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Relativistic quantum vorticity of the quadratic form of the Dirac equation

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Abstract

We explore the fluid version of the quadratic form of the Dirac equation, sometimes called the Feynman–Gell-Mann equation. The dynamics of the quantum spinor field is represented by equations of motion for the fluid density, the velocity field, and the spin field. In analogy with classical relativistic and non-relativistic quantum theories, the fully relativistic fluid formulation of this equation allows a vortex dynamics. The vortical form is described by a total tensor field that is the weighted combination of the inertial, electromagnetic and quantum forces. The dynamics contrives the quadratic form of the Dirac equation as a total vorticity free system.

Keywords: relativistic quantum mechanics, hydrodynamical version, Feynman–GellMann equation

Vorticity is a key concept in fluid theories. The unifying role of canonical vorticity was highlighted in an unified magnetofluid theory for a fully relativistic (in temperature as well as in directed energy) charged fluid [1]. On the other hand, there has also been additional conceptual advancement in the sense that quantum effects could be cast in mathematical forms that are very similar to the the canonical vortex dynamics of classical charged fluids [2, 3]. This required the emergence of a 'vorticity-like' quantum object that could seemlessly combine with the canonical vorticity. As a natural extension of this approach, quantum vorticity was introduced by Takabayasi in [3] in the vortex dynamical formulation of a non-relativistic quantum fluid whose constituents are described by the Pauli–Schrödinger equation.

The aim of this paper is to construct a fully relativistic quantum vortex dynamics of the hydrodynamical version of the quadratic form of the Dirac equation. This equation is obtained by squaring the Dirac operator, and it is sometimes known as the Feynman–Gell-Mann equation (FGE) [4]. The quadratic form of the Dirac equation was first used as a possible model for studying β-decay [4, 5]. It has also been invoked to construct relativistic quantum kinetic theories of spinor fields in external abelian and non-abelian gauge fields including, for instance, in deriving the transport equations for quantum chromodynamic Wigner operator [6–8].

Along this work we demonstrate that the vortex hydrodynamics of the FGE allows an unification of forces; the inertial, the electromagnetic and the quantum one. The final expression will be the construction of a total fully antisymmetric second order tensor mathematically akin to the Faraday tensor.

Ideally one should begin and manipulate the Dirac equation for such an enterprise. Our attempts at such a construction have not yet led to a transparent and simple theory. We have been, however, successful in appropriately transforming the FGE. The basic motivation was to construct a relativistic equation for a massive spin-1/2 particle in terms of a two-component spinor satisfying a second-order wave equation rather than the standard Dirac version of a four-component spinor obeying a first order differential equation. Thus, the quadratic form of the Dirac equation is closer to a Klein–Gordon equation, modified to have an explicit spin term [4, 9, 10].

The hydrodynamical formulation of the FGE presented here differs from previous studies [2, 11, 12] because of its emphasis on a unifying vortex dynamical formulation. For the emergence of such a theory for a relativistic quantum fluid, one must find an appropriate relativistic generalization of the quantum vorticity (originating in the gradients of the spin field) already discussed in the non-relativistic quantum plasma formulations [3, 13].

We begin with the FGE for the four-component spinor Ψ [4]

$$\left(i\hbar\partial_\mu + \frac{q}{c}A_\mu\right)\left(i\hbar\partial^\mu + \frac{q}{c}A^\mu\right)\Psi + \frac{q\hbar}{2c}\sigma_{\alpha\beta}F^{\alpha\beta}\Psi = m^2c^2\Psi \ , \quad (1)$$

where $\hbar$ is the reduced Planck constant, q (m) is the charge (mass) of the particle, c is the speed of light, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic tensor with the electromagnetic four-potential A^μ, $\sigma_{\alpha\beta} = i(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha)/2$ is the antisymmetric spin tensor, and γ_α are the Dirac matrices. Throughout this paper, greek indices (0–3) will be used to label four vectors and tensors, while latin indices (1–3) will denote ordinary vector components. The indices will be raised and lowered by the Minkowski signature tensor $(+,-,-,-)$.

The dynamical equation (1) may be obtained from the Lagrangian

$$\mathcal{L} = \frac{1}{2m}\left(i\hbar\partial_\mu\Psi^\dagger + \frac{e}{c}A_\mu\Psi^\dagger\right)\left(-i\hbar\partial^\mu\Psi + \frac{e}{c}A^\mu\Psi\right) - \frac{mc^2}{2}\Psi^\dagger\Psi - \frac{e\hbar}{4mc}M_{\alpha\beta}F^{\alpha\beta}\Psi^\dagger\Psi, \quad (2)$$

where we have used $q = -e$ as the electron charge, $\Psi^\dagger$ is the conjugate transpose of the spinor Ψ, and the antisymmetric tensor of a local measure of spin

$$M_{\alpha\beta} = \frac{\Psi^\dagger\sigma_{\alpha\beta}\Psi}{\Psi^\dagger\Psi} \ , \quad (3)$$

has been introduced.

As it stands, equation (1), could create confusion; it is a second order operator working on a a four-component spinor. One would think that we have doubled the number of degrees of freedom from the Dirac equation. That, however, is not the case, as we recapture the arguments in [4]. Since each term of FGE (1) commutes with γ_5, the wavefunction satisfy $\gamma_5\Psi = \pm\Psi$ [4, 10]. If one were to choose the positive solution, for instance, the four-component spinor may be represented as

$$\Psi = \begin{pmatrix} \psi \\ -\psi \end{pmatrix}, \quad (4)$$

where ψ is a two-component spinor [4, 10]. Thus, there is only one independent two-component spinor obeying a second order equation. In terms of ψ we find that $\Psi^\dagger\Psi = 2\psi^\dagger\psi$, $M_{j0} = -M_{0j} = i\Sigma_j$, and $M_{ij} = \varepsilon_{ijk}\Sigma_k$, where $\Sigma_i = \psi^\dagger\sigma_i\psi/(\psi^\dagger\psi)$ is the usual definition for the components of a unit spin, with the Pauli matrices σ_i.

The fluidization of FGE is advanced by invoking the Madelung decomposition for the spinor [22]

$$\psi = \sqrt{n}\ e^{iS/\hbar}\begin{pmatrix} \cos(\theta/2)e^{i\eta/2} \\ i\sin(\theta/2)e^{-i\eta/2} \end{pmatrix}, \quad (5)$$

where $\sqrt{n}$ is the amplitude of the wavefunction, such that $n = \psi^\dagger\psi$ can be considered as the fluid density. S is an action-like phase that will, eventually, be related to the 'fluid velocity'. Notice that our decomposition conserves the degrees of freedom; a two-component spinor is expressed explicitly in terms of two spin fields: θ and η. The fields θ and η constitute a parametric representation of the unimodular spin vector with spin components $\Sigma_1 = \sin\theta\sin\eta$, $\Sigma_2 = \sin\theta\cos\eta$, and $\Sigma_3 = \cos\theta$. Notice that, therefore, $\eta = \arctan(\Sigma_1/\Sigma_2)$.

The fluid version of FGE can be achieved easily at Lagrangian level. Employing the decomposition (5), we may rewrite the Lagrangian (2) in terms of the new variables as [23]

$$\mathcal{L} = \frac{n}{m}\left(\partial_\mu S + \zeta\partial_\mu\eta + \frac{e}{c}A_\mu\right)\left(\partial^\mu S + \zeta\partial^\mu\eta + \frac{e}{c}A^\mu\right) + \frac{\hbar^2}{4mn}\partial_\mu n\,\partial^\mu n - mc^2 n - \frac{e\hbar n}{2mc}M_{\alpha\beta}F^{\alpha\beta} + \frac{\hbar^2 n}{4m\kappa}\partial^\mu\zeta\,\partial_\mu\zeta + \frac{n\kappa}{m}\partial^\mu\eta\,\partial_\mu\eta \ , \quad (6)$$

where $\kappa = \hbar^2/4 - \zeta^2$, and we have defined $\zeta = (\hbar/2)\Sigma_3$. The spin variables ζ and η have been used previously in fluid versions of non-relativistic quantum mechanics to define the Clebsch potentials for the quantum vorticity [3, 13].

Classically, the Clebsch potentials were introduced as a method for integration of the hydrodynamical equations [14–16]. The Clebsch potentials has a similar role in fluid dynamics as the one played by the electromagnetic potential in electrodynamics. For this reason, they are important in the study of vortex dynamics [17, 18] and in variational formulations of hydrodynamics [19–21].

Taking the variations for S, n, η and ζ, we can obtain the equivalent set of fluid equations. Defining the fluid four-velocity as

$$v^\mu = \frac{1}{m}\left(\partial^\mu S + \zeta\partial^\mu\eta + \frac{e}{c}A^\mu\right) \quad (7)$$

then we get a set of equations formed by

$$\partial_\mu\left(nv^\mu\right) = 0 \ , \quad (8)$$

$$v^\mu v_\mu = c^2 + \frac{e\hbar}{2m^2c}M_{\alpha\beta}F^{\alpha\beta} - \frac{\hbar^2}{8m^2}\partial_\mu M^{\alpha\beta}\partial^\mu M^*_{\alpha\beta} + \frac{\hbar^2\Box\sqrt{n}}{m^2\sqrt{n}}, \quad (9)$$

$$v^\mu\partial_\mu\zeta + \frac{1}{mn}\partial_\mu\left(n\kappa\partial^\mu\eta\right) + \frac{e\hbar}{4mc}\frac{\partial M_{\alpha\beta}}{\partial\eta}F^{\alpha\beta} = 0 \quad (10)$$

and

$$v^\mu\partial_\mu\eta + \frac{\hbar^2\zeta}{4m\kappa^2}\partial^\mu\zeta\,\partial_\mu\zeta - \frac{\zeta}{m}\partial_\mu\eta\,\partial^\mu\eta - \frac{\hbar^2}{4mn}\partial_\mu\left(\frac{n}{\kappa}\partial^\mu\zeta\right) - \frac{e\hbar}{4mc}\frac{\partial M_{\alpha\beta}}{\partial\zeta}F^{\alpha\beta} = 0 \ , \quad (11)$$

where $\Box = \partial_\mu\partial^\mu$ is the D'Alembertian operator, and $M^*_{\alpha\beta}$ is the conjugate of $M_{\alpha\beta}$. Explanatory remarks are in order. Equation (7) is the usual definition for velocity in hydrodynamical versions of quantum mechanics, and in this case, it is the covariant generalization of the fluid velocity that appears in the fluidization of the Pauli equation (see for example [3, 22, 24]).

The above set is completely equivalent to the FGE (1). Equation (8) is the continuity equation and can be derived directly from (1), as it was shown in [11]. On the other hand, equation (9) represents the normalization of the velocity for the fluid version of the quantum system. The second term in the right-hand side represents the spin-magnetic interaction energy, while the third one is the energy stored in the effective spin pressure. Both the second and third terms are the covariant version of the spin energies of the fluid version of spin quantum systems [3, 13, 25]. The fourth term, proportional to the square of Compton length, is the relativistic form of the Bohm potential [26, 27]. The combination of the Bohm potential and the potential energy due to the spin gradients (the third term) is the total quantum potential of the fluid with spin dependence [28]. Notice that in absence of quantum effects, the classical relativistic normalization for the velocity $v^\mu v_\mu = c^2$ is recovered. Finally, equations (10) and (11) represent the dynamics of the two internal degree of freedom of the fluid. From these equations the vortical dynamics of the spin field can be found.

A standard equation of motion may be derived by manipulating the convective derivative of equation (9)

$$m v^\mu \partial_\mu v^\nu = \frac{e}{c} v_\mu F^{\mu\nu} + \frac{e\hbar}{4mc} M_{\alpha\beta} \partial^\nu \hat{F}^{\alpha\beta} + \frac{\hbar^2}{2m} \partial^\nu \left(\frac{\Box\sqrt{n}}{\sqrt{n}} + \frac{1}{8} \partial_\mu M_{\alpha\beta} \partial^\mu M^*_{\alpha\beta} \right), \tag{12}$$

where we introduce the effective electromagnetic tensor

$$\hat{F}^{\alpha\beta} = F^{\alpha\beta} + \frac{\hbar c}{2en} \partial_\mu \left(n \partial^\mu M^{*\alpha\beta} \right). \tag{13}$$

Equation (12) reflects how the quantum effects enter in the covariant fluid description of the FGE. Notice the resemblance with the classical relativistic fluid equation [1] when quantum effects are ignored. The first term in the right-hand side of (12) is the classical Lorentz force. The second term appears as the effective generalization of the spin interaction with the magnetic field. The spin interact with the effective electromagnetic field (13) that contains a spin self-produced magnetic field due to its possible space-time variations [3, 13, 22]. The last two terms represent the combination of the force due to the relativistic Bohm potential and the spin gradient force. The relativistic form of the Bohm potential has been already derived for the Klein–Gordon equation [26, 27]. The other term is the effective spin pressure stored by the spin gradients [28]. Both are the effective quantum potential for spinning particles. We would like to remark that equation (12) is the covariant generalization of the equation of motion found for the charged fluid version of non-relativistic spin quantum mechanics [13].

We now can construct the vortex dynamics of system. We first show that the equations for the evolution of the two spin fields (equations (10) and (11)) can be combined in an equation for an antisymmetric quantum vorticity tensor $\Omega^{\mu\nu}$ defined as

$$\frac{\hbar}{2} \Omega^{\mu\nu} \equiv \partial^\mu (\zeta\, \partial^\nu \eta) - \partial^\nu (\zeta\, \partial^\mu \eta) = \partial^\mu \zeta\, \partial^\nu \eta - \partial^\nu \zeta\, \partial^\mu \eta\,, \tag{14}$$

depending only on spin derivatives; it is the covariant generalization of the quantum vorticity introduced in [3] and explored in [13]. Also, doing a straightforward algebra, it can be shown that the quantum vorticity tensor (14) may be cast into an alternative form

$$\Omega^{\mu\nu} = \frac{1}{4} M^{\beta\alpha} \partial^\mu M_{\alpha\gamma} \partial^\nu M_\beta{}^\gamma\,. \tag{15}$$

The non-relativistic limit corresponds to the quantum vorticity of Pauli equation [3, 13]. To show it explicitly it is necessary to define (in analogy with the Pauli–Lubanski pseudovector) the pseudovectorial spin vorticity of the fluid $\Omega_\mu = (1/2)\varepsilon_{\mu\nu\alpha\beta} v^\nu \Omega^{\alpha\beta}$. In the non-relativistic limit, this pseudovector correspond to the spin vorticity found in Pauli theory [13].

The dynamics of the quantum vorticity (14) (or (15)) can be obtained from manipulating equations (10) and (11). It is not difficult to show that the quantum vorticity evolves according to

$$v_\mu \Omega^{\mu\nu} = -\frac{e}{2mc} \partial^\nu M_{\alpha\beta} F^{\alpha\beta} - \frac{\hbar}{4mn} \partial_\mu \left(n \partial^\nu M^{\alpha\beta} \partial^\mu M^*_{\alpha\beta} \right) + \frac{\hbar}{8m} \partial^\nu \left(\partial_\mu M^{\alpha\beta} \partial^\mu M^*_{\alpha\beta} \right). \tag{16}$$

Thereby the dynamical evolution of the quantum vorticity is coupled to the evolution of the momentum equation.

The FGE is equivalent to the system of equations formed by (8), (12) and (16), along with the constraint (9). However, a much deeper insight could be achieved by reorganizing the system of equations. Let us define an antisymmetric tensor that will express the dynamical content of the fluid as in [1]. Introducing the dynamical fluid vorticity tensor $S^{\mu\nu} = \partial^\mu v^\nu - \partial^\nu v^\mu$, we can see that equations (12) and (16) lead to the equation $v_\mu \Xi^{\mu\nu} = 0$, or

$$\Xi^{\mu\nu} \equiv m S^{\mu\nu} - \frac{e}{c} F^{\mu\nu} - \frac{\hbar}{2} \Omega^{\mu\nu} = 0 \tag{17}$$

which is consistent with the velocity definition (7). The antisymmetric tensor $\Xi^{\mu\nu}$ may be interpreted as the total relativistic quantum vorticity tensor. This tensor combines (unifies) three fundamental fields (forces) in the fluid description: the electromagnetic field with its charge e, the fluid vortical dynamics with its inertial 'charge' m, and the spin vorticity of the fluid with the effective 'charge' $\hbar/2$.

We can understand the appearance of the spin evolution equation as the necessary dynamical equation that evolves along the momentum equation to preserve the null value of the total vorticity (17) of the system.

In conclusion we have shown that the FGE can be completely described in a hydrodynamical form, where the spinor field cast into the dynamical fluid variables: density, fluid velocity and spin field. This description allows a vortical representation where a relativistic quantum spin vorticity tensor emerges. The spin vorticity depends on the space-time variations of the spin fields (Clebsch potentials). We shown that the FGE fluid behaves as a system in which the inertial, the electromagnetic and the quantum forces are unified through a total relativistic quantum vorticity tensor (17); the latter being the quantum generalization of the classical unified vorticity tensor [1]. The ideal dynamics demands that the total vorticity tensor of the FGE system be zero. Thus the spin fields must evolve synchronously with the canonical fields, so that the combination of them with the spin vorticity tensor always vanishes. The similarities of the spin vorticity tensor with its non-relativistic analogues can suggest a robust structure for this dynamical quantity that must be explored in a deeper way for other quantum theories.

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