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Comparison of the electron-spin force and radiation reaction force

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ABSTRACT

It is shown that the forces that originate from the electron-spin interacting with the electromagnetic field can play, along with the Lorentz force, a fundamentally important role in determining the electron motion in a high energy density plasma embedded in strong high-frequency radiation, a situation that pertains to both laser-produced and astrophysical systems. These forces, for instance, dominate the standard radiation reaction force as long as there is a 'sufficiently' strong ambient magnetic field for affecting spin alignment. The inclusion of spin forces in any advanced modelling of electron dynamics pertaining to high energy density systems (for instance in particle-in-cell codes), therefore, is a must.

Key words: magnetic fields – plasmas – radiation mechanisms: general.

1 INTRODUCTION

Working out the dynamics of a plasma interacting with very strong electromagnetic (EM) fields in extremely high energy setups existing in nature or created by ultrahigh intensity lasers- becomes more and more challenging as the EM fields gain in intensity, and in frequency. Not only does a relativistically description become a must, a whole slew of other effects – such as pair production–annihilation, emission of radiation, radiation reaction, and a variety of quantum electrodynamic effects (QED; Dirac 1938; Bell & Kirk 2008; Kirk, Bell & Arka 2009; Di Piazza, Hatsagortsyan & Keitel 2010; Sokolov et al. 2010; Duclous, Kirk & Bell 2011; Elkina et al. 2011; Di Piazza et al. 2012; Ridgers et al. 2014) will need to be incorporated for a proper description.

Notice that even when a plasma is hit with a very intense optical frequency laser, it can drive the electrons to extreme energies which can, then, emit very energetic gamma rays. One can imagine that the back reaction of these copious and energetic gamma rays can become an important factor in determining the plasmas (electron) dynamics. This general phenomenon of radiation reaction, in fact, has been investigated for a very long time.

But even before one begins to worry about QED-like effects, it makes sense to introduce in the equation of motion an effective force coming from the interaction of the electron spin with the EM field. Since the spin force, in general, depends upon the gradients of the EM field, its contribution can become large if the system is immersed in a very high frequency (short wavelength) radiation field. The model-spin-dependent force we invoke here comes from the semiclassical covariant formulation worked out by Frenkel (1926a,b) a while ago. The Frenkel form, qualitatively similar to the spin-dependent force that can be derived from more recent formulations (Pesci, Goldstein & Uys 2005; Silenko 2008), has been found useful in a wide variety of situations (Tamm 1929; Ternov & Bordovitsyn 1980), and captures the essential features of interest to this conceptual study. For the reader's convenience, a very accessible derivation is given in appendix.

We will study the semiclassical equation of motion of the electron, mostly qualitatively, to bring out the importance of the spin-dependent force in the presence of high-frequency radiation. Similar studies have been performed using particle-in-cell codes (Tamburini et al. 2010). The most interesting part of this work is the demonstration that the spin force can dominate the conventional radiation reaction force in laboratory as well as astrophysical systems.

The spin force can also contribute a new addition to the radiation reaction force. In a future publication, we plan to dwell on this contribution along with treating spin as a dynamic variable in a more detailed setting. This new contribution compares to the standard radiation reaction force just as the spin force does to the Lorentz force.

2 SINGLE-PARTICLE EQUATION OF MOTION FOR A SPINNING PARTICLE

2.1 Conventional radiation reaction

We begin with the semiclassical equation for a spinning electron that may be written as in Bagrov & Bordovitsyn (1980) and Pesci et al. (2005)

$$\dot{p}^\mu = \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = \frac{e}{m}\left(F^{\mu\nu}p_\nu + Q^\mu\right) + R^\mu, \tag{1}$$

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where p^μ is the four-momentum of an electron with charge e and mass m; τ denotes proper time, $F^{\mu\nu}$ is the Faraday tensor, R^μ is the radiation reaction force

$$R^\mu = \frac{2e^2}{3m}\zeta^\mu_\nu \frac{d^2 p^\nu}{d\tau^2}, \tag{2}$$

and

$$Q^\mu \equiv \frac{m\mu}{2e}\zeta^\mu_\nu \partial^\nu \Phi \tag{3}$$

is the force that the electron experiences when its spin magnetic moment interacts with the inhomogeneous EM field. In the preceding equations,

$$\zeta^\mu_\nu \equiv \eta^\mu_\nu + m^{-2} p^\mu p_\nu$$

is the projector ($\eta^\mu_\nu = [-1, 1, 1, 1]$ is the Minkowsky signature tensor), and

$$\Phi \equiv F^{\mu\nu}\Pi_{\mu\nu}$$

is the scalar measure of the spin–field interaction, easily recognized from its rest-frame expression $\Phi_R = 2\boldsymbol{s}\cdot\boldsymbol{B}$, where $\boldsymbol{s}$ is the spin vector (see appendix for details).

A semiclassical limit of the force Q^μ is equivalent with the results derived by Silenko (2008) directly from the Dirac equation through Foldy–Wouthyusen transformations (see discussion in appendix). Furthermore, Bordovitsyn & Ternov (1983) have proved that the Frenkel force has an exact quantum analogue when the even terms are separated in the operators of the Dirac equation.

The fluid equations of a plasma of electrons, evolving under the combined influence of the Lorentz force and the radiation reaction, were derived and discussed in detail in Berezhiani, Hazeltine & Mahajan (2004) by taking the moments of the relativistic kinetic equation.

In principle, spin could be treated as a dynamical variable with its own evolution equation to supplement equation (1). For the purposes of this more conceptual paper, we will, however, treat spin as a given. We will find later, that though for the single-particle motion, $|\boldsymbol{s}| = 1$, for a plasma of particles the average spin could be much less than unity. The appropriate magnitude will be arrived at from statistical arguments.

For the radiation reaction, we first use the well-known Landau–Rohrlich formalism, in which the double-time derivative of the electron motion is computed in the absence of both the radiative reaction and the spin force (in the context of this paper). This method was introduced by Landau & Lifshitz (1962) as an approximation, but Rohrlich (2001) has more recently argued that the result is exact. Using this prescription, the electron dynamics of a spinning electron with standard radiation reaction is controlled by

$$\dot{p}^\mu = \frac{e}{m}\left(F^{\mu\nu}p_\nu + Q^\mu\right) + \frac{2e^3}{3m^2}\zeta^\mu_\nu\left(\frac{e}{m}F^{\nu\alpha}F_{\alpha\kappa}p^\kappa + \frac{dF^{\nu\alpha}}{d\tau}p_\alpha\right). \tag{4}$$

A qualitative analysis of this equation will constitute the first part of this paper. Let us split the EM field into two distinct components: $F^{\mu\nu} = F^{\mu\nu}_{\rm amb} + F^{\mu\nu}_{\rm hf}$, the ambient and the high-frequency parts. Their effective measures may be symbolically denoted as

$$|F^{\mu\nu}_{\rm amb}| \sim |\boldsymbol{B}_0|, \quad |F^{\mu\nu}_{\rm hf}| \sim |\boldsymbol{B}_1|, \quad |F^{\mu\nu}| \sim |\boldsymbol{B}_1 + \boldsymbol{B}_0|, \tag{5}$$

where B_1/B_0 can be arbitrary; B_0 and B_1 denote, for example, the laboratory frame magnitudes of the two contending fields (we have used $|\boldsymbol{B}| = B$ to simplify the notation). The ambient field is assumed to vary on a time-scale T and spatial scale L, while the photon field has characteristic frequency $\omega = 1/\lambda$, where λ is a typical photon wavelength. The principal reason for this split is the assumption

$$\omega T \sim L/\lambda \gg 1. \tag{6}$$

In order to put the spin force in perspective, we will first compare it to the Lorentz force. The Lorentz force responds directly to the EM field while the spin force resounds to its derivatives. For simplicity, we will further assume that $\omega B_1 \gg B_0/T$, and $B_1/\lambda \gg B_0/L$, allowing us to retain only the radiation field contribution from terms that involve field derivatives.

In the context of the preceding discussion, the spin force is estimated as $|Q| \sim \hbar\omega B_1$, and we have (using $|p| \sim \gamma m$ for a relativistic particle)

$$\frac{\text{Spin force}}{\text{Lorentz force}} \sim \left(\frac{\hbar\omega}{\gamma m}\right)\delta_1, \tag{7}$$

where

$$\delta_1 = \frac{B_1}{|\boldsymbol{B}_1 + \boldsymbol{B}_0|}, \tag{8}$$

comparing the strength of the radiation field as compared to the total field (ambient and radiation). Clearly, δ_1 should be viewed only as an approximate ratio. If the particle is a part of a plasma, then δ_1 can serve as an index of how strongly the plasma is magnetized: it is strongly magnetized, if $\delta_1 \ll 1$. Equation (7) is easy to read. If the photon is less energetic than the electron, and its intensity is such that the radiation magnetic field is smaller than the ambient one, then the spin term will be a small correction. However, as $\hbar\omega \to \gamma m$ and $B_1 \to B_0$, it can be of the order of unity. These results are, surely, quite obvious and unsurprising.

However, the next comparison of the spin force (whose dominant part is from the spin–radiation interaction) with the conventional radiation reaction force is new, interesting, and perhaps, of considerable importance. Far a sharper focus, we will assume in the following discussion that $B_1 < B_0$ so that $|\boldsymbol{B}_1 + \boldsymbol{B}_0| \approx B_0$ and $\delta_1 \approx B_1/B_0 < 1$.

From equation (4), we can see that the conventional radiation reaction force has two components $R = R_1 + R_2$, with

$$R^\mu_1 = \frac{2e^4}{3m^3}\zeta^\mu_\alpha F^{\alpha\nu}F_{\nu\kappa}p^\kappa, \qquad \text{and} \quad R^\mu_2 = \frac{2e^3}{3m^2}\zeta^\mu_\nu \frac{dF^{\nu\alpha}}{d\tau}p_\alpha. \tag{9}$$

They compare as

$$\frac{R_2}{R_1} \equiv \delta_2 \sim \frac{\omega}{\Omega}\delta_1, \tag{10}$$

where Ω is the cyclotron frequency corresponding to the dominant ambient field B_0.

Comparing each part, independently, with the Spin force Q, yields the ratios

$$\frac{Q}{R_1} \equiv \Delta_1 \sim \frac{\delta_2}{\gamma\alpha}, \tag{11}$$

where α is the fine structure constant, and

$$\frac{Q}{R_2} \equiv \Delta_2 \sim \frac{1}{\gamma\alpha}. \tag{12}$$

Equations (10)–(12) are rather revealing:

(1) The second part of radiation reaction (stemming from the field gradients) is small compared to the spin force for moderately relativistic electron. But for very highly relativistic particles $\gamma\alpha > 1$, the radiation reaction term can, again, begin to dominate spin force. This is independent of the radiation frequency. A similar result was already obtained in Tamburini et al. (2010).

(2) Unless the ambient fields are very strong compared to the typical field associated with radiation, even the first part of the radiation could be considerably smaller than the spin force for moderately relativistic electrons.

Thus, we conclude that as far as the single-particle equation of motion is concerned, the spin force, depending on the electron energy, could be considerably greater than the radiation reaction force. Its inclusion, therefore, is a must.

It is time to repeat the important caveat we stated earlier; in all the preceding estimates, the spin vector's magnitude $|\boldsymbol{s}|$ was taken to be unity as it ought to be for a single particle. However, we normally will be dealing with a collection of such particles (a plasma) and then one should expect thermal randomization and the average value of $|\boldsymbol{s}|$ could fall far below unity and if there was no aligning mechanism like an ambient magnetic field, average $|\boldsymbol{s}|$ could even go to zero.

It is thus important to compare not just the single-particle spin force and the radiation reaction force but also the average $\langle Q\rangle$ and $\langle R\rangle$ over an appropriate ensemble of particles.

3 AVERAGE SPIN FORCE, CONCLUSIONS

The fluid equations of a plasma of electrons, evolving under the combined influence of the Lorentz force and the radiation reaction, were derived and discussed in detail in Berezhiani et al. (2004) by taking the moments of the relativistic kinetic equation. This system can be readily analysed by including the spin-dependent forces. We will deal with the kinetic theory as well as the evolution of the spin vector as a dynamical variable in a later paper, but for this qualitative paper, we will simply estimate what fluid-averaged $\boldsymbol{s}\cdot\boldsymbol{B}$ will be by a simple argument. We will focus in the magnetized plasma case $B_0 > B_1$ as it is the simplest system we can study (other different cases will require a more complicated procedure for the statistical ensembles). We will see that even in this simplest case, the spin forces can be equally relevant than the radiation reaction ones. In the presence of the spin–field interaction, the standard Boltzman probability factor (in the rest frame) will change to

$$P = \mathrm{e}^{-(E+\mu \boldsymbol{s}\cdot\boldsymbol{B}_0)/T}, \tag{13}$$

where $\boldsymbol{B}_0$ is the ambient magnetic field (essentially uniform on the radiation scalelength). The electron spin has two eigenstates (spin dynamics is neglected) such that $\mu\boldsymbol{s}\cdot\boldsymbol{B}_0 = \pm\mu B_0$. Thus, the average value of spin could be approximated by

$$|\boldsymbol{s}| = \frac{P_+ - P_-}{P_+ + P_-} = \frac{\mathrm{e}^{\mu B_0/T} - \mathrm{e}^{-\mu B_0/T}}{\mathrm{e}^{\mu B_0/T} + \mathrm{e}^{-\mu B_0/T}} = \tanh(\mu B_0/T), \tag{14}$$

with the $\tanh(\mu B_0/T)$ measuring the spin alignment energy versus the thermal energy of the system.

The simplest estimation of the effective thermal statistical averaged strength of the spin contribution relative to the standard radiation reactions (11) and (12) can be calculated to be

$$\langle\Delta_1\rangle \sim \frac{\delta_2}{\gamma\alpha}\frac{\tanh(\mu B_0/T)}{f_1(m/T)} \tag{15}$$

and

$$\langle\Delta_2\rangle \sim \frac{1}{\gamma\alpha}\frac{\tanh(\mu B_0/T)}{f_2(m/T)}, \tag{16}$$

where f_1 and f_2 are thermodynamical functions appearing from the statistical averages of the radiation forces (for details see appendix A of Berezhiani et al. 2004).

We believe that results epitomized in equations (15)–(16) are of fundamental importance indicating that in more advanced modelling of high energy density electron plasmas immersed in strong high-frequency radiation fields (in the laboratory and astrophysical settings), the spin–radiation interaction must be included. Although $f_1 \geq 1$ and $f_2 \geq 1$ (Berezhiani et al. 2004), there are many physical systems where, even for moderately small spin alignments (caused by an ambient magnetic field working against thermal randomization), the average spin force could dominate the radiation reaction force. There will be many scenarios where the spin forces are certainly likely to be considerably larger than the more exotic QED effects.

4 DISCUSSION

In the end, we estimate $\langle\Delta_1\rangle$ and $\langle\Delta_2\rangle$ for the electrons involved in a gamma-ray burster. A typical gamma-ray burst has an energy equivalent to that of the solar energy output over 10^{10} yr, produced by a system of highly relativistic electrons with following parameters: system size $\approx$10 Km, the ambient magnetic field $B_0 \approx 10^{15}$ G, temperature $\approx 10^9$ K (Golonetskii et al. 1983; Piran 2005; Ride 2008). It is, then, straightforward to show: $\mu B_0 \gg T, f_1 \sim 1, f_2 \sim 1$, and $\delta_2 \approx B_1/B_0 \approx 0.1$, where B_1 is obtained by equating $B_1^2/8\pi$ with the energy density of emission. Thus, $\tanh(\mu B_0/T) \approx 1$, and both $\langle\Delta_1\rangle$ and $\langle\Delta_2\rangle$ are comparable to Δ_1 and Δ_2 and could be much greater than unity for a range of electron energies. The dynamics of an electron, participating in a gamma-ray burst, must take cognizance of the spin force.

As it can be seen, in high-energy plasmas, the spin can play a fundamental role under specific conditions. We believe this could be of great importance in laser–plasmas interactions. To have a complete understanding of the role of the spin force, a dynamical model needs to be developed where the spin force evolves along a general radiation reaction force produced by the EM and spin fields. That model will be presented in a forthcoming work.

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APPENDIX A: FRENKEL SPIN FORCE

Here, we review the derivation, due to of the classical representation of spin. Despite its classical heritage and form, the Frenkel spin formulation is physically predictive in a wide variety of situations. Following the presentation of Bagrov & Bordovitsyn (1980), we consider a point-particle with spin vector $\boldsymbol{s}$, velocity $\boldsymbol{v}$, charge e, and magnetic moment μ. Its four-vector velocity is denoted by $u^\mu = \gamma(1, \boldsymbol{v})$, where γ is the Lorentz factor; similarly, we define its spin four-vector by $s^\mu = (\boldsymbol{v}\cdot\boldsymbol{s}, \boldsymbol{s})$, thus satisfying the condition

$$u_\mu s^\mu = 0$$

first enunciated by Tamm (1929). Then, the Frenkel spin tensor is given by

$$\Pi_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} s^\rho v^\sigma. \quad \text{(A1)}$$

Here, $\epsilon_{\mu\nu\rho\sigma}$ is the unit antisymmetrical tensor, with convention $\epsilon^{0123} = 1 = -\epsilon_{0123}$.

An alternative representation of Π is sometimes useful; we define the two vectors

$$\boldsymbol{K} = \gamma[\boldsymbol{s} - \boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{s})], \qquad \boldsymbol{L} = \boldsymbol{v}\times\boldsymbol{K}$$

and find that equation (A1) can be expressed as

$$\Pi_{\mu\nu} = \begin{pmatrix} 0 & L_x & L_y & L_z \\ -L_x & 0 & K_z & K_y \\ -L_y & -K_z & 0 & K_x \\ -L_z & K_y & -K_x & 0 \end{pmatrix}.$$

Note that $\Pi_{\mu\nu}$ is constructed out of the three-vectors $\boldsymbol{K}$ and $\boldsymbol{L}$ in the same way that the Faraday tensor $F_{\mu\nu}$ is constructed from $\boldsymbol{B}$ and $\boldsymbol{E}$.

Finally, we construct from Π and the Faraday tensor the magnetic potential energy function Φ, given by

$$\Phi \equiv F^{\mu\nu}\Pi_{\mu\nu}.$$

This quantity, first introduced by Frenkel, can also be written as

$$\Phi = 2(\boldsymbol{E}\cdot\boldsymbol{L} + \boldsymbol{B}\cdot\boldsymbol{K})$$

or

$$\Phi = 2\gamma[\boldsymbol{B}\cdot(\boldsymbol{s} - \boldsymbol{v}\boldsymbol{v}\cdot\boldsymbol{s}) + \boldsymbol{E}\cdot\boldsymbol{v}\times\boldsymbol{s}]. \quad \text{(A2)}$$

Its rest-frame version is denoted by $\Phi_{\rm R}$:

$$\Phi_{\rm R} = 2\boldsymbol{B}\cdot\boldsymbol{s}.$$

Also, in the semiclassical limit $v^2 \ll 1$ and $\gamma \approx 1$, the potential (equation A2) coincides with the result found in equation (33) of Silenko (2008).

Since the rest-frame force on a magnetic dipole moment $\boldsymbol{m} = \mu\boldsymbol{s}$ is given by $\nabla(\boldsymbol{m}\cdot\boldsymbol{B})$, we expect the contravariant spin force to have the form

$$\frac{1}{2}\mu\partial^\mu\Phi = \frac{1}{2}\mu\eta^{\mu\nu}\partial_\nu\Phi,$$

where $\eta^{\mu\nu}$ is the Minkowski metric. However, every four-vector force f^μ must satisfy $p_\mu f^\mu = 0$, where $p_\mu = mu_\mu$ is the four-momentum, in order to conserve $p_\mu p^\mu = -m^2$. Thus, we introduce the projector

$$\zeta^\mu_\nu \equiv \eta^\mu_\nu + m^{-2}p^\mu p_\nu$$

which satisfies

$$p^\nu\zeta^\mu_\nu = 0 \quad \text{(A3)}$$

as well as

$$\zeta^\alpha_\nu\zeta^\nu_\mu = \zeta^\alpha_\mu. \quad \text{(A4)}$$

The spin four-force is then given by eQ^μ/m, where

$$Q^\mu \equiv \frac{m\mu}{2e}\zeta^\mu_\nu\partial^\nu\Phi. \quad \text{(A5)}$$

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