$$\varphi = \begin{pmatrix} \cos(\theta/2) e^{i\omega/2} \\ i \sin(\theta/2) e^{-i\omega/2} \end{pmatrix} \tag{63}$$

is a two-component spinor leading to the normalizations

$$\psi^\dagger \psi = g, \quad \Psi^\dagger \Psi = 2g \tag{64}$$

The field  $\theta$  is a parametric replacement of  $\varsigma$ ,

$$\varsigma = \frac{\hbar}{2} \cos \theta \tag{65}$$

Using (61)-(65), we identify various intermediate variables in terms of the spinor  $\varphi$

$$\varsigma \partial^\mu \omega = -i \hbar \varphi^\dagger \partial^\mu \varphi, \tag{66}$$

$$M_{\alpha\beta} = \frac{\psi^\dagger \sigma_{\alpha\beta} \psi}{g} = \varphi^\dagger \sigma_{\alpha\beta} \varphi, \tag{67}$$

$$\frac{1}{8} \partial_\mu M_{\alpha\beta} \partial^\mu M^{*\alpha\beta} = \partial_\mu \varphi^\dagger \partial^\mu \varphi + (\varphi^\dagger \partial_\mu \varphi) (\varphi^\dagger \partial^\mu \varphi), \tag{68}$$

and

$$P^\mu = \partial^\mu S - qA^\mu - i\hbar\varphi^\dagger\partial^\mu\varphi. \quad (69)$$

With the somewhat elaborate machinery, contained in (61)–(69), (57), and the accompanying continuity equation

$$0 = \partial_\mu(gP^\mu) = \partial_\mu\zeta(\partial^\mu S - qA^\mu + \zeta\partial^\mu\omega) + \partial_\mu(\partial^\mu S - qA^\mu + \zeta\partial^\mu\omega), \quad (70)$$

can be manipulated to, finally, derive the nonlinear Feynman-Gell-mann equation

$$\left[ D_\mu D^\mu + \frac{q\hbar}{2c}\sigma_{\alpha\beta}F^{\alpha\beta} + m^2 + \lambda\bar{\Pi}(\Psi^\dagger\Psi) \right] \Psi = 0, \quad (71)$$

where

$$D^\mu = i\hbar\partial^\mu - qA^\mu$$

is the standard gauge derivative, and where the constant of integration was identified with the fermion mass square  $m^2$ . The nonlinear term  $\lambda\bar{\Pi}(\Psi^\dagger\Psi)$ , as before, is fluid specific. The spinor Fluidons, obeying the nonlinear (71), are, then, the quantum objects that embody many body interactions of a thermal fermion assembly. It is totally remarkable that their dynamics is relatively simple- almost like that of an elementary particle.

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