Applied Stochastic Models in Business and Industry

Research Article

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A methodology for stochastic inventory models based on a zero-adjusted Birnbaum-Saunders distribution

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The Birnbaum–Saunders (BS) distribution is receiving considerable attention. We propose a methodology for inventory logistics that allows demand data with zeros to be modeled by means of a new discrete–continuous mixture distribution, which is constructed by using a probability mass at zero and a continuous component related to the BS distribution. We obtain some properties of the new mixture distribution and conduct a simulation study to evaluate the performance of the estimators of its parameters. The methodology for stochastic inventory models considers also financial indicators. We illustrate the proposed methodology with two real-world demand data sets. It shows its potential, highlighting the convenience of using it by improving the contribution margins of a Chilean food industry. Copyright © 2015 John Wiley & Sons, Ltd.

Keywords: demand data; financial indicators; maximum likelihood method; mixture distributions; Monte Carlo simulation; R software

1. Introduction

Diverse types of studies in business and industry require an appropriate knowledge of the demand behavior for products. Because this demand often occurs in a stochastic fashion, the corresponding demanded quantity (DQ) is assumed to be a random variable (RV) following a statistical distribution. When the demand for products is described by an adequate distribution and an optimized inventory model is used, the contribution margins (CMs) of the companies can be increased.

Although data of demand for products have a discrete nature, usually, these are modeled by continuous distributions. Particularly, the normal distribution has been used for this modeling (e.g. [1]), although some authors criticize its use (e.g. [2]). This is because the DQ for products only admits nonnegative values and often follows an asymmetric probabilistic model with positive skewness. In some cases, such as in the illustrations showed in this paper, the product to be stored can be packaged by means of a volume or weight unit, so that it can be considered as a continuous RV. In any case, although demand data for products may be discrete or continuous, continuous probabilistic models are often used for describing them. Some selected studies that support this assertion are [1–8], who used the gamma, inverse Gaussian (IG), log-normal, triangular, uniform, and Weibull distributions to describe demand data for products.

A probability model that is receiving considerable attention is the Birnbaum–Saunders (BS) distribution, which is unimodal, positively skewed, of two parameters and allows data greater than zero to be described. This attention is because of its attractive properties. For example, the BS distribution is a transformation of the normal distribution. Recently, Vanegas and Paula [9] presented a more general class of distributions including the BS model. Extensive work has been performed on the BS distribution with regard to its properties, inference, methodology, and modeling (e.g., [10–19]). Although the BS distribution has its genesis from engineering, its applications also include business and industry (e.g., [18,20–29]).

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Based on its genesis, the BS distribution considers the duration of a counting period (daily or weekly), which may be switched without collecting extra data, among other properties. It permits the BS distribution to have theoretical arguments for modeling demand data (see details in [30]). Rojas *et al*. [5] carried out an empirical study in which the BS distribution shows to be a good model to describe the demand for different food products.

Despite the wide use of the BS distribution, it is well defined only for positive values. This makes that data sets that contain zero values, such as demand data, cannot be modeled with this distribution. Thus, in the presence of zero values, we should adapt the BS distribution to allow these values to be captured in the modeling. A way for deriving a model suitable to data on the interval $[0, \infty)$ is using a mixture distribution of two components: a BS distribution (continuous component) and a degenerate distribution at a zero value (discrete component). It is well known that the mixture models are powerful and popular tools to generate flexible distributions with good properties [31, 32].

Santos-Neto *et al*. [33] proposed a reparameterized BS (RBS) distribution, which has several interesting properties (see also [25, 34]). For example, one of the two parameters of the RBS distribution is its mean and the other one a precision parameter. This allows us to make different statistical analyses mimicking the case of the normal distribution, often used to model demand data, but now in an asymmetric framework, which is closer to reality of this kind of data. Then, we use the RBS distribution and a degenerate distribution at zero to construct the demand semi-continuous mixture model. We name this new class of models as the zero-adjusted RBS (ZARBS) distribution, because they are in the line of zero-adjusted models ([35–38]).

The main objectives of this paper are as follows: (i) to propose the new ZARBS distribution and (ii) to introduce a methodology for inventory models based on this distribution including financial indicators. We derive several features of the ZARBS distribution and estimate its parameters. We use this distribution because it allows positive data in the presence of zeros to be modeled, which occurs often when daily demand data for a product are analyzed. To the best of our knowledge, until now, zero-adjusted continuous distributions have not been used for modeling demand data. We implement this methodology in the `R` software ([39] and www.r-project.org).

The paper is organized as follows. In Section 2, we introduce the ZARBS distribution, present some of its mathematical and statistical properties, propose maximum likelihood (ML) estimators for its parameters, and conduct Monte Carlo (MC) simulations to evaluate the behavior of the proposed estimators. In Section 3, we detail a new methodology for stochastic inventory models based on the ZARBS distribution and financial indicators. In Section 4, we illustrate this methodology with two real-world demand data sets, which shows its potential by improving the CMs of a Chilean company. In Section 5, we provide conclusions and possible future works.

2. A zero-adjusted Birnbaum–Saunders distribution

2.1. Distributional properties and features

Let Y_1 be a RV with RBS distribution (reparameterized on the mean) (see details in [33]). Then, its cumulative distribution function (CDF) is given by

$$F_{Y_1}(y;\mu,\delta) = \Phi\left(\sqrt{\delta/2}\left[\sqrt{\{\delta+1\}\,y/\{\mu\,\delta\}} - \sqrt{\mu\,\delta/\{(\delta+1)\,y\}}\right]\right), y>0, \mu>0, \delta>0, \tag{1}$$

where $\Phi(\cdot)$ is the standard normal CDF. If a RV Y_1 follows a RBS distribution with mean μ and precision parameter δ, then the notation $Y_1 \sim \text{RBS}(\mu,\delta)$ is used. From (1), note that standard normal (Z) and RBS (Y_1) RVs are related by $Z = \sqrt{\delta/2}[\sqrt{\{\delta+1\}\,Y_1/\{\mu\,\delta\}} - \sqrt{\mu\,\delta/\{(\delta+1)\,Y_1\}}]$ and $Y_1 = \delta\,\mu/[\delta+1][Z/\sqrt{2\,\delta} + \sqrt{\{Z/(2\,\delta)^{1/2}\}^2+1}]^2$. Also, it is easy to note that the RBS distribution holds with proportionality and reciprocation properties, that is, if $Y_1 \sim \text{RBS}(\mu,\delta)$, then (P1) $aY_1 \sim \text{RBS}(a\mu,\delta)$, for $a>0$, and (P2) $1/Y_1 \sim \text{RBS}(\mu^*,\delta)$, where $\mu^* = [\delta+1]^2/[\delta^2\mu]$. The quantile function (QF) of Y_1 is given by

$$y_1(q;\mu,\delta) = F_{Y_1}^{-1}(q;\mu,\delta) = \frac{\delta\,\mu}{\delta+1}\left[z(q)/\sqrt{2\,\delta} + \sqrt{\left\{z(q)/\sqrt{2\,\delta}\right\}^2+1}\right]^2, \; 0<q<1, \tag{2}$$

where $z(q)$ is the q-th quantile of the standard normal distribution and $F_{Y_1}^{-1}(\cdot)$ is the inverse RBS CDF. The four first moments around zero of $Y_1 \sim \text{RBS}(\mu,\delta)$ are $\mu_1 = \mu$, $\mu_2 = \mu^2\,[\delta^2+4\,\delta+6]/[\delta+1]^2$, $\mu_3 = \mu^3\,[\delta^3+9\,\delta^2+36\,\delta+60]/[\delta+1]^3$, and $\mu_4 = \mu^4\,[\delta^4+16\,\delta^3+120\,\delta^2+460\,\delta+840]/[\delta+1]^4$. Then, the mean and variance of Y_1 are given by $\text{E}[Y_1] = \mu$ and $\text{Var}[Y_1] = \mu^2[2\delta+5]/[\delta+1]^2$, respectively (for more details, see [34]). Let Y be a mixture between two RVs Y_1 and Y_2, with a continuous probability density function (PDF) and a Dirac mass at zero, respectively. Then, if $Y_1 \sim \text{RBS}(\mu,\delta)$, the PDF of Y is given by $f_Y(y;\mu,\delta,p) = [1-p]f_{Y_1}(y)1_{(0,\infty)}(y) + p\,f_{Y_2}(y)1_{\{0\}}(y)$, for $y \geqslant 0$, where $0<p<1$ is the mixture parameter, $f_{Y_1}(\cdot)$ is the PDF obtained from (1), and $1_B(\cdot)$ is the indicator function of the set B. Thus, if a RV Y follows a ZARBS distribution, the notation $Y \sim \text{ZARBS}(\mu, \delta, p)$ is used, and its PDF expressed previously can be written as

$$f_Y(y; \mu, \delta, p) = \frac{[1-p]\sqrt{\delta+1}}{4\, y^{3/2}\sqrt{\pi\mu}} \left[y + \frac{\delta\mu}{\delta+1}\right] \exp\left(-\frac{\delta}{4}\left[\frac{y\{\delta+1\}}{\delta\mu} + \frac{\delta\mu}{y\{\delta+1\}} - 2\right]\right) 1_{(0,\infty)}(y) + p\, 1_{\{0\}}(y),$$

with $y \geqslant 0$, $\delta > 0$, $\mu > 0$ and $0 < p < 1$. Hence, the CDF of Y is given by

$$F_Y(y; \mu, \delta, p) = \begin{cases} p, & \text{if } y = 0; \\ p + \big[1-p\big]\, F_{Y_1}(y; \mu, \delta), & \text{if } y > 0; \end{cases} \tag{3}$$

where $F_{Y_1}(\cdot)$ is the CDF of the RBS distribution given in (1). From (3), we have that (P3) if $Y \sim \text{ZARBS}(\mu, \delta, p)$, then $V = aY \sim \text{ZARBS}(a\,\mu, \delta, p)$, for $a > 0$. Property (P3) is obtained from $\text{P}(V \leqslant v) = \text{P}(aY \leqslant v) = \text{P}(Y \leqslant v/a)$. For $Y = 0$, $\text{P}(V \leqslant v) = p$ and, for $Y > 0$, $\text{P}[V \leqslant v] = p + [1-p]\, F_{Y_1}(v/a; \mu, \delta)$, where $F_{Y_1}(v/a; \mu, \delta)$ is the RBS CDF of parameters $a\mu$ and δ. The QF of $Y \sim \text{ZARBS}(\mu, \delta, p)$ can be obtained by the inverse CDF defined in (3) as

$$y(u; \mu, \delta, p) = F_Y^{-1}(u; \mu, \delta, p) = \begin{cases} 0, & \text{if } p \geqslant u; \\ y_1([u-p]/[1-p]; \mu, \delta), & \text{if } p < u; \end{cases} \tag{4}$$

where $0 < u < 1$ and $y_1(\cdot)$ is the QF of the RBS distribution given in (2). Expression provided in (4) can be obtained from Castellacci [40, Proposition 2.1]. A random number generator from $Y \sim \text{ZARBS}(\mu, \delta, p)$ is given by Algorithm 1.

Algorithm 1 Random number generator for the ZARBS distribution

1: Generate a random number u from $U \sim \text{U}(0, 1)$.
2: Set values for μ, δ, and p of $Y \sim \text{ZARBS}(\mu, \delta, p)$.
3: Compute a random number $y = y_1$ or $y = y_2$ from $Y \sim \text{ZARBS}(\mu, \delta, p)$ using (4), that is,
  3.1: If $u \leqslant p$, then $y_1 = 0$;
  3.2: Else, $y_2 = y([u-p]/[1-p]; \mu, \delta)$.
4: Repeat steps 1 to 3 until the required amount of random numbers to be completed.

The k-th moment about zero of $Y \sim \text{ZARBS}(\mu, \delta, p)$ is given by $\text{E}[Y^k] = [1-p]\mu_k$, where

$$\mu_k = \frac{\mu^k \delta^2 \exp\left(\frac{\delta}{2}\right)\left[\{\delta^{k-1/2} + \delta^{k-3/2}\}\,\{\delta+1\}^{-k+1/2}\, K_{k+1/2}\left(\frac{\delta}{2}\right) + \delta^{k-3/2}\,\{\delta+1\}^{-k+3/2}\, K_{k-1/2}\left(\frac{\delta}{2}\right)\right]}{2\sqrt{\pi}\,[\delta+1]^3/2}, \tag{5}$$

with $K_\nu(\cdot)$ being the modified Bessel function of second type [41]. Expressions for the function $K_\nu(\cdot)$, which are useful for calculating (5), can be found in [34]. Thus, we have the mean and variance of $Y \sim \text{ZARBS}(\mu, \delta, p)$ are $\text{E}[Y] = \lambda = [1-p]\,\mu$ and $\text{Var}[Y] = \sigma_Y^2 = p[1-p]\mu^2 + [1-p]\mu^2[2\,\delta+5]/[\delta+1]^2$, respectively.

2.2. Parameter estimation

Let $\boldsymbol{Y} = (Y_1, \ldots, Y_n)^\top$ be a random sample from $Y \sim \text{ZARBS}(\mu, \delta, p)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)^\top$ its observations (data). Then, the corresponding likelihood function for $\boldsymbol{\theta} = (\mu, \delta, p)^\top$ is given by

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f_Y(y_i; \mu, \delta, p) = p^{n_0}[1-p]^{n-n_0} \prod_{i=1}^{n} f_{Y_1}(y_i; \mu, \delta)^{1-1_{\{0\}}(y_i)}, \tag{6}$$

where $n_0 = \sum_{i=1}^n 1_{\{0\}}(y_i)$ is the number of zeros in the sample. Hence, the respective log-likelihood function obtained from (6) can be expressed as $\ell(\boldsymbol{\theta}) = \ell(p) + \ell(\mu, \delta)$, where $\ell(p) = n_0 \log(p) + [n - n_0]\log(1-p)$ and $\ell(\mu, \delta) = [n - n_0]c(\mu, \delta) - [3/2]\sum_{y_i>0}\log(y_i) - [\{\delta+1\}/\{4\mu\}]\sum_{y_i>0} y_i - \sum_{y_i>0}[\mu\delta^2]/[4\{\delta+1\}y_i] + \sum_{y_i>0}\log(y_i + \mu\delta/[\delta+1])$, with $c(\mu, \delta) = -[1/2]\log(16\pi) + [\delta/2] - [1/2]\log(\mu) + [1/2]\log(\delta+1)$. Note that the log-likelihood function factorizes in a term that depends only on p and other that depends on δ and μ. Thus, the ML method can be applied separately

([42, p. 128]). We obtain the elements of the score vector by taking derivatives of the corresponding log-likelihood function with respect to the unknown parameters as

$$\dot{\ell}_\mu = -\frac{[n-n_0]}{2\mu} + \frac{[\delta+1]}{4\mu^2}\sum_{y_i>0} y_i - \frac{\delta^2}{4[\delta+1]}\sum_{y_i>0}\frac{1}{y_i} + \frac{\delta}{[\delta+1]}\sum_{y_i>0}\frac{1}{\left[y_i+\frac{\mu\delta}{\delta+1}\right]}, \quad \dot{\ell}_p = \frac{n_0}{p} - \frac{n-n_0}{1-p},$$
$$\dot{\ell}_\delta = \frac{[n-n_0][\delta+2]}{2[\delta+1]} - \frac{1}{4\mu}\sum_{y_i>0} y_i - \frac{\mu\delta[\delta+2]}{4[\delta+1]^2}\sum_{y_i>0}\frac{1}{y_i} + \frac{\mu}{[\delta+1]^2}\sum_{y_i>0}\frac{1}{\left[y_i+\frac{\mu\delta}{\{\delta+1\}}\right]}. \tag{7}$$

From (7), the ML estimate of p is $\widehat{p} = n_0/n$, which represents the proportion of zeros in the sample. Because the solution of the equations $\dot{\ell}_\mu = 0$ and $\dot{\ell}_\delta = 0$ obtained from (7) does not have a closed form, we maximize the log-likelihood function obtained from (6) on μ and δ by using a nonlinear optimization algorithm for determining the ML estimates of μ and δ. A nonlinear optimization algorithm called Cole–Green (CG) can be used in this case [43]. This algorithm works well for distributions with correlated parameter estimators, as is the case of the ZARBS distribution. To obtain starting values for the numerical optimization algorithm, consider the following. Note that, if we have only the continuous component of the distribution of Y, that is, the RBS distribution, then $\mathrm{E}[Y_1] = \mu$ and $\mathrm{Var}[Y_1] = \mu^2[2\delta+5]/[\delta+1]^2$. Hence, the moment estimates of μ and δ are given by $\breve{\mu} = \bar{y}_*$ and $\breve{\delta} = [\bar{y}_*^2 - s_*^2 + \sqrt{\bar{y}_*^4 + 3\,\bar{y}_*^2\,s_*^2}]/s_*^2$, respectively, where $\bar{y}_* = [1/\{n-n_0\}]\sum_{y_i>0} y_i$ and $s_*^2 = [1/\{n-n_0\}]\sum_{y_i>0}\{y_i - \bar{y}_*\}^2$. In addition, we have that $\breve{\delta}$ is well defined when the sample coefficient of variation (CV) is less than $\sqrt{5}$ [34]. Moment estimates given previously can be used as starting values for the CG algorithm.

Now, we obtain the respective expected Fisher information matrix by taking derivatives of the elements of the score vector given in (7) with respect to the unknown parameters as

$$\mathcal{I}(\boldsymbol{\theta}) = -\mathrm{E}\left[\frac{\partial \ell(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}\right] = \begin{pmatrix} \mathcal{I}_{\mu\mu} & \mathcal{I}_{\mu\delta} & 0 \\ \mathcal{I}_{\delta\mu} & \mathcal{I}_{\delta\delta} & 0 \\ 0 & 0 & \mathcal{I}_{pp} \end{pmatrix}, \quad i,j = 1,2,3, \tag{8}$$

where $\theta_1 = \mu, \theta_2 = \delta$ and $\theta_3 = p$, whereas $\mathcal{I}_{\mu\mu} = n[1-p][\delta/\{2\mu^2\} + \delta^2/\{\delta+1\}^2 I(\mu,\delta)]$, $\mathcal{I}_{\mu\delta} = \mathcal{I}_{\delta\mu} = n[1-p][\{2\mu(\delta+1)\}^{-1} + \{\mu\delta/(\delta+1)^3\}I(\mu,\delta)]$, $\mathcal{I}_{\delta\delta} = n[1-p][\{\delta^2+3\delta+1\}/\{2\delta^2(\delta+1)^2\} + \{\mu^2/(\delta+1)^4\}\,I(\mu,\delta)]$ and $\mathcal{I}_{pp} = n/[p\{1-p\}]$, with $I(\mu,\delta) = \int_0^\infty [y + \{\mu\delta\}/\{\delta+1\}]^{-2} f_{Y_1}(y;\mu,\delta)\,\mathrm{d}y$ and $f_{Y_1}(\cdot)$ being the PDF of $Y_1 \sim \mathrm{RBS}(\mu,\delta)$. From (8), note that the parameter p is orthogonal to the parameters μ and δ.

Under some regularity conditions [44], $\widehat{\boldsymbol{\theta}}$ is a consistent estimator of $\boldsymbol{\theta}$ and it has a normal asymptotic distribution. Then, $\sqrt{n}\,[\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}] \xrightarrow{\mathcal{D}} \mathrm{N}_3(0, \mathcal{J}(\boldsymbol{\theta})^{-1})$, as $n \to \infty$, where $\mathcal{J}(\boldsymbol{\theta}) = \lim_{n\to\infty}[1/n]\mathcal{I}(\boldsymbol{\theta})$, with $\mathcal{I}(\boldsymbol{\theta})$ being the expected Fisher information matrix given in (8), and $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution to. Note that $\widehat{\mathcal{I}}(\boldsymbol{\theta})^{-1}$ is a consistent estimator of the asymptotic variance–covariance matrix of $\widehat{\boldsymbol{\theta}}$. In practice, one may approximate the expected Fisher information matrix by its observed version, whereas the elements of the diagonal of the inverse of this matrix can be used to approximate the corresponding standard errors (see [45] for details about the use of observed versus expected Fisher information matrices). Asymptotic confidence intervals (CIs) for μ, δ, and p, with level of $100 \times [1-\xi]\%$, can be obtained from the mentioned asymptotic normality as

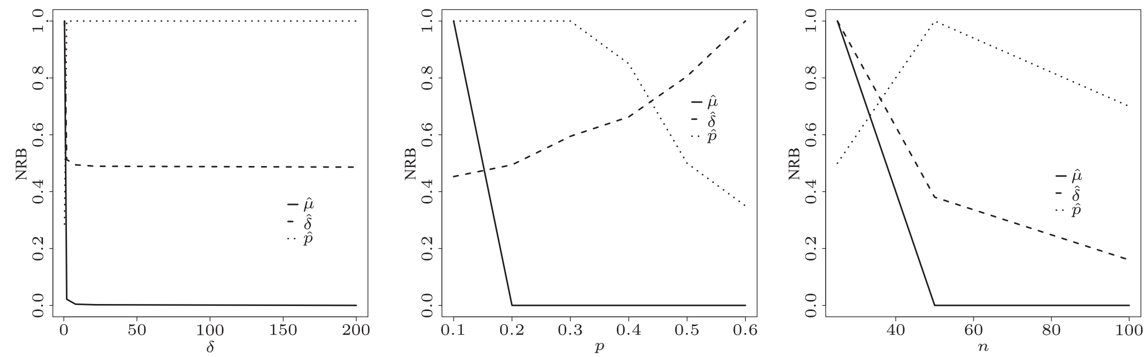
$$\mathrm{CI}(\theta; [1-\xi]\times 100\%) = \left[\exp\left(\widehat{\theta^*} - z(1-\xi/2)\,\widehat{\mathrm{SE}}(\widehat{\theta^*})\right), \exp\left(\widehat{\theta^*} + z(1-\xi/2)\,\widehat{\mathrm{SE}}(\widehat{\theta^*})\right)\right], \tag{9}$$

$$\mathrm{CI}(p; [1-\xi]\times 100\%) = \left[1/\left\{1+\exp\left(-\widehat{p^*} - z(1-\xi/2)\,\widehat{\mathrm{SE}}(\widehat{p^*})\right)\right\}, 1/\left\{1+\exp\left(-\widehat{p^*} + z(1-\xi/2)\,\widehat{\mathrm{SE}}(\widehat{p^*})\right)\right\}\right], \tag{10}$$

where $\widehat{\theta^*}$ is the ML estimate of $\theta^* = \log(\mu)$ or $\theta^* = \log(\delta)$ and $\widehat{\mathrm{SE}}(\widehat{\theta^*})$ is the estimated asymptotic SE of the ML estimator of θ^*. We also define $p^* = \mathrm{logit}(p)$, with the corresponding SEs of these estimators being obtained by using the delta method. Confidence bounds (CBs) can also be obtained in a similar way to the CIs given in (9) and (10).

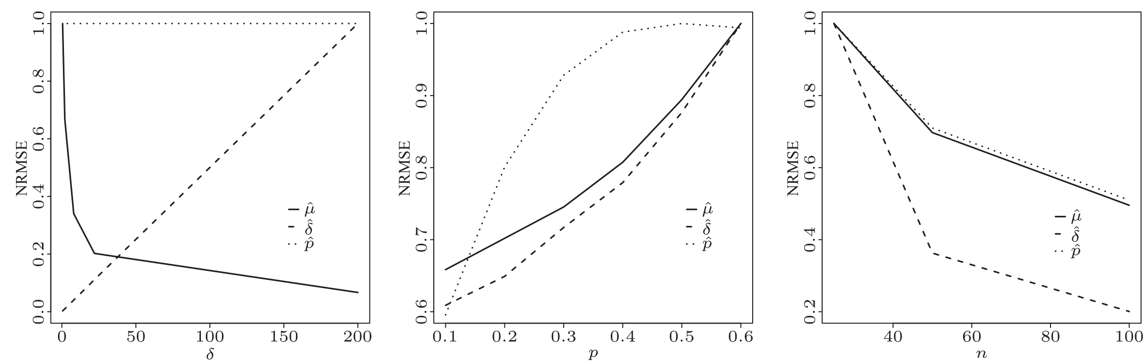
2.3. Simulation study

We use MC simulations to evaluate the behavior of the ML estimators of the parameters of the ZARBS distribution and Algorithm 1 to generate random numbers of a RV with this distribution. The scenario of the simulation considers a sample



(a) $p = 0.6$ and $n = 100$ (b) $\delta = 200$ and $n = 100$ (c) $\delta = 200$ and $p = 0.6$

Figure 1. Normalized relative bias (NRB) of $\hat{\mu}$, $\hat{\delta}$, and $\hat{p}$ for the indicated combination of n, p, and δ with simulated data.



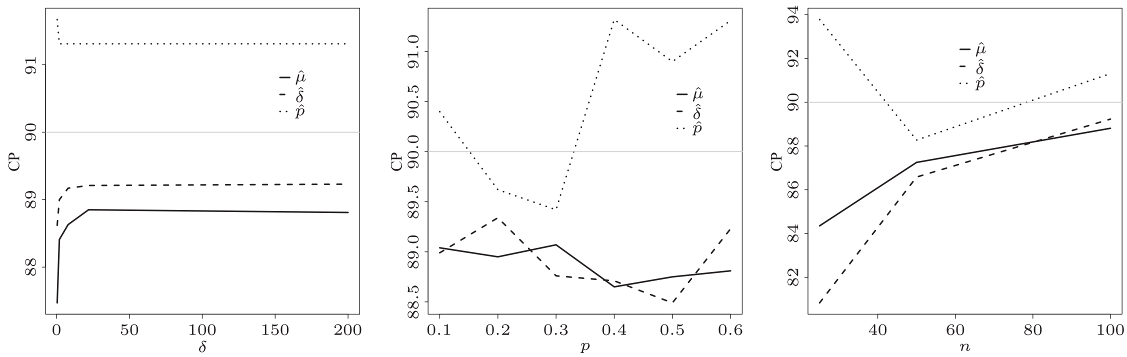
(a) $p = 0.6$ and $n = 100$ (b) $\delta = 200$ and $n = 100$ (c) $\delta = 200$ and $p = 0.6$

Figure 2. Normalized root mean square error (RMSE) of $\hat{\mu}$, $\hat{\delta}$, and $\hat{p}$ for the indicated combination of n, p, and δ with simulated data.

size $n \in \{25, 50, 100\}$, shape parameter $\delta \in \{0.5, 2.0, 8.0, 22, 200\}$, and proportion of zeros $p \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6\}$ for the ZARBS distribution. The parameter μ is fixed at 1.0, without loss of any generality, because μ is, besides the mean, a scale parameter. All the results are based on 10000 MC experiments.

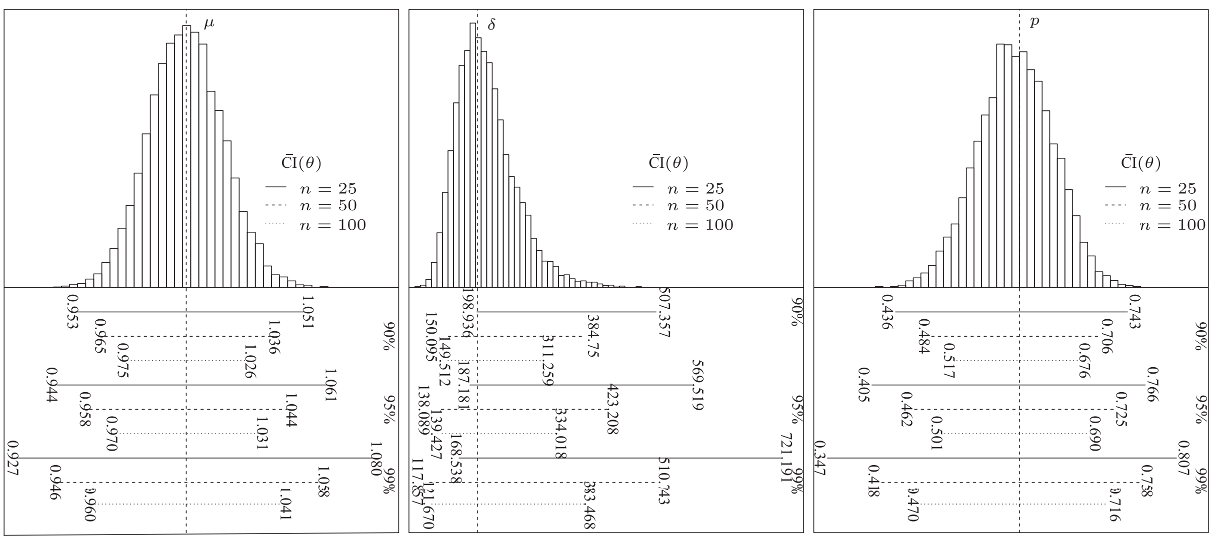
The simulations are performed using the `R` programming language. The ML estimates of the parameters of the ZARBS distribution are computed with the `gamlss` function of an `R` package of the same name, which uses the CG algorithm ([36] and www.gamlss.org). We have incorporated the ZARBS family in the `gamlss` package following the instructions provided in Section 4.2 of [46]. The `R` codes used in this article are available from the authors under request.

To analyze the point estimation results, we compute, for each sample size and parameter value, the empirical mean, relative bias (RB), SE, and root mean square error (RMSE) of the corresponding estimators. Concerning to interval estimation, we present the empirical coverage probabilities (CPs) obtained from the relative frequencies, at which the true parameter value belongs to the CI. The CIs for μ, δ, and p, given in (9) and (10), aim to ensure that the estimates of the limits are within the parameter space that indexes the ZARBS distribution. Furthermore, we compute the percentages of the 10000 CIs in which the true parameter value is smaller than the lower limit of the CIs (in short %L) and greater than its upper limit (in short %U). We consider confidence levels of 90%, 95%, and 99%, but some results for 95% and 99% levels have been omitted here because of space restrictions. To make the RMSEs and RBs of the estimators $\widehat{\mu}$, $\widehat{\delta}$, and $\widehat{p}$ comparable, we normalize these measures and abbreviate them as NMSE and normalized relative bias (NRB). The normalization is performed as follows: $\text{NRMSE}_i = \text{RMSE}_i/\max\{\text{RMSE}_i\}$ and $\text{NRB}_i = \text{RB}_i/\max\{\text{RB}_i\}$. In Figures 1–4, note that (i) in general, NRB and NRMSE decrease as n increases, whereas they increase as p increases; (ii) the estimator of μ is seemingly unbiased; (iii) the empirical CP of the CIs gets closer to the nominal level as n increases. When δ increases, the empirical CP tends toward the nominal level, whereas the variation of p causes an instability in the empirical CP; and (iv) the empirical distribution of $\widehat{\mu}$ and $\widehat{p}$ are apparently symmetric, but the empirical distribution of $\widehat{\delta}$ presents a positive skewness. These results show the good performance of the corresponding ML estimators and are in agreement with the expected results.



(a) $p = 0.6$ and $n = 100$ (b) $\delta = 200$ and $n = 100$ (c) $\delta = 200$ and $p = 0.6$

Figure 3. Coverage probabilities (CP) for confidence interval of μ, δ, and p for the indicated combination of n, p, and δ with simulated data.



(a) (b) (c)

Figure 4. Empirical distributions of (a) $\widehat{\mu}$, (b) $\widehat{\delta}$, and (c) $\widehat{p}$, where $\bar{\text{C}}\text{I}(\theta)$ is a confidence interval (CI) whose limits are means of CI limits, for $\theta = \mu, \delta, p$ and the indicated value of n and confidence level of $100 \times [1 - \xi]\%$, with $\mu = 1.0$, $\delta = 200$ and $p = 0.6$, respectively, using simulated data.

3. A methodology for inventory logistics

3.1. Relevant costs

Inventory logistics allow companies to minimize its costs and to reduce management inefficiencies. The total cost of an inventory is function of purchasing cost (PC), ordering cost (OC), and storing cost (SC) [47]. In addition to these costs, other costs could be present in inventory models, such as the unsatisfied demand shortage cost per unit of product, which could include lost revenue and cost of loss of customer goodwill, and the holding cost per unit/day of product, which could include the storing cost minus a salvage value of a unit of product.

3.2. How to collect the data

Our methodology requires the mentioned relevant costs, which must be fixed by the company. In addition, a record system for all the products comprising the inventory assortment of food service raw materials must be designed and implemented. Such a system should rely on identification code, unit cost, DQ, price, date and arrival and delivery times of products employed in the preparation of food rations. The record system should be conducted with an individual identification by means of bar codes and developed for product demand profiles of the inventory assortment, during the period scheduled for the study.

3.3. Suppositions and restrictions

Suppositions must be considered for our methodology. For example, the demand must be an independent RV and seasonality and trend factors must are missing. In addition, the lead time (LT) must be constant and known. Our methodology is valid for perishable and non-perishable products. Moreover, no penalties that are imposed to a product with unsatisfied demand exist in our methodology. In practice, for food services, a product is replaced by other if it is missing. We optimize an only product, instead of carrying out a simultaneous optimization of the assortment of inventory products. In addition, a service level based on a safety factor (SF) must be established and the economic order quantity and SF are individually optimized.

3.4. Stochastic inventory models

After the costs are fixed, the demand data must be collected and the most appropriate demand distribution is selected. For example, the ZARBS distribution can be considered. In addition, the suitable stochastic inventory model must be used depending on whether the product is (i) non-perishable under multiple periods or (ii) perishable under a single period. Then, according to the kind of product, we calculate the quantity to be replenished minimizing costs (PC, OC, and SC) and considering the (Q, r) or critical ratio (CR) inventory models according to considerations detailed in the succeeding texts.

3.4.1. Model for non-perishable products (M1). In this case, the quantity needed to optimize the OCs and SCs is calculated from the economic order quantity model defined as

$$Q = \sqrt{\frac{2\,\lambda\,c_o}{c_s}}, \tag{11}$$

where c_o is the OC, c_s the SC per unit/day, and λ is the demand rate in units of the product per time unit, obtained from the expected value of the distribution considered as suitable for the demand data. For model M1, we must keep in mind the reorder point (ROP), corresponding to the stock level of an inventory when a purchase order is placed. The ROP is computed as

$$r = \mu_{D_l} + \mathrm{SS}, \tag{12}$$

where $\mu_{D_l} = \mathrm{E}[D_l] = l\,\lambda$ is the mean of the demand during the LT (D_l), with λ defined in (11) and l being the constant and known LT. Here, $\mathrm{SS} = k_q\,\sigma_{D_l}$ is the safety stock (SS), with k_q being the SF for a service level $q \times 100\%$ and $\sigma_{D_l} = \sqrt{\mathrm{Var}[D_l]} = \sqrt{l}\,\sigma$ being the standard deviation of the demand during the LT. From (12), note that it is necessary to know the demand distribution during the LT for determining the SF [4]. The SF may be obtained from some percentile of the demand during the LT, which is often the 95th percentile, that is, $q = 0.95$. Therefore, $k_{0.95}$ must be determined from a model that fits the demand data during LT. For more details about this model, see Hillier and Lieberman [47, pp. 956–961].

3.4.2. Model for perishable products (M2). In this case, the quantity necessary for optimizing the cost of ordering one unit less in contrast to ordering one unit more, based on the CR, is given by

$$\mathrm{CR} = \frac{c_u - c_p}{c_u + c_h}, \tag{13}$$

where c_u is the unsatisfied demand shortage cost per unit, c_p is the PC of the product, and c_h is the holding cost per unit/day. Thus, the single period model for perishable products allows the optimum stored quantity of units (y^0) to be obtained from the expression $F_Y(y^0) = \mathrm{CR}$, known as optimum service level, where $F_Y(\cdot)$ is the CDF of the demand and CR is defined in (13). For more details about this model, see Hillier & Lieberman [47, pp. 961-975].

3.5. Financial indicators

After an appropriate inventory management model is selected from M1 or M2, the CMs for each product of the inventory assortment utilized in the preparation of a food ration must be computed. This is obtained from the incomes generated during each of n_w weeks by the company due to the food ration sold. The quantities for each of n_p products used in the preparation of the food ration are established by means of its corresponding consumption measuring unit.

The prorated demand (PD) of the product i in the j-th week may be calculated with the proportion that each product of the food ration holds weekly according to

$$\mathrm{PD}_{i,j} = \mathrm{DQ}_{i,j}/\mathrm{DQ}_j, \quad i = 1, \dots, n_p, \quad j = 1, \dots, n_w, \tag{14}$$

Table I. Costs for generating a purchase order (OC^h).	
Cost	Description
OC^1	Administrative costs associated with the order movements (input and general service costs with respect to order generation).
OC^2	Inspection and receiving costs (social security contributions and warehouseman wages) of movements associated with an order.
OC^3	Transportation costs related solely to order generation.

OC, ordering cost.

where $DQ_{i,j}$ is the DQ of the product i in the j-th week and DQ_j is the DQ for all the products in the j-th week. The income of the company for all food rations sold during the j-th week is given by

$$I_j = N_j\, S_j, \quad j = 1, \dots, n_w, \tag{15}$$

where N_j is the number of food rations sold and S_j is its price during the j-th week. Hence, the prorated income (PI) of the product i in the j-th week is expressed as

$$PI_{i,j} = I_j\, PD_{i,j}, \quad i = 1, \dots, n_p, \quad j = 1, \dots, n_w, \tag{16}$$

where $PD_{i,j}$ and I_j are defined in (14) and (15), respectively.

The PC of the product i in the j-th week is defined as

$$PC_{i,j} = NC_{i,j}\, PQ_{i,j}, \quad i = 1, \dots, n_p, \quad j = 1, \dots, n_w, \tag{17}$$

where $NC_{i,j}$ and $PQ_{i,j}$ are the unit net cost and the purchased quantity of the product i in the j-th week, respectively. Notice that, in the case of the optimized system with model M1, $PQ_{i,j}$ is estimated from Q defined in (11), while in the case of perishable products based on model M2, $PQ_{i,j}$ is estimated from $y^0 - s_j$ given in (13), with s_j being the stock level at the beginning of the j-th week. For the non-optimized system, this value may be empirically obtained. When financial indicators $PI_{i,j}$ and PC_j given in (16) and (17) are computed, the variable contribution margin (VCM) of the product i in the j-th week is computed by

$$VCM_{i,j} = PI_{i,j} - PC_{i,j}, \quad i = 1, \dots, n_p, \quad j = 1, \dots, n_w. \tag{18}$$

The OC for the product i in the j-th week is given by

$$OC_{i,j} = OC_i/52, \quad i = 1, \dots, n_p, \quad j = 1, \dots, n_w, \tag{19}$$

where OC_i is the annual OC of the product i expressed by

$$OC_i = \sum_{h=1}^{3} OC_i^h\, OQ_i,$$

with OQ_i being the annual order quantity and OC_i^h the cost of kind h given in Table I. Notice that, for the optimized system with model M1, OQ_i is estimated from Q defined in (11) employing λ/Q for each product (with λ being expressed as a demand rate per year), while in the case of perishable products with model M2, $OQ_i = 52$, for all $i = 1, \dots, n_p$. For the non-optimized system, this value may be empirically obtained.

The SC for the product i in the j-th week is given by

$$SC_{i,j} = [SC_i/52]\, SQ_{i,j}, \quad i = 1, \dots, n_p, \quad j = 1, \dots, n_w, \tag{20}$$

where SC_i is the annual SC of the product i defined by $SC_i = \sum_{k=1}^{5} SC_i^k/SQ_i$, with SC_i^k being the annual SC of kind k defined in Table II, and $SQ_i = \sum_{j=1}^{52} SQ_{i,j}$ the annual stored quantity, both for the product i, whereas $SQ_{i,j}$ is the stored quantity of the product i during the j-th week. Notice that, for the optimized system with model M1, $SQ_{i,j}$ is estimated from the average stock level per cycle defined as

$$SQ = Q/2 + SS, \tag{21}$$

Table II. Annual costs involved in the storage of a product (SC^k).

Cost	Description
SC^1	Annual cost of amortization of buildings and networks for air conditioning, handling equipment, information processing, receiving, storage media, and weighing, among others.
SC^2	Annual cost of damage, losses, obsolescence, and product losses incurred in the storage period.
SC^3	Annual cost of cleaning materials and storehouse, containers, packaging, and printed matter.
SC^4	Annual cost of energy spent on the storehouse, including battery charging necessary for handling, data processing equipment, and lighting.
SC_5	Annual cost of rental of equipment and facilities, during insurance, storage and communications, and taxes.

SC, storing cost.

where Q and SS are given in (11) and (12), respectively. However, in the case of perishable products with model M2, $SQ_{i,j}$ is estimated from the expected inventory level by single period. For the non-optimized system, this value may be empirically obtained.

The CMs are absorbable by the sales with respect to indirect costs, which are subtracted from the VCM given in (18) to obtain the total CM of the product i in the j-th week by

$$\mathrm{CM}_{i,j} = \mathrm{VCM}_{i,j} - \left[\mathrm{OC}_{i,j} + \mathrm{SC}_{i,j}\right], \quad i = 1, \ldots, n_p, \quad j = 1, \ldots, n_w, \tag{22}$$

where $\mathrm{VCM}_{i,j}$, $\mathrm{OC}_{i,j}$, and $\mathrm{SC}_{i,j}$ are defined in (18)–(20), respectively. Hence, a series of CMs for n_p products (one for each of them) is collected. Thus, the CM of all the products of the inventory assortment in the j-th week is

$$\mathrm{CM}_j = \sum_{i=1}^{n_p} \mathrm{CM}_{i,j}, \quad j = 1 \ldots, n_w,$$

where $\mathrm{CM}_{i,j}$ is defined in (22). Therefore, the total CM of the inventory system is expressed by

$$\mathrm{CM} = \sum_{j=1}^{n_w} \mathrm{CM}_j.$$

Notice that the objective function to be maximized is the sum of $\mathrm{CM}_{i,j}$ for the product i in the j-th week, during all the period of study totalizing n_w weeks, for the food ration composed by n_p products with independent demand. The $\mathrm{VCM}_{i,j}$ and costs OC_j, $\mathrm{SC}_{i,j}$ depend on the inventory model of the product i. Therefore, the objective function to be maximized is

$$\sum_{i=1}^{n_p}\sum_{j=1}^{n_w} \mathrm{CM}_{i,j} = \sum_{i=1}^{n_p}\sum_{j=1}^{n_w} \left[\mathrm{VCM}_{i,j} - \mathrm{OC}_{i,j} - \mathrm{SC}_{i,j}\right].$$

To calculate (i) CMs from the differential revenues and (ii) costs from the movements in and out of the inventory assortment, our methodology considers independent products and not to all the food ration. Then, absorbable costs to order and store are also obtained with the same criteria of independence. This method turns out to be more streamlined, because it does not consider the correlations that could exist between products of the food ration, which is a source for a future work (see Section 5).

3.6. Summary of the methodology

Algorithm 2 summarizes our methodology detailed in Sections 3.2 to 3.5.

4. Illustrative example

4.1. Description of the example

We illustrate our methodology with two real-world demand data sets. One of them is associated with a perishable product and other one with a non-perishable product. We select these two products from the assortment of inventory because they allow us to show adequately the proposed methodology. A detailed study for all the products can be found in [5]. The

Algorithm 2 Methodology for stochastic inventory models based on the ZARBS distribution

1: Fix costs c_h, c_o, c_p, c_s, and c_u.
2: Collect demand data for the product i, with $i = 1, \ldots, n_p$, in each day of n_w weeks.
3: Carry out a correlation study for data collected in step 2 to detect any type of possible dependence. If some dependence is detected, it must be removed using suitable techniques and then to continue with step 4.
4: Propose distributions for the demand data analyzed in step 3 based on an exploratory data analysis.
5: Estimate the parameters of the distributions proposed in step 4.
6: Apply goodness-of-fit tests to establish the most adequate distribution.
7: Select the suitable inventory model depending on the kind of product i.
8: Find the optimal inventory elements (Q, r, y^0) based on the distribution established in step 6 and the costs fixed in step 1.
9: Compute the VCM for the product i during the j-th week of the optimal policy obtained in step 8.
10: Determine the corresponding OC for the product i in the j-th week.
11: Calculate the SC for the product i during the j-th week.
12: Obtain the CM for the product i in the j-th week.
13: Repeat steps 2 to 12 until completing n_p products.
14: Establish the optimized total CM and compare it with the non-optimized total CM.

illustration is divided in three parts. First, we carry out the statistical analysis for the two demand data sets of the products under study and describe the computational implementation developed for the proposed methodology. We perform an exploratory data analysis and, based on it, we postulate the ZARBS distribution for analyzing the data. We test if this distribution provides a good fitting to both of the data sets. Second, we use inventory management models for responding questions about what quantity of the product should be required in each order and when such an order should be placed. Third and finally, we carry out a financial analysis, which can be helpful in making decisions of inventory management, demonstrating the potential of our methodology summarized in Algorithm 2.

4.2. *Description of the problem*

Currently, most of the Chilean food service companies are not optimizing their supply of raw materials. These materials form the inventory assortment of such food companies, which are divided in perishable products under single period models (suitable for fruits, meats, and vegetables), and non-perishable products under multiple period models (with greater storage capacity and suitable for grocery). These companies have administrative deficiencies, because they do not consider an optimized inventory policy in their organizational practices. They rather rely the management of their raw materials on the monthly planning of a food ration. This ration takes into account technical regulations for procurement contracts related to nutritional contributions and adequacy, frequency, and amounts of food consumption, with respect to the people who consume such a ration. In this way, because of these regulations, several nutritional recommendations must be met, but the administrative deficiencies aforementioned remain. Such deficiencies affect negatively the CMs of the company, which can be improved by a scientific management of inventory logistics. In this illustration, we carry out a statistical analysis of the demand for two products (one perishable and the other not) utilized to prepare a food ration of an anonymous Chilean company. It allows an optimized inventory system for this company to be established and its CMs to be improved.

4.3. *Statistical analysis*

The data to be analyzed correspond to the daily DQ of two products. The non-perishable or stable food product is hair noodles, whose demand data set is denoted as set I. It was collected since 20 November 2011 and until 26 May 2012 (27 weeks), during a study of inventory management conducted by Fernando Rojas and Victor Leiva in the University of Valparaiso, Chile. There were 189 days, in which 168 (88.89%) of them, the product was not demanded, so that during 21 days (11.11%), at least one unit of the product was requested in the period of observation. The perishable product is ground beef, whose demand data set is denoted as set II. It was collected during the same period and for the same study and that the product hair noodles. For set II, 121 (64.02%) of 189 days the product was not demanded and during 68 days (35.98%), at least one product was requested. The quantity of each product used in the preparation of a food ration has its respective measuring unit, which possesses an equivalence with respect to the other products of the inventory assortment. For hair noodles, 1 unit = 1 package of 1 kg, whereas for ground beef, 1 unit = 1kg. Data sets I and I are available from the authors under request.

Because demand for a product can have some dependence over time, we study the autocorrelation function (ACF) and partial ACF (PACF) of sets I and II. Figure 5 displays plots of such ACFs and PACFs, from which no evidence of a large

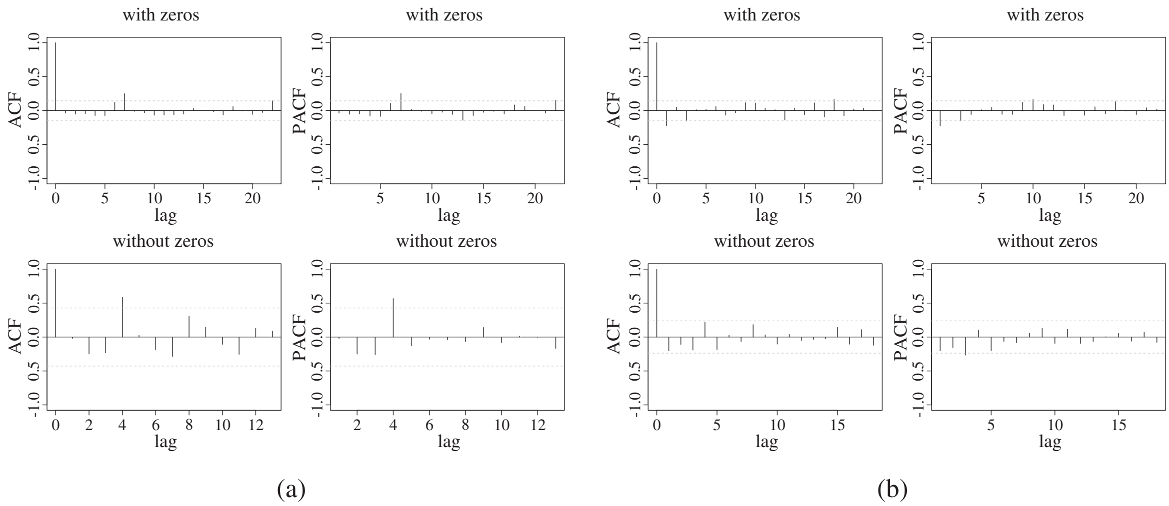


Figure 5. Autocorrelation function (ACF) and partial ACF plots for (a) set I and (b) set II.

Table III. Descriptive measures for the indicated daily demand data set.

Set	With zeros	n	Min.	Median	$\bar{y}$	SD	CV	CS	CK	Max.
Hair noodles	Yes	189	0.000	0.000	0.259	0.972	3.75	4.556	21.419	6.800
	No	21	0.500	1.000	2.333	1.951	0.836	0.856	−0.820	6.800
Ground beef	Yes	189	0.000	0.000	7.329	13.416	1.83	2.157	4.368	65.000
	No	68	1.000	17.000	20.391	15.344	0.753	1.075	0.429	65.000

Min., minimum; SD, standard deviation; CV, coefficient of variation; CS, coefficient of skewness; CK, coefficient of kurtosis; Max., maximum.

autocorrelation is detected. Only in the case of the product hair noodles we detect a small autocorrelation each 7 days (1 week), which is explained because this product is often used weekly, but such a correlation is negligible. Furthermore, our autocorrelation analysis is carried out with and without the presence of zeros and, in both of these cases, no evidence of autocorrelation is detected in the data. Therefore, our analysis is centered in a sample for independent data, modeling the positive part of the data with a suitable demand distribution.

Table III presents some descriptive measures of the data sets under analysis, such as the sample size (n), minimum and maximum values, mean ($\bar{y}$), median, standard deviation, CV, coefficient of skewness (CS), and coefficient of kurtosis (CK). Note that the empirical demand distributions of both products have positive skewness (asymmetry) and moderate kurtosis.

Castro-Kuriss *et al.* [48] discussed several goodness-of-fit tests and proposed graphical tools based on these tests with uncensored and censored data for non-location-scale distributions, which is summarized in Algorithm 3 for the positive part of the ZARBS distribution. This algorithm is based on the Kolmogorov–Smirnov (KS) test to evaluate whether the RBS distribution fits a data set or not. As it is known, the KS test can be associated with the probability versus probability (PP) plot. We use Algorithm 3 to determine whether the ZARBS distribution is suitable for modeling the positive part of the analyzed data. We compare the fit of the normal, IG, and RBS distributions to the positive part of the data. The corresponding KS p-values are as follows: (i) 0.0710 (normal), 0.1215 (IG), and 0.1017 (RBS) for set I; and (ii) 0.0498 (normal), 0.1228 (IG), and 0.1666 (RBS) for set II. Note that the IG and RBS distributions are suitable for modeling both demand data sets (I and II), because no significative statistical evidence exists against these two distributions (at a significance level of 10%), but in the case of the normal distribution, it exists. For set I, the IG distribution fits these data slightly better than the RBS distribution, but the latter one is still a good model. For set II, the RBS distribution is now who fits these data slightly better than the IG distribution. Figures 6 and 7 show a good agreement of the mentioned distributions to sets I and II by PP plots with 95% acceptance bands constructed using the KS test. To corroborate the fitting of the RBS distribution, Figure 8 displays CDF plots for sets I and II. Therefore, we employ the RBS distribution for modeling the demand data and for carrying out the subsequent inventory and financial analyses.

Table IV presents point and interval estimates (at a confidence level of 95%) of the parameters and the corresponding estimated SEs for the ZARBS distribution based on sets I and II. The estimation of parameters and fit of the ZARBS

Algorithm 3 Goodness-of-fit test for the RBS distribution with demand data

1: Collect data $y_1, \dots, y_n$ and order them as $y_{1:n}, \dots, y_{n:n}$.
2: Estimate μ and δ of the RBS distribution by $\hat{\mu}$ and $\hat{\delta}$, respectively, with $y_1, \dots, y_n$.
3: Compute $\hat{v}_{j:n} = F_{Y_1}(y_{j:n}; \hat{\mu}, \hat{\delta})$, for $j = 1, \dots, n$, with $F_{Y_1}(\cdot)$ being the RBS CDF.
4: Calculate $\hat{t}_j = \Phi^{-1}(\hat{v}_{j:n})$, where $\Phi^{-1}(\cdot)$ is the N(0, 1) inverse CDF.
5: Obtain $\hat{u}_{j:n} = \Phi(\hat{z}_j)$, with $\hat{z}_j = [\hat{t}_j - \bar{t}]/s_t$, $\bar{t} = \sum_{j=1}^{n} \hat{t}_j/n$ and $s_t = [\sum_{j=1}^{n} \{\hat{t}_j - \bar{t}\}^2/\{n-1\}]^{1/2}$.
6: Draw the PP plot with points $w_{j:n} = [2j-1]/[2n]$ versus $\hat{u}_{j:n}$, for $j = 1, \dots, n$.
7: Specify a significance level $1 - \zeta$.
8: Construct acceptance bands according to $(\max\{w - d_\zeta + 1/[2n], 0\}, \min\{w + d_\zeta - 1/[2n], 1\})$, where d_ζ is the $100\times\zeta$-th percentile of the KS distribution and w is a continuous version of $w_{j:n}$.
9: Determine the p-value of the KS statistic and reject the null hypothesis of a RBS distribution for the specified significance level based on this p-value.
10: Corroborate coherence between steps 8 and 9.

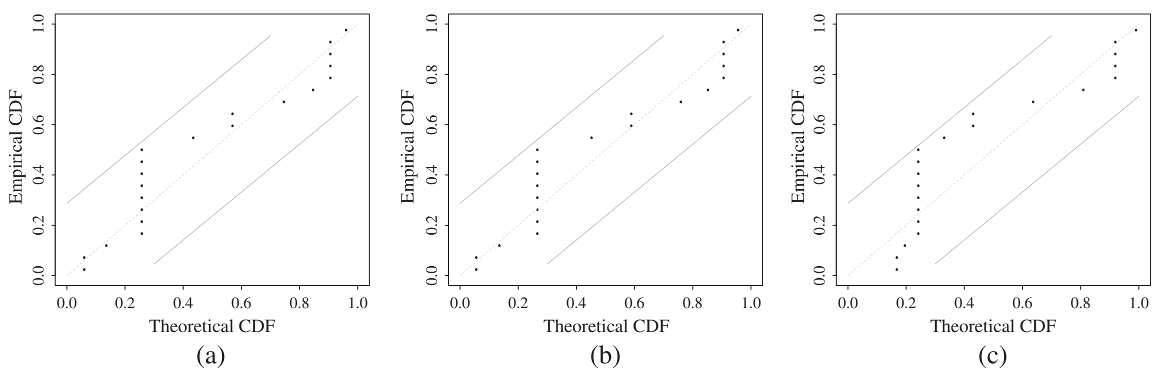


Figure 6. Probability versus probability plots with 95% acceptance bands for (a) reparameterized Birnbaum–Saunders (RBS), (b) inverse Gaussian (IG), and (c) normal distributions with set I.

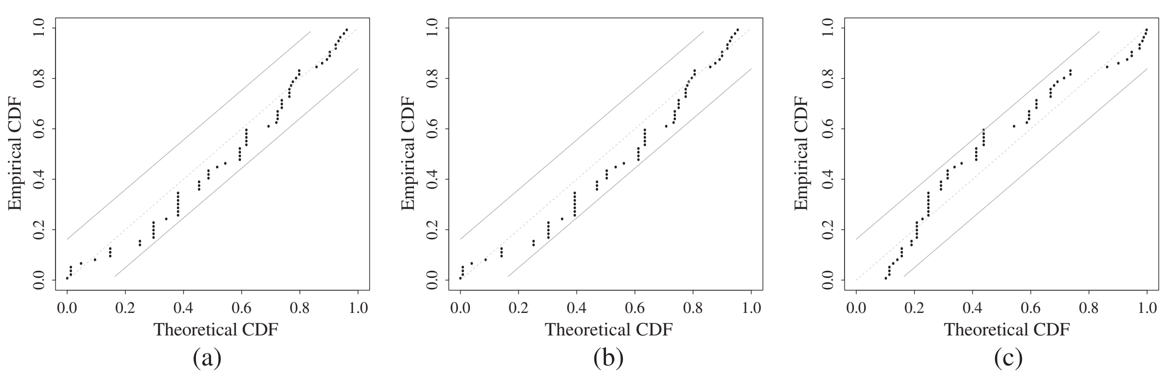


Figure 7. Probability versus probability plots with 95% acceptance bands for (a) reparameterized Birnbaum–Saunders (RBS), (b) inverse Gaussian (IG), and (c) normal distributions with set II.

distribution to the data under analysis is performed by using the `gamlss` package. With the results in Table IV, we estimate the demand rate (mean of the demand Y, λ say) of the product ground beef per time unit, whose point estimate and corresponding 95% CI and CB, using the delta method and asymptotic independence of the estimators, are given by

$$\widehat{\mathrm{E}[Y]} = \hat{\lambda} = [1 - \hat{p}]\hat{\mu} = [1 - 0.640] \times 20.034 = 7.208 \text{ units/day},$$

$$\mathrm{CI}(\lambda; 95\%) = \left[\hat{\lambda} \pm z(0.975) \times \widehat{\mathrm{SE}}(\hat{\lambda})\right] = [7.208 \pm 1.960 \times 1.111] = [5.030; 9.386] \text{ units/day},$$

$$\mathrm{CB}(\lambda; 95\%) = \hat{\lambda} + z(0.95) \times \widehat{\mathrm{SE}}(\hat{\lambda}) = 7.208 + 1.645 \times 1.111 = 9.036 \text{ units/day},$$

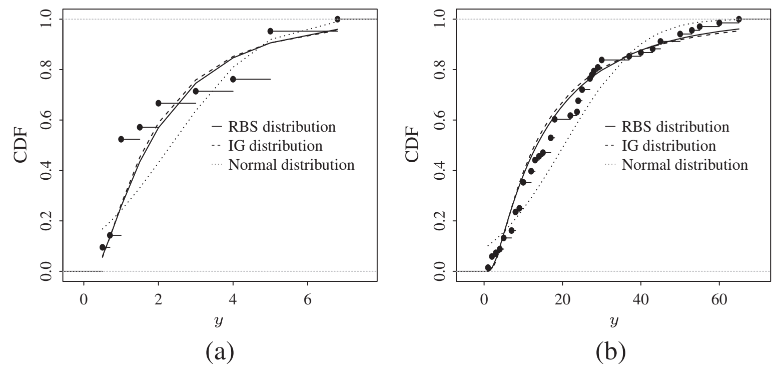


Figure 8. Cumulative distribution function (CDF) plots for (a) set I and (b) set II.

Table IV. Point and interval estimates of the ZARBS distribution parameters for the indicated data set.

Data set	θ	θ_1	$\widehat{\theta}$	$\widehat{\theta}_1$	$\widehat{\mathrm{SE}}(\widehat{\theta})$	$\widehat{\mathrm{SE}}(\widehat{\theta}_1)$	CI(θ)	CI(θ_1)
Ground beef	μ	log(μ)	20.03	2.997	2.372	0.118	[15.89;25.27]	[2.765;3.229]
	δ	log(δ)	2.051	0.718	0.352	0.172	[1.465;2.871]	[0.382;1.055]
	p	logit(p)	0.640	0.575	0.035	0.152	[0.569;0.705]	[0.278;0.873]
Hair noodles	μ	log(μ)	2.347	0.853	0.442	0.188	[1.623;3.395]	[0.484;1.222]
	δ	log(σ)	2.782	1.023	0.859	0.309	[1.519;5.095]	[0.418;1.628]
	p	logit(p)	0.889	2.081	0.023	0.233	[0.835;0.927]	[1.624;2.537]

ZARBS, zero-adjusted reparameterized Birnbaum–Saunders; SE, standard error; CI, confidence interval.

respectively, where $\widehat{\mathrm{SE}}(\widehat{\lambda}) = [\widehat{\mu}^2\,\widehat{\mathrm{Var}}[\widehat{p}] + \{1-\widehat{p}\}^2\widehat{\mathrm{Var}}[\widehat{\mu}]]^{1/2}$. Analogously, for the product hair noodles per time unit, we have

$$\widehat{\mathrm{E}[Y]} = \widehat{\lambda} = [1-\widehat{p}]\widehat{\mu} = [1-0.889]\times 2.347 = 0.261 \text{ units/day},$$

$$\mathrm{CI}(\lambda; 95\%) = [\widehat{\lambda} \pm z(0.975)\times\widehat{\mathrm{SE}}(\widehat{\lambda})] = [0.261 \pm 1.960\times 0.073] = [0.118; 0.404] \text{ units/day},$$

$$\mathrm{CB}(\lambda; 95\%) = \widehat{\lambda} + z(0.95)\times\widehat{\mathrm{SE}}(\widehat{\lambda}) = 0.261 + 1.645\times 0.073 = 0.381 \text{ units/day}.$$

4.4. *Inventory management analysis*

We select the ZARBS distribution as the suitable model among the proposed distributions to describe the demand data of the two products (ground beef and hair noodles). Then, we use the adequate inventory management model to determine the optimum stock and to minimize the corresponding total cost, depending on whether the product is non-perishable (hair noodles) or perishable (ground beef). The relevant cost are c_o = US\$0.880 and c_s = US\$0.085, for model M1, and c_u = US\$1.08, c_p = US\$0.032 and c_h = US\$0.088, for model M2, all of them fixed by the manager of the food service, whereas the LT is fixed at $l = 3$ days.

To calculate the ROP based on the ZARBS distribution for the product hair noodles, we need the mean and variance of $Y \sim \mathrm{ZARBS}(\mu, \delta, p)$. By using model M1 and the ZARBS QF provided in (4), we estimate the ROP for the product hair noodles as $\widehat{r} = \widehat{\lambda}\times 3 + (\widehat{k}_{95}\times\sqrt{3}\times\widehat{\sigma}) = 0.261\times 3 + (5.761\times\sqrt{3\times 0.996}) = 10.741$ units. The optimum quantity of stock required in each order is estimated as $\widehat{Q} = [2\times\widehat{\lambda}\times c_o/c_s]^{1/2} = \sqrt{2\times 0.261\times 0.880/0.085} = 2.325$ units. Thus, we must place an order of the hair noodles product of $\widehat{Q} = 2.325$ units, when the stock level is $\widehat{r} = 10.741$ units, for assuring a service level of 95%. Therefore, from (21), the average stock level per cycle is $\widehat{Q}_a = \widehat{Q}/2 + \widehat{\kappa}_{95}\times\widehat{\sigma}_{D_l} = 2.325/2 + 5.761\times\sqrt{3\times 0.996} = 11.121$ units.

For the product ground beef, we consider model M2 based on the CR given in (13). To calculate the optimum quantity of units to be stored, from the ZARBS distribution for set II, we need the CDF of $Y \sim \mathrm{ZARBS}(\mu, \delta, p)$ given in (3). By using expression given in (13), we estimate y^0 for the product ground beef from $F_Y(y^0) = 0.8968$, that is, we need to determinate the estimated 89.68-th quantile of the ZARBS distribution. Then, the optimum quantity of units of ground beef to be stored, according to the ZARBS QF given in (4), is $\widehat{y}^0 = \widehat{y}(0.8968; \widehat{\mu}, \widehat{\delta}) = 18.653$ units. Thus, at the beginning of each single period (1 week, in our case), the stock level must be checked and then a quantity of $18.653 - s_j$, for $j = 1, \ldots, 27$ (weeks), must be ordered. Therefore, in this case, the average stock level is $\widehat{y}_a = \int_0^{18.653}[18.653 - x]f_Y(x)\,\mathrm{d}x = 2.182$ units, where $f_Y(\cdot)$ is the ZARBS PDF.

4.5. Financial analysis

To compare the optimized system with the current (non-optimized) system used by the food company, we determine the total CM of the product ground beef employed in the preparation of the food ration, during the time period under study (27 weeks). Note that this product is only one out of 89 products of the inventory assortment of the food company. An analogous procedure can be used for hair noodles and any other product of the inventory assortment. Optimized and non-optimized CMs are obtained according to Algorithm 2. Specifically, to determine the CMs generated by the product ground beef, we use the expression given in (22). First, we obtain the VCMs based on (18) following the definitions and the sequence of equations given in (14)–(17). Second, we calculate the corresponding OC and SC using the formulas displayed in (19) and (20). Based on this information, we obtain a reduction of 6.61% for the product ground beef. Therefore, an improving in the CMs of the product ground beef is reached, when the proposed optimized inventory management system is used, in relation to the current system (non-optimized) by the food company. We recall the results reported here are only an illustration with one of the 89 products of the inventory assortment of the food company studied in this paper. Thus, the profits in the CMs of each product must be totalized. In any case, the overall CM for all the inventory assortment should be positive and improved with the optimized inventory management system (see details in [5]).

5. Conclusions and future works

We have proposed a methodology for stochastic inventory models based on the new zero-adjusted Birnbaum–Saunders distribution. This study was divided in two parts. First, several properties and features of the new distribution were obtained and simulations were conducted for evaluating the behavior of the proposed estimators for its parameters. These simulations have shown the good performance of the proposed estimators. We have developed a function in the `R` statistical software incorporating the new distribution in the family of generalized additive models of location, shape, and scale. It has allowed a real-world demand data analysis to be carried out. Second, we have detailed a methodology for inventory logistics discussing its relevant costs, how the data must be collected, its suppositions and restrictions, the associated stochastic inventory models, and indicators that must be used to evaluate financially the results. We have illustrated the methodology with real-world demand data showing the potential and convenience of using it by improving the contribution margins of a Chilean food company.

It is noteworthy that although we have shown good results using the proposed methodology, still some methodological and theoretical aspects can be improved. For example, it is possible to explore the statistical dependence over time and among products considering time series and multivariate models. In fact, we are planning to collect real-world data of this type in a future study. In addition, from the practical point of view, other future studies considering demand data for a food ration instead of its components is being considered by the authors.

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