Torsion induces gravity

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In this work the Poincaré–Chern-Simons and anti-de Sitter–Chern-Simons gravities are studied. For both, a solution that can be cast as a black hole with manifest torsion is found. Those solutions resemble Schwarzschild and Schwarzschild-AdS solutions, respectively.

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I. INTRODUCTION

The theory of gravity can be constructed under the idea of a generic covariant theory of fields with second order equations of motion for the metric. In four dimensions this restriction almost univocally leads to the Einstein-Hilbert action (EH) action.

To include fermions requires us to introduce a local orthonormal basis (called vielbein) on the manifold and a connection for the local Lorentz group. If vierbein e^a and Lorentz connection ω^{ab} are understood as independent fields [1] then first order gravity ([2]) arises. Usually in first order gravity the fields are written in terms of differential forms. This theory of gravity is a generalization of the metric gravity, for instance in four dimensions the EH action reads

$$I_{\rm EH} = \int_{\mathcal{M}} R^{ab} \wedge e^c \wedge e^d \varepsilon_{abcd}, \tag{1}$$

where $R^{ab} = d\omega^{ab} + \omega^a_c \wedge \omega^{cb} = \frac{1}{2} R^{ab}{}_{cd} e^c \wedge e^d$. $R^{ab}{}_{cd}$ is the Riemann tensor. $\varepsilon_{abcd} = \pm 1, 0$ stands for the complete antisymmetric symbol.

The equations of motion yielded by the action (1) leads to the vanishing of $T^a = de^a + \omega^a{}_b \wedge e^b = \frac{1}{2} T^a{}_{bc} e^b \wedge e^c$ called the torsion two form with $T^a{}_{bc}$ the torsion tensor. In this way on-shell standard Einstein equations are recovered. From now on the $\wedge$ product will be understood between differential forms.

In higher dimensions the premise of the second order equation of motion for the metric does not restrict the action to EH. In fact there are several sensible theories of gravities whose Lagrangians can be constructed in terms of only R^{ab}, e^a, and $\epsilon_{a_1 \ldots a_d}$. These theories of gravity are known as Lovelock gravities [3] and by construction are manifestly invariant under local Lorentz transformations [1,2]. Although the equations of motion of this gravity do not force a vanishing torsion, the consistency of them, e.g. their covariant derivatives, in general introduce overdeterminant conditions unless torsion vanishes [2].

However, in odd dimensions there is a subfamily of Lovelock gravities whose consistency conditions do not introduce new conditions to be satisfied, thus in general nonvanishing torsion solutions are permitted. These are known as Chern-Simons gravities, and are called this because they coincide with a Chern-Simons gauge theory, respectively, for the Poincaré group for a vanishing cosmological constant ($\Lambda = 0$), and for the (A)dS group if $\Lambda \neq 0$ [4].

Unfortunately only a few solutions of Chern-Simons gravity are known. For $\Lambda = 0$ and $T^a = 0$ the solutions are merely flat spaces with angular defects. For $\Lambda < 0$ and $T^a = 0$ black hole solutions are known [5,6]. The solutions in Ref. [5] have constant curvature, and are obtained through identification of the AdS space. On the other hand, the solution in Ref. [6] in $2n + 1$ dimensions reads

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 d\Omega^2_{2n-1}$$

with $F(r) = 1 - (M + 1)^{1/n} + \frac{r^2}{l^2}$. One can check that this solution is not a perturbation of the $2n + 1$ dimensional Schwarzschild-AdS solution in any sense.

The sections below show that the presence of torsion in the five dimensional Chern-Simons gravity, as an example for higher dimensions, allows solutions that resemble the Schwarzschild solution with either vanishing or negative cosmological constant. In addition in Sec. V it is argued that these solutions can be cast as black holes in a geometric sense. This work is intended as a first step to study the role of torsion in (Chern-Simons) gravity.

In five dimensions the torsion tensor does not appear explicitly in the action since the most general—pure gravitational—Lagrangian invariant under either the Poincaré or (A)dS groups is actually the already mentioned Chern-Simons Lagrangian plus the term [7]

$$K^3 = T_a R^a{}_b e^b. \tag{2}$$

K^3 is, however, an exact differential form, so it does not affect the equations of motion of the Chern-Simons Lagrangian in Eqs. (3) and (10). In this sense Eq. (2) is only relevant up to boundary conditions.

Finally, it is worthwhile to stress that in another context, usually called teleparallel gravity, the role of torsion as an inducer of gravity is well known [8]. This approach, as seen below, is essentially different.

A. The space

The spaces in this work, are topologically cylinders. In general the space $\mathcal{M} = \mathbb{R} \times \Sigma$ where Σ corresponds to a 3-dimensional spacelike hypersurface and $\mathbb{R}$ stands for the time direction. In addition, $\partial\Sigma$ will be considered the union of an exterior and an interior surface, thus $\partial\Sigma = \partial\Sigma_\infty \oplus \partial\Sigma_H$.

II. $\Lambda = 0$ CHERN-SIMONS GRAVITY

The Poincaré Chern-Simons action in five dimensions reads

$$I = \int_{\mathcal{M}} R^{ab} R^{cd} e^f \varepsilon_{abcdf}, \tag{3}$$

yielding the two set of equations of motion

$$R^{ab} R^{cd} \varepsilon_{abcdf} = 0 \quad \text{and} \quad R^{ab} T^c \varepsilon_{abcdf} = 0. \tag{4}$$

A. Spherical symmetric solution

Considering a spherical symmetric solution, one is led to the ansatz

$$e^0 = f(r)dt e^1 = g(r)dr e^m = r\tilde{e}^m, \tag{5a}$$

$$\omega^{01} = c(r)dt \qquad \omega^{mn} = \tilde{\omega}^{mn},$$
$$\omega^{0m} = a(r)\tilde{e}^m \qquad \omega^{1m} = b(r)\tilde{e}^m, \tag{5b}$$

where $\tilde{e}^m$ is a dreivein (with $m = 2, 3, 4$) for the spherical transverse section and $\tilde{\omega}^{mn}$ is its associated Levi Civita (torsion free) connection.

In this ansatz the curvature tensor reads

$$\begin{aligned} R^{01} &= -\frac{1}{g(r)f(r)} \frac{dc(r)}{dr} e^0 e^1 \\ R^{0m} &= b(r)\frac{c(r)}{f(r)r} e^0 e^m + \frac{1}{rg(r)} \frac{da(r)}{dr} e^1 e^m \\ R^{1m} &= \frac{1}{rg(r)} \frac{db(r)}{dr} e^1 e^m + a(r)\frac{c(r)}{f(r)r} e^0 e^m \\ R^{mn} &= (1 + a(r)^2 - b(r)^2)\tilde{e}^m \tilde{e}^n. \end{aligned} \tag{6}$$

The torsion tensor on the other hand reads

$$\begin{aligned} T^0 &= \frac{1}{g(r)f(r)}\left(c(r)g(r) - \frac{df(r)}{dr}\right)e^0 e^1, \\ T^m &= \frac{1}{rg(r)}(1 + b(r)g(r))e^1 e^m - \frac{a(r)}{r} e^0 e^m. \end{aligned} \tag{7}$$

B. Boundary conditions

A boundary condition that $\mathcal{M}$ is asymptotically flat is imposed. Equation (5) satisfies

$$\begin{aligned} &f(r)_{x\to\partial\Sigma_\infty} \to 1, \qquad c(r)_{x\to\partial\Sigma_\infty} \to 0, \qquad g(r)_{x\to\partial\Sigma_\infty} \to 1, \\ &a(r)_{x\to\partial\Sigma_\infty} \to 0, \qquad b(r)_{x\to\partial\Sigma_\infty} \to -1. \end{aligned} \tag{8}$$

Looking for a solution that resembles a five-dimensional Schwarzschild solution, here it is imposed that $f(r) = 1/g(r)$ has a simple zero at certain value $r = r_+$. This condition in manifold with vanishing torsion would determine the presence of a Killing horizon at $r = r_+$, in this case, nonetheless, this must be proven. The boundary condition at $r \to r_+$ is the regularity of R^{ab}.

C. Solution

A solution can be found by imposing that $a(r)^2 = b(r)^2 - 1$ which determines that $b(r)^2 \geq 1$ since $a(r)$ must be real.

Next, imposing the boundary conditions in Sec. II B arises the solution

$$\begin{aligned} f(r) &= \sqrt{1 - \frac{r_+^2}{r^2}}, \qquad g(r) = f(r)^{-1}, \\ c(r) &= \frac{r_+^2}{r^3} - \frac{r_+^3}{r^4}, \qquad b(r) = -f(r) + b_2(r), \end{aligned} \tag{9}$$

where

$$b_2(r) = F(r)\left(C_1 + \int_{r_+}^r d\rho \frac{1}{F(\rho)} \frac{d}{d\rho} f(\rho)\right),$$
$$F(r) = r e^{r_+/r}(r + r_+)^{-1}.$$

It must be stressed that the vanishing of the torsion at $\partial\Sigma_\infty$ determines that

$$C_1 = -\int_{r=r_+}^{\infty} d\rho \frac{1}{F(\rho)} \frac{d}{d\rho} f(\rho),$$

which also satisfies the bound $b(r)^2 > 1$ for $r \in [r_+ \ldots \infty]$. In Sec. V we sketch a demonstration that this solution can be cast as a black hole.

III. $\Lambda < 0$ CHERN-SIMONS GRAVITY

In this section the problem is restudied for $\Lambda < 0$. In five dimensions the AdS–Chern-Simons Lagrangian reads

$$\mathbf{L} = \left(R^{ab} R^{cd} e^f + \frac{2}{3l^2} R^{ab} e^c e^d e^f + \frac{1}{5l^4} e^a e^b e^c e^d e^f\right)\varepsilon_{abcdf}, \tag{10}$$

where the cosmological constant $\Lambda = -6l^{-2}$. This Lagrangian (13) generates the two sets of equations

$$\bar{R}^{ab} \bar{R}^{cd} \varepsilon_{abcdf} = 0 \quad \text{and} \quad \bar{R}^{ab} T^c \varepsilon_{abcdf} = 0, \tag{11}$$

where $\bar{R}^{ab} = R^{ab} + l^{-2} e^a e^b$.

A. Boundary conditions and a solution

To obtain an explicit solution it is still necessary to define boundary conditions. In this case the negative cosmological constant suggests to impose that $\mathcal{M}$ be asymptotically AdS, which implies that

$$f(r)_{x\to\partial\Sigma_\infty}\to\frac{r}{l},\qquad c(r)_{x\to\partial\Sigma_\infty}\to\frac{r}{l},$$
$$g(r)_{x\to\partial\Sigma_\infty}\to\frac{l}{r},\qquad a(r)_{x\to\partial\Sigma_\infty}\to 0,\tag{12}$$
$$b(r)_{x\to\partial\Sigma_\infty}\to-\frac{r}{l}.$$

The presence of a *horizon* will be introduced through imposing that $f(r)=1/g(r)$ has at least a zero, called r_+. The boundary condition as $r\to r_+$ is regularity of $\bar{R}^{ab}$.

Using the spherical symmetric ansatz in Eqs. (5) and imposing $a(r)^2=b(r)^2-1-\frac{r^2}{l^2}$ arises the Schwarzschild-like solution

$$\begin{aligned}f(r)^2&=1-\frac{\mu}{r^2}+\frac{r^2}{l^2}=\frac{(r-r_+)(r+r_+)}{r^2}\left(1+\frac{r^2}{l^2}+\frac{r_+^2}{l^2}\right)\\ g(r)&=f(r)^{-1}\\ c(r)&=\frac{r_+^2}{r^3}\left(1+\frac{r_+^2}{l^2}\right)+\frac{r}{l^2}-\frac{1}{r}\left(1+\frac{2r_+^2}{l^2}\right)\\ b(r)&=-f(r)+b_2(r)\end{aligned}\tag{13}$$

with

$$b_2(r)=F(r)\left(C_1+\int_{r_+}^{r}d\rho\frac{1}{F(\rho)}\left[\frac{d}{d\rho}f(\rho)-\frac{\rho}{l^2f(\rho)}\right]\right),\tag{14}$$

where

$$C_1<-\sqrt{1+\frac{r_+^2}{l^2}}\left(\frac{2r_+^2+l^2}{r_+l}\right),$$

and

$$F(r)=\frac{rl}{r^2+r_+^2+l^2}.\tag{15}$$

Remarkably, unlike $\Lambda=0$, the torsion vanishes asymptotically without further constraints on the integration constants; for instance, C_1 is bounded by the existence of the horizon.

IV. NEAR $r=r_+$ LIMIT

The asymptotical spatial behavior, namely, as $r\to\infty$, is smooth because of the boundary conditions. Although this behavior has been proven to be very relevant [9], for instance for the computation entropy [10], its analysis in detail will be done elsewhere. Here the near $r=r_+$ behavior will be done.

The spin connection ω_c^{ab} is regular everywhere, in particular, at $r=r_+$, thus the minimal coupling is in principle well defined.

Near $r=r_+$, either for $\Lambda<0$ or $\Lambda=0$, the Riemann tensor is smooth, in fact $R^{ab}{}_{cd}\sim\kappa_1+\kappa_2(r-r_+)^q+\cdots$ where κ_1, κ_2 are constants and $q=1,\frac{1}{2}$. Therefore, any scalar constructed exclusively from the Riemann tensor is well defined everywhere.

The components of the torsion tensor satisfy a similar behavior near $r=r_+$ except for

$$T_{01}^0\sim\frac{1}{\sqrt{r-r_+}}+\cdots,\tag{16}$$

which diverges. This divergent behavior determines that any scalar constructed in terms of the torsion tensor must diverge.

V. A HORIZON

The determination of the presence of a horizon in a manifold with vanishing torsion relays basically in the geodesic curves structure. In general, one can argue that $r=r_+$ is a genuine horizon, roughly speaking, only if there no lightlike outward curves connecting $r<r_+$ to $r>r_+$. In principle, one could apply the same definition in a manifold with nonvanishing torsion.

The influence of torsion needs to be discussed. That one component of the torsion tensor diverges at $r=r_+$ could indicate an unphysical behavior of the solutions above, however one must recall that it is not unusual that at a horizon some matter fields could diverge. To proceed one can note that at tree level, integer spin fields, as scalar or gauge fields, couple gravity only through the vielbein (or the metric), therefore the presence of torsion at tree level is irrelevant for these kinds of fields. Half-spin fields, on the other hand, couple gravity through the spin connection which, for the solutions above, are finite as $r\to r_+$. Thus, in a first approximation one can conjecture that the divergence of the torsion is rather benign and, furthermore, it does not alter the geometric definition of a horizon in the usual geometric terms described above.

A divergency in the Riemann curvature R^{ab} is forbidden because otherwise there would be unbounded Hawking radiation from it. In fact, they are only allowed provided they are covered by a horizon. This is the usually called the cosmic censorship conjecture. A divergent torsion tensor is not clear what its effects would be. The analysis of nonvanishing torsion manifolds in an extension of the Hawking mechanism is a very interesting direction to continue this work.

Since the asymptotical region is locally maximally symmetric to determine the classical geometric structure of both solutions above, it is enough to study the behavior near $r=r_+$. Furthermore, both solutions can be studied in a general ground.

In any manifold there are two parallel ways to address the movement of point particles, the straightest and the shortest (or longest) curve (geodesic curve) [11]. Unfortunately, in a manifold with torsion, both definitions disagree. The tangent vector of the geodesic curves satisfies

$$l_g^\mu\left(\partial_\mu l_g^\nu + \begin{Bmatrix} \nu \\ \alpha\mu \end{Bmatrix} l_g^\alpha\right) = 0,$$

where {} is the Christoffel symbol, while the straightest curve satisfies $l_s^\mu(\partial_\mu l_s^\nu + \Gamma^\nu_{\alpha\mu} l_s^\alpha) = 0$ or explicitly

$$l_s^\mu\left(\partial_\mu l_s^\nu + \left(\begin{Bmatrix} \nu \\ \alpha\mu \end{Bmatrix} + \frac{1}{2}(T_\mu{}^\nu{}_\alpha + T_\alpha{}^\nu{}_\mu)\right) l_s^\alpha\right) = 0. \quad (17)$$

Note that the straightest curve satisfies a nonhomogenous geodesic equation in a term proportional to the torsion tensor. Sometimes Eq. (17) is interpreted as a geodesic equation with an effective electric field due to torsion, although this is only apparent [12].

To proceed one can restrict the analysis only to the radial curves, i.e. the $(t - r)$ plane. After solving both equations, the tangent vector of the light like radial geodesic curves reads

$$l_g = \frac{1}{f(r)^2}\frac{\partial}{\partial t} \pm \frac{\partial}{\partial r},$$

while the tangent vector of the lightlike radial straightest curves is

$$l_s = e^{-\int c(r)/f^2}\left(\frac{1}{f(r)}\frac{\partial}{\partial t} \pm f(r)\frac{\partial}{\partial r}\right) = (f(r)e^{-\int c(r)/f^2}) l_g. \quad (18)$$

Since the tangent vectors are parallel one can show that both curves are equivalent modulo a redefinition of the affine parameter.

The timelike curves, on the other hand, are not equivalent. Nonetheless the boundary conditions ensure that as $r \longrightarrow \infty$ both curves must converge.

Since radial lightlike curves are equivalent and l_g depends only on the metric, $r = r_+$ is a horizon in the sense that $r = r_+$ is a lightlike surface. This result grants that the name *black hole* can be applied to the solutions presented in this work.

To determine additional characteristics of the solutions one probably needs to study fields evolving on them, if not the backreaction too. The presence of torsion affects, for instance, the propagation of the electromagnetic fields at higher loops [13].

IV. CONCLUSIONS AND DISCUSSION

In this work it is proven that Chern-Simons gravity in five dimensions has solutions that resemble Schwarzschild black holes. These solutions have nonvanishing torsion, but the standard geodesic structure of a black hole is preserved. In both solutions [Eqs. (9) and (13)], torsion vanishes fast enough, in particular, for $\Lambda < 0$, that its influence can be neglected in a region still far from $\mathbb{R} \times \partial\Sigma_\infty$. Thus both solutions represent interesting generalizations of the Schwarzschild solution, preserving its general behavior.

One can expect that the results in this work can be readily extended to higher odd dimensions, thus Schwarzschild-like black holes with torsion should exist in any odd dimensions within Chern-Simons gravity. This can be particularly interesting in 11 dimensions, where a supergravity theory in terms of a Chern-Simons action is known [14].

Chern-Simons theories, and, in particular, Chern-Simons gravities, are known to have complicated phase spaces [15,16]. These solutions probably can be a contribution to understanding the role of torsion in Chern-Simons gravities, in particular, because the presence of torsion *activates* degrees of freedom usually *turned off* in the torsion-free solutions.

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