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Inventory management for new products with triangularly distributed demand and lead-time

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ABSTRACT

This paper proposes a computational methodology to deal with the inventory management of new products by using the triangular distribution for both demand per unit time and lead-time. The distribution for demand during lead-time (or lead-time demand) corresponds to the sum of demands per unit time, which is difficult to obtain. We consider the triangular distribution because it is useful when a distribution is unknown due to data unavailability or problems to collect them. We provide an approach to estimate the probability density function of the unknown lead-time demand distribution and use it to establish the suitable inventory model for new products by optimizing the associated costs. We evaluate the performance of the proposed methodology with simulated and real-world demand data. This methodology may be a decision support tool for managers dealing with the measurement of demand uncertainty in new products.

1. Introduction

Studying uncertainty of demand during lead-time or lead-time demand (LTD) is a key aspect not only for retailing and manufacturing, but also for supply chain planning [9]. This uncertainty is present because demand per unit time (DPUT) and lead-time (LT) usually occur in a stochastic fashion. Therefore, DPUT, LT and LTD are random variables (RVs) following statistical distributions, which can be characterized by their corresponding probability density functions (PDFs).

We assume that the LTD is a random sum until the LT of independent DPUTs, that is, the corresponding DPUT time series is uncorrelated. The PDF of the LTD distribution is useful to determine the components of probabilistic inventory models. A model that is often used for inventory supply planning is the (Q, r) model, which is based on the order quantity or lot size (Q) and reorder point (r). Note that Q corresponds to the quantity to be ordered when the stock achieves a certain amount of products r. The reorder point often includes a safety stock (SS) corresponding to a buffer stock used to mitigate the risk of a stock-out. The model components Q and r must be determined to minimize the total cost of the inventory management. Such a cost is function of the holding, ordering and shortage costs. When calculating the reorder point for a fixed service level, the LTD PDF is used. If the LTD distribution is unknown, this PDF can be approximated by some suitable approach. We simultaneously optimize Q and r as detailed in Section 2.4; see also [33].

The Gaussian (or normal) distribution is often employed to describe the RVs DPUT, LT and LTD due to its attractive properties. However, assuming normality is not always suitable to model these RVs [23,29,36]. When historical DPUT data for a single-product are available, a pool of non-normal distributions can be considered as candidates for modeling these data. To obtain the inventory management model, the suitable DPUT distribution must be selected by standard goodness-of-fit (GOF) methods [1]. Nevertheless, there are cases where the associated LTD distribution is difficult to obtain. Under such circumstances, empirical distributions generated from raw data may be helpful for decision making [35]. In the case of new products, modeling DPUT, LT and LTD is difficult because historical data are unavailable, but business decisions must be made prior to the availability of these data [14,22]. Ref. [5] studied inventory models with a lognormal DPUT distribution and indicated the LTD distribution under different DPUT and LT distributions.

Demand uncertainty for new products has been handled by learning-based and non-learning-based approaches [7]. Under learning-based approaches, multiple production or purchasing commitments are decided first in such a way that sales data should be further obtained to update the demand forecasts and,

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then, to review these production/purchasing commitments [3,6,28]. Under non-learning-based approaches, [8] used judgmental forecasts to establish demand uncertainty, whereas [36] employed a uniform (UNI) distribution in that same case. Demand uncertainty of new products could also benefit from other non-learning-based approaches, such as approximations of untractable LTD distributions by considering tractable DPUT and LT distributions.

The triangular (TRI) distribution is tractable and known to be useful when data are unavailable, difficult to obtain or expensive to collect [10]. This distribution can be used for involving managers in the analytical process by considering their subjective estimates of the minimum, most likely (mode) and maximum values. According to [15], the TRI distribution has the advantage of being intuitively plausible to practitioners. However, despite its long history dating back to [32], its recognition as a user-friendly tool is more recent [19]. Assuming triangularity may help managers in dealing with new products, which have no historical data, and therefore, offer no possibility of establishing analogies with similar products. Based on this assumption, managers may decide the first lot size to be ordered and the reorder point.

The objective of the present paper is to propose a novel computational methodology for inventory management of new products. Specifically, we consider TRI distributions for modeling both DPUT and LT. In this case, the LTD distribution is unknown. Based on the (Q, r) inventory model, we need the LTD PDF to determine the components Q and r that minimize the expected inventory total cost. We provide an approach to estimate the actual PDF of the unknown LTD distribution obtained from triangularly distributed DPUT and LT by using polynomials and a mixture of truncated exponentials (MTEs). We evaluate the quality of the proposed approach with the Kullback–Leibler (KL) divergence [20] using the kernel non-parametric method to estimate the unknown LTD PDF such as [21]. Then, we employ the approach to the unknown PDF for establishing a computational solution to the (Q, r) inventory model for new products optimizing the associated costs. Components Q and r are found by using the bisection method on the partial derivatives of the total cost function with expected shortages per cycle, which are studied under different scenarios [13]. Managerial implications for inventory decision-making are also addressed.

Section 2 proposes the novel methodology; Section 3 discusses a computational framework for this methodology and conducts simulations to evaluate its performance; Section 4 illustrates its potential applications with real-world data; and Section 5 provides our conclusions and possible future works.

2. Methodology

In this section, we propose a methodology for inventory management of new products. In Section 2.1, we present a background on the TRI distribution, which is helpful to model both DPUT and LT, when their distributions are unknown due to data unavailability or difficulties to collect them. In Section 2.2, we provide some details on LTD distributions obtained from the sum of independent RVs, which are useful for determining the LTD PDF. In Section 2.4, we approximate the unknown LTD PDF resulting from triangularly distributed DPUT and LT by using polynomials and MTEs. The LTD PDF is needed to determine Q and r when minimizing the inventory cost. In Section 2.5, we define the KL divergence to evaluate the quality of the proposed approximations in relation to an actual PDF, which is obtained with the kernel method described in Section 2.3. At last, in Section 2.6, we compute an analytical solution of the (Q, r) model considering the polynomial approximation for the LTD PDF.

2.1. Triangular distribution

Let T be a continuous RV following a TRI distribution with parameters $a, b, c \in \mathbb{R}$, where a and b are the minimum and maximum values of T, respectively, and c is the mode of the distribution. This is denoted by $T \sim \text{TRI}(a, b, c)$. Then, the PDF, cumulative distribution function (CDF) and quantile function (QF) of T are, respectively, given by

$$f_T(t) = \frac{\mathrm{d}F_T(t)}{\mathrm{d}t} = \begin{cases} \dfrac{2(t-a)}{(b-a)(c-a)} & \text{if } a \le t \le c; \\ \dfrac{2(b-t)}{(b-a)(b-c)} & \text{if } c \le t \le b; \\ 0 & \text{otherwise;} \end{cases}$$

$$F_T(t) = \mathrm{P}(T \le t) = \int_{-\infty}^{t} f_T(v)\mathrm{d}v = \begin{cases} 0 & \text{if } t < a; \\ \dfrac{(c-a)}{(c-b)}\dfrac{(t-a)^2}{(c-a)^2}, & \text{if } a \le t \le c; \\ 1 - \dfrac{(b-c)}{(b-a)}\dfrac{(b-t)^2}{(b-c)^2}, & \text{if } c \le t \le b; \\ 1 & \text{if } t > b; \end{cases}$$

$$t(q) = F_T^{-1}(q) = \begin{cases} a + \sqrt{q(c-a)(b-a)} & \text{if } 0 \le q \le (c-a)/(b-a); \\ b - \sqrt{(1-q)(b-c)(b-a)} & \text{if } (c-a)/(b-a) \le q \le 1. \end{cases} \quad (1)$$

A random number generator for $T \sim \text{TRI}(a, b, c)$ is provided in Algorithm 1 based on (1).

Algorithm 1. Random number generator for the TRI distribution.

1: Generate a uniform value u from $U \sim \text{UNI}(0, 1)$.
2: Set values for a, b and c of $T \sim \text{TRI}(a, b, c)$;
3: Compute a random number $t = t_1$ or $t = t_2$ from $T \sim \text{TRI}(a, b, c)$ using (1), that is,
  3.1: If $0 \le u \le (c-a)/(b-a)$, then $t_1 = a + \sqrt{u(c-a)(b-a)}$;
  3.2: Else $t_2 = b - \sqrt{(1-u)(b-c)(b-a)}$;
4: Repeat Steps 1–3 until the required number of LTD observations has been generated.

The mean and variance of $T \sim \text{TRI}(a, b, c)$ are, respectively,

$$\lambda = \mathrm{E}(T) = \frac{a+b+c}{3}, \quad \sigma^2 = \mathrm{Var}(T) = \frac{(b-a)^2}{18}\left(1 - \frac{(c-a)(b-c)}{(b-a)^2}\right). \quad (2)$$

2.2. Demand distribution during lead-time

Let X be a RV corresponding to the DPUT, which has mean $\mathrm{E}(X) = \lambda_X$ and variance $\mathrm{Var}(X) = \sigma_X^2$. In addition, let the RV L be the LT between the ordering of a product and its delivery, which has mean $\mathrm{E}(L) = \lambda_L$ and variance $\mathrm{Var}(L) = \sigma_L^2$. Furthermore, L is assumed to be independent from each element of the sequence of independent identically distributed RVs $\{X_1, X_2, \ldots, X_L\}$ obtained from the RV X. Moreover, assume that orders do not cross [12]. Therefore, the LTD for a product is the random sum given by

$$Y = X_1 + X_2 + \cdots + X_L, \quad (3)$$

with PDF $f_Y(\cdot)$ defined on $[0, \infty)$ (non-negative support), CDF

$$F_Y(y) = \int_0^y f_Y(v)\,\mathrm{d}v, \quad (4)$$

and QF $y(q) = F_Y^{-1}(q)$, for $0 < q < 1$. The expectation and variance of Y are, respectively, expressed as

$$\mathrm{E}(Y) = \mathrm{E}(L)\mathrm{E}(X) = \lambda_L \lambda_X, \quad (5)$$

$$\mathrm{Var}(Y) = \mathrm{Var}(L)(\mathrm{E}(X))^2 + \mathrm{E}(L)\mathrm{Var}(X) = \sigma_L^2 \lambda_X^2 + \lambda_L \sigma_X^2. \quad (6)$$

Note that, in general, the LT and DPUT can be modeled by any discrete or continuous distribution. However, since in this work we assume that managers are planning to order the first lot size and reorder point for a new product, we consider TRI distributions for both LT and DPUT.

2.3. Kernel estimation

By fixing minimum, maximum and mode values for the RVs LT and DPUT with TRI distributions, using Algorithm 1 to generate LT and DPUT data, and the expression given in (3), we are able to generate a sequence $\{y_1,\ldots,y_n\}$ of n LTD observations (data). Then, based on this sequence, we can define a kernel estimate of the unknown PDF $f_Y(\cdot)$ by

$$\widehat{f}_Y(y)=\frac{1}{n\,h}\sum_{i=1}^{n}K\left(\frac{y_i-y}{h}\right),\ y>0, \tag{7}$$

where $K(\cdot)$ is a kernel function satisfying $\int_0^\infty K(y)\,\mathrm{d}y=1$, h a smoothing parameter (or bandwidth) and y the point at which the PDF is estimated. The Gaussian kernel with support in $\mathbb{R}$ is often assumed for $K(\cdot)$ given in (7). However, we are modeling demand data with support in $[a,b]$. Thus, instead of the Gaussian kernel, it seems more natural to estimate the unknown PDF with a TRI kernel by using

$$\widehat{f}_Y(y)=\frac{1}{n}\sum_{i=1}^{n}K_{h,y}(y_i), \tag{8}$$

where $K_{h,y}$ is a TRI kernel of parameters h (bandwidth) and y (point at which the PDF is estimated) [31].

2.4. Density approximations

As mentioned, we consider TRI distributions to model both DPUT and LT. Then, the LTD distribution of the new product is unknown. We approximate the LTD PDF $f_Y(\cdot)$ with some suitable functions by using the PDF estimated from the kernel method as the actual LTD PDF. Consider real-valued basis functions given by

$$g_k(y)=\sum_{i=0}^{k}\alpha_i\Psi_i(y), \tag{9}$$

where $\alpha_i\in\mathbb{R}$ are coefficients of the function $\Psi_i(\cdot)$, for $i=0,1,\ldots,k$. Particular cases of $\Psi_i(\cdot)$ defined in (9) correspond to polynomial and exponential functions [21]. In order to approximate the PDF $f_Y(\cdot)$ from (9), the polynomial function of order k given by

$$\tilde{f}_Y(y)=\sum_{i=0}^{k}\alpha_i y^i \tag{10}$$

may be assumed. Thus, the LTD CDF given in (4) is approximated from (10) by

$$\tilde{F}_Y(y)=\int_0^y\sum_{i=0}^{k}\alpha_i v^i\,\mathrm{d}v.$$

In addition, also from (9), a second approximation for the PDF $f_Y(\cdot)$ can be established by a function of MTEs with k terms as

$$\overline{f}_{Y,j}(y)=\alpha_{1j}+\sum_{i=1}^{k}\alpha_{2i,j}\exp(\alpha_{2i+1,j}\,y),\quad j=1,\ldots,m, \tag{11}$$

where j corresponds to each m-piece, k-term interval; readers are referred to [30] for more details about approximation (11). Thus, the LTD CDF is obtained from (11) by

$$\overline{F}_{Y,j}(y)=\int_0^y\left(\alpha_{1j}+\sum_{i=1}^{k}\alpha_{2i,j}\exp(\alpha_{2i+1,j}\,v)\right)\mathrm{d}v,\quad j=1,\ldots,m.$$

2.5. Evaluation of the approximation

To evaluate the quality of the approximation provided in (10), we use the KL divergence [4] given in general by

$$\mathrm{KL}=\int_{-\infty}^{\infty}\log\left(f(y)/\tilde{f}(y)\right)f(y)\,\mathrm{d}y, \tag{12}$$

where $f(\cdot)$ is an actual PDF and $\tilde{f}(\cdot)$ its approximation.

Such as in [21], to compute the KL divergence given in (12) in practice, we consider a kernel estimate as actual PDF obtaining

$$\mathrm{KL}=\int_a^b\log\left(\widehat{f}_Y(y)/\tilde{f}_Y(y)\right)\widehat{f}_Y(y)\,\mathrm{d}y, \tag{13}$$

where $\widehat{f}_Y(\cdot)$ is the TRI kernel estimate given in (8) and $\tilde{f}_Y(\cdot)$ the approximation provided in (10) (or equivalently in (11)). We select the approximation whose KL value given (13) is the smallest one. We have empirically detected that the computational burden to calculate (13) is negligible, which is less than one second.

2.6. Inventory management models

The expected annual total cost of inventory assuming shortage is expressed as a sum of (i) the holding cost per product unit per year, denoted by $\mathrm{C_h}$, multiplied by the expected quantity in stock of product units; (ii) the ordering cost, denoted by $\mathrm{C_o}$, multiplied by the number of orders per year, and (iii) the penalty cost whenever there are stock-outs, denoted by $\mathrm{C_p}$, that is, the penalty cost per shortage product unit per year multiplied by the number of orders per year and by the expected quantity of shortage product units per year. We are assuming a business has demand 365 days a year. Thus, for the $(Q,\ r)$ model, the expected total cost per year is

$$\mathrm{C_T}=G(Q,\ r)=\left(\frac{Q}{2}+r-\mathrm{E}(Y)\right)\mathrm{C_h}+\frac{365\ \lambda_X}{Q}\mathrm{C_o}+S(r)\frac{365\ \lambda_X}{Q}\mathrm{C_p}, \tag{14}$$

where λ_X and $\mathrm{E}(Y)$ are defined as in (2) and (5), respectively [11,16,33]. On the one hand, note that λ_X is multiplied by 365, because the total cost given in (14) is defined on an annual basis and λ_X on a daily basis. On the other hand, $\mathrm{E}(Y)$ is not altered, because its scope is verified within each safety inventory cycle. For the inventory total cost given in (14), $r-\mathrm{E}(Y)=\mathrm{SS}=k_q\,\sqrt{\mathrm{Var}(Y)}$, with the standard deviation (SD) of the LTD $\sqrt{\mathrm{Var}(Y)}$ being given from (6) and k_q the amount of SDs of the LTD or safety factor (SF) associated with a service level of $q\times 100\%$, for $0<q<1$. Note that k_q corresponds to the $q\times 100$ th standardized quantile, often fixed at the 95th position for assuring a service level of 95%. In addition, in (14), $S(r)$ is the expected shortage per cycle given by

$$S(r)=\int_r^{y_{\max}}(y-r)f_Y(y)\,\mathrm{d}y, \tag{15}$$

where $y_{\max}$ is the maximum value of the LTD, r the already mentioned reorder point and, as also mentioned, $f_Y(\cdot)$ the LTD PDF. From (10), expression in (15) can be approximated by

$$\tilde{S}(r)=\int_r^{y_{\max}}(y-r)\sum_{i=0}^{k}\alpha_i y^i\,\mathrm{d}y. \tag{16}$$

Hence, by following the well-known sum rule in integration – where summation and integral can be reversed, (16) is given by

$$\tilde{S}(r)=\sum_{i=0}^{k}\alpha_i\int_r^{y_{\max}}(y-r)y^i\,\mathrm{d}y=\sum_{i=0}^{k}\alpha_i\left(\frac{y_{\max}^{i+2}-r^{i+2}}{i+2}-\frac{r(y_{\max}^{i+1}-r^{i+1})}{i+1}\right). \tag{17}$$

Note in (17) that we integrate first, expressing this approximate expected shortage per cycle by a polynomial summation in r (the reorder point – still a decision variable) and $y_{\max}$ (the maximum value of the LTD). To minimize the expected total cost given in (14), we insert (17) into it, take derivatives of $G(Q,r)$ with respect to Q and r, equate both derivatives to zero and obtain the optimal values of Q and r of the inventory model with shortages from the solutions in Q to these two equations given by

$$Q_1=\sqrt{\frac{2*365\lambda_X}{\mathrm{C_h}}\left(\mathrm{C_o}+\mathrm{C_p}\left(\sum_{i=0}^{k}\frac{\alpha_i(y_{\max}^{i+2}-r^{i+2})}{i+2}-\sum_{i=0}^{k}\frac{\alpha_i(y_{\max}^{i+1}-r^{i+1})}{i+1}\right)\right)}, \tag{18}$$

$$Q_2=\frac{365\lambda_X\mathrm{C_p}}{\mathrm{C_h}}\sum_{i=0}^{k}\frac{\alpha_i(y_{\max}^{i+1}-r^{i+1})}{i+1}. \tag{19}$$

Therefore, to find the optimal values of Q and r of the inventory model, we consider the equation

$$Q_1-Q_2=0. \tag{20}$$

The equation given in (20) can be solved in r by applying the bisection method. Substituting r into (18) or (19), we obtain the optimal value of Q. A similar treatment may be applied to equations obtained in (16)–(19) when the MTE approximation given in (11) is used. We recall the approximation (10) or (11) can be selected from the KL value given (13) to be the smallest one. We use the bisection method because it is the simplest and most robust algorithm for finding the root of an one-dimensional continuous function within a closed interval. One of its properties is that it always converges. Also, it is preferable to the Newton–Raphson method when the function coefficients are unknown. In our case, it is not simple to find coefficients from Eqs. (18) and (19). For details about the bisection method, see [26].

3. Computational framework

In this section, we detail the steps of the methodology proposed in Section 2 with four algorithms. Then, we discuss a computational framework developed for implementing the four algorithms and study the performance of this methodology by means of a simulation study.

3.1. Algorithms for the methodology

The sequence of algorithms below shows how inventory management of new products can be planned by companies using the methodology proposed in Section 2.

Algorithm 2. Simulation of LTD data.

1: Fix values a_2, b_2 and c_2 of the RV LT $L\sim \mathrm{TRI}(a_2,b_2,c_2)$.
2: Generate one LT value l from $L\sim \mathrm{TRI}(a_2,b_2,c_2)$ by using Algorithm 1.
3: Set values for the minimum (a_1), maximum (b_1) and mode (c_1) of the RV DPUT $X\sim \mathrm{TRI}(a_1,b_1,c_1)$.
4: Simulate a number l of DPUT data $x_1,\dots,x_l$ from $X\sim \mathrm{TRI}(a_1, b_1,c_1)$ by using Algorithm 1.
5: Compute one LTD value y summing DPUT data $x_1,\dots,x_l$ such as in expression (3).
6: Repeat Steps 1–5 to complete a number n of LTD data.

Algorithm 3. Kernel estimation of the LTD PDF.

1: Generate n LTD data $y_1,\dots,y_n$ by using Algorithm 2.
2: Estimate the LTD PDF with a TRI kernel by using expression (8) and the data $\boldsymbol{y}=(y_1,\dots,y_n)^\top$ by means of the R code `density(y, kernel=''triangular'')`.

Algorithm 4. Approximation of the unknown LTD PDF.

1: Generate n LTD data $y_1,\dots,y_n$ with Algorithm 2.
2: Estimate the unknown LTD PDF with the kernel method by using Algorithm 3.
3: Approximate the unknown LTD PDF estimated with the kernel method by the polynomial function defined in (10) as follows:
3.1: For k=2 to k=20 by 2, fit a polynomial function to the actual PDF and calculate the KL divergence value between it and the kernel estimate;
3.2: Choose the value of k with the smallest KL divergence by using as stopping criteria a difference of less than 0.01% in the KL value of the fit between two consecutive values of k.
4: Use the polynomial of order k obtained in Step 3 as an approximation for the unknown LTD PDF.

Algorithm 5. Optimization of the total cost for the (Q, r) model.

1: Replace the polynomial coefficients of order k obtained by Algorithm 4 in derivatives (18) and (19).
2: Estimate λ_X and λ_L defined in (2) and (5) with DPUT and LT data sets, respectively, and then replace them in derivatives (18) and (19).
3: Fix holding (per unit) –$\mathrm{C_h}$–, ordering –$\mathrm{C_o}$– and penalty (per unit) –$\mathrm{C_p}$– costs.
4: Insert the cost values fixed in Step 3 into derivatives (18) and (19).
5: Find optimal values of Q and r that minimize the total cost using (20), Steps 1–4 and the bisection method.

3.2. Computational implementation

R is a non-commercial and open source software for statistics and graphs, which can be obtained at no cost from http://www.r-project.org. The statistical software R is currently very popular in the international scientific community. For use of this software in inventory models, see [29] and [37]. We implement the methodology introduced in this paper in the R software by using Algorithms 1–5. A computational framework for analyzing data utilizing this approach is being developed by the authors in an R package whose "in progress" version is available upon request. Its more important functions are detailed in Table 1, whereas analogous functions can be considered for the MTE approximation. Some R packages related to statistical distributions that may be useful in inventory models are available at http://CRAN.R-project.org and [24,34].

3.3. Simulation results

First, we use Algorithm 2 (command: `data<-LTD(10000,tD,tLT)`) and the Monte Carlo method to simulate data in nine different scenarios for the RVs DPUT $X\sim \mathrm{TRI}(a_1,b_1,c_1)$ and LT $L\sim \mathrm{TRI}(a_2,b_2,c_2)$. Each of these nine scenarios is a combination of $a_1,a_2\in\{0.25,0.5,0.75\}$ and $b_1,b_2\in\{1.25,1.5,1.75\}$, which both are multiples of $c_1=\lambda_X$ and $c_2=\lambda_L$. Table 2 summarizes each TRI distribution parameter and the corresponding scenario position presented in Fig. 1. In this figure, first row represents scenarios with negative skewness for DPUT distribution; second row sketches scenarios with no skewness for the DPUT distribution; whereas third row displays scenarios with positive skewness for the DPUT distribution.

Second, with the n=10,000 LTD data generated with Algorithm 2, we now use Algorithm 3 (command: `kernelTRI(data$(LTD))`) to estimate the unknown LTD PDF with the kernel

Table 1
Basic functions of an R package for the proposed methodology.

Function usage	Arguments	Description
LTD(Nsim,tD,tLT)	Nsim: number of simulated data tD: TRI(a1,b1,c1) DPUT distribution tLT: TRI(a2,b2,c2) LT distribution	It simulates LTD data from TRI distributions.
kernelTRI(d)	d: LTD data obtained from the function LTD()	It returns the kernel estimate of the PDF.
appPoly(d)	d: LTD data obtained from the function LTD()	It approximates the LTD PDF by a polynomial.
appMTE(d)	d: LTD data obtained from the function LTD()	It approximates the LTD PDF by an MTE.
KL(f,g)	f: the PDF estimated by TRI kernel g: the approximate PDF	It calculates the KL divergence of g compared to f.
optimTC(kPOL,De,Ch,Co,Cp)	kPOL: vector obtained from the function appPoly() De: annual demand rate for a new product Ch: holding cost Co: ordering cost Cp: penalty cost	It calculates the inventory total cost.
printReport(Report, ltd)	Report: an environment containing all the information of an evaluated scenario ltd: LTD	It prints report for a given scenario.

Table 2
Summary of TRI distribution parameters for simulation scenarios.

Scenario	Row	Column	a_1	b_1	c_1	a_2	b_2	c_2
1	1	1	2.5	12.5	10	2.5	12.5	10
2	1	2	5.0	15.0	10	2.5	12.5	10
3	1	3	7.5	17.5	10	2.5	12.5	10
4	2	1	2.5	12.5	10	5.0	15.0	10
5	2	2	5.0	15.0	10	5.0	15.0	10
6	2	3	7.5	17.5	10	5.0	15.0	10
7	3	1	2.5	12.5	10	7.5	17.5	10
8	3	2	5.0	15.0	10	7.5	17.5	10
9	3	3	7.5	17.5	10	7.5	17.5	10

method for each scenario. Then, by using Algorithm 4, we approximate the estimated PDF with the best polynomial function (command: `appPoly(data$LTD)`) determined by the KL divergence (command: `KL(f,g)`); see Fig. 2. From Fig. 1, note that an excellent agreement exists between the kernel method and the approximation based on TRI distributions, but the quality of the approximation decreases as the order of the polynomial decreases, as expected. Although the polynomial adjustment for $k=2$ does not present the smallest KL values, the real roots of the second order polynomial function are considered as proxies for the integration limits of the respective polynomial function. We also perform a robustness analysis on these results, by comparing the polynomial fit with the MTE approximation (command: `appMTE(data$LTD)`). The MTE approximation provides an average KL value of 0.0419 and a maximum KL value of 0.1400 for scenario 1; see Table 3. These values reflect the approximation method of [5] with four intervals; see Fig. 4. Detailed results for the MTE approximation are omitted here due to restrictions of space, but they are available under request. Comparing the achieved KL results for both polynomial and MTE approximations, the provided approach presents better results for the polynomial approximation. Therefore, we decide to adopt it for our empirical illustration.

Third, we replace the polynomials of order k obtained with Algorithm 4 in the expressions of the inventory total cost derivatives with shortages given in (18) and (19). In addition, we consider (i) a capital cost of 24% per year and a cost of one product unit of \$47; (ii) an opportunity (holding) cost per product unit per year of $C_h = 0.24 \times \$47 = \$11.28$; (iii) an ordering cost (for placing each order) of $C_o = \$25$; and (iv) a penalty cost per shortage product unit per year of $C_p = \$10$. With this, we have the total cost function to be optimized. Now, we use Algorithm 5 (command: $\texttt{optimal} < -\texttt{optimTC}(\texttt{kPOL}, \texttt{De}, \texttt{Ch}, \texttt{Co}, \texttt{Cp})$) to obtain optimal values of the (Q, r) model by applying the bisection method for each of the nine scenarios, which allows us to find the roots of the equation given in (20) after inserting the respective polynomial coefficients.

The optimal r values are depicted in Fig. 3 (command: `printReport(optimal,ltd)`). The final results for each of the nine scenarios considered are presented in Table 3. As expected, the skewness of the resulting LTD distribution significantly impacts the optimal Q and r values, although none of the polynomial fits provides KL values above 0.04. The tested scenarios show that this computational solution is robust enough to handle with strong asymmetries in both DPUT and LT distributions.

4. A real-world empirical illustration

In this section, we illustrate potential applications of the approach provided in Section 2 by using a real-world problem and the computational framework discussed in Section 3. First, we deal with the inventory management of new products by employing the proposed methodology considering DPUT and LT following TRI distributions. Second, we consider the inventory management of new products with (i) standard and (ii) equivalent product methodologies, both analyzing real-world demand data. Standard methodology consists of a known demand data analysis of a new product, with data collected one year after its launching. Equivalent methodology consists of a known demand data analysis for an equivalent product, with data collected one year before its launching.

4.1. Description of the problem

The drug supply in pharmacy units of Chilean primary health centers is channeled through their central warehouse, which acts as an intermediary between suppliers and output units (OU). The OUs receive the demand for drugs, including its own pharmacy, which performs dispensing of prescriptions to patients. This warehouse needs the storage, conservation and distribution of such drugs. Supply of warehouse is carried out by different suppliers, each of them with different delivery periods. The suppliers

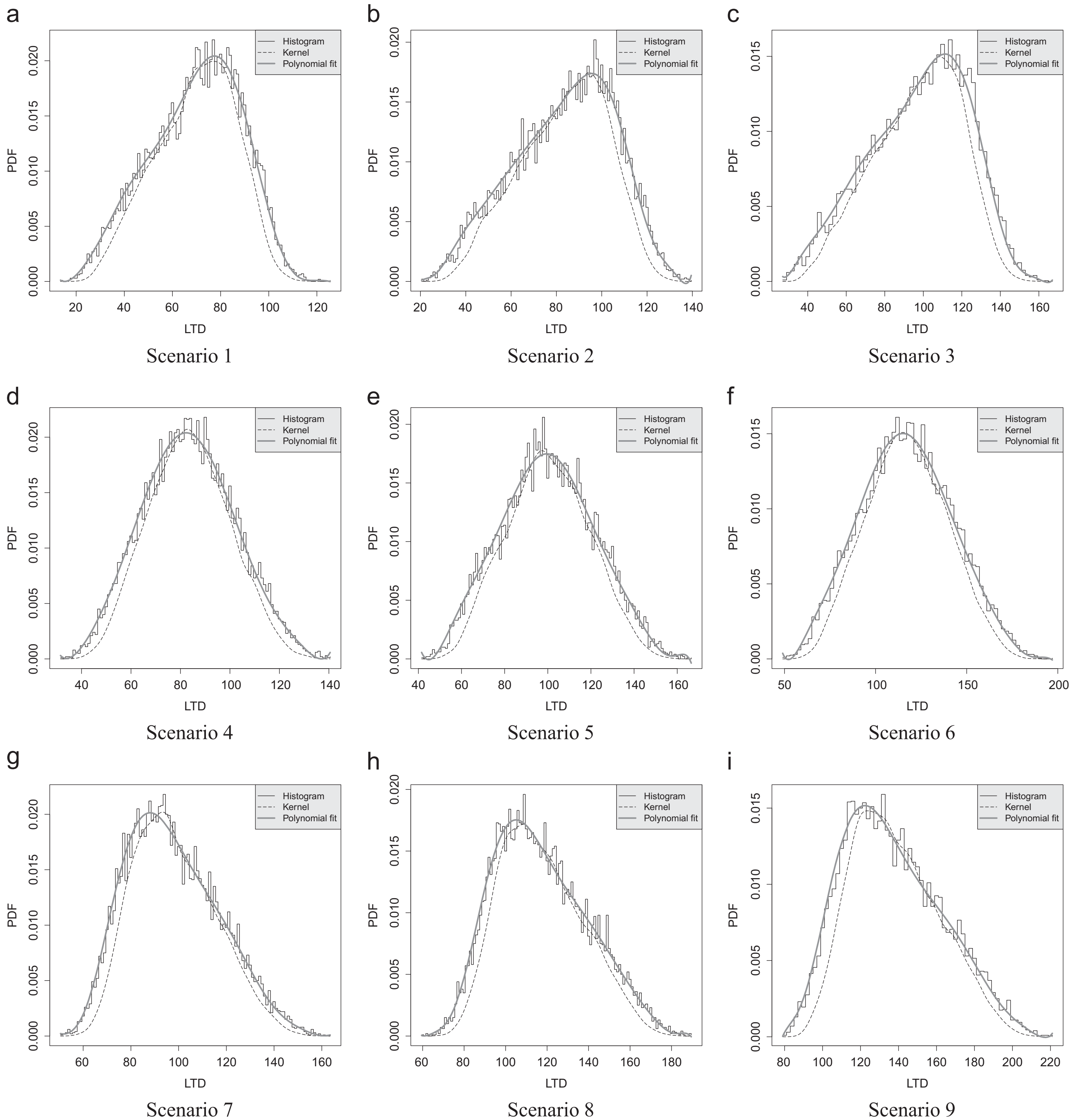

Fig. 1. Histogram, kernel estimate and polynomial fit for the LTD PDF in the indicated scenario. (a) Scenario 1. (b) Scenario 2. (c) Scenario 3. (d) Scenario 4. (e) Scenario 5. (f) Scenario 6. (g) Scenario 7. (h) Scenario 8. (i) Scenario 9.

are selected according to technical criteria or direct negotiation, based on the needs for that time [27], which produces LT uncertainty. The warehouse delivers products on a weekly basis to all OUs by using aggregated demand requirements for each of them in the same period. When introducing a new product to the therapeutic arsenal, a problem is generated because the behavior of demand in OUs is unknown, making the determination of the lot sizes to be ordered difficult for an adequate supply. Empirically, comparisons of the behavior of aggregated demand of OUs are done based on products of similar therapeutic characteristics (equivalent) in previous periods, in order to determine the rate of demand and lot sizes that satisfy the requirements. For details illustrating the supply system, see Fig. 5. We consider the weekly actual demand of two pharmaceutical products. One of them is an innovative product named Losartan Potassium, whose unit of measurement per coated tablet is 50 mg. This pharmaceutical product replaces a similar therapeutic use product named Acetyl Salicylic Acid, whose unit of measurement per tablet is 100 mg. We consider a family health center located at the city of Concon, Chile, whose data were collected for a study of supply policy conducted by Fernando Rojas in the University of Valparaíso, Chile, during 52 weeks of the year 2012 (from January 1 to December 31). The products are shipped from the warehouse to this family health center.

4.2. The proposed methodology

By using an expert judgment for the new product Losartan Potassium, we consider the TRI DPUT parameters to be $a_1 = 0, b_1 = 45,000$ and $c_1 = 10,000$, and the TRI LT parameters to be $a_2 = 1$, $b_2 = 4$ and $c_2 = 2$. Therefore, the mean DPUT of the new product is $\lambda_X = 18,333.33$ units and the mean LT of the new product is $\lambda_L = 2.33$ weeks. In addition, we consider $C_h = \$0.22$ per product unit per year, $C_o = \$138,603.84$ per order and $C_p = \$9.44$ per shortage product unit per year. All of these values were provided by the manager of pharmacy unit of the primary health care center. Thus, based on the proposed methodology, the optimal values for the (Q, r) model of the new product obtained from the expected total cost per year given in (14) are $Q^\star = 885,187$ units and $r^\star = 66,917$ units, whereas the optimum total cost is $189,343.90.

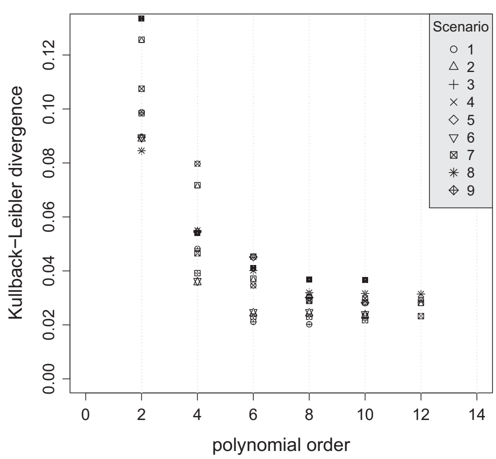

Fig. 2. KL analysis for the polynomial fit in the indicated scenario.

4.3. The standard methodology

Now, assume a data set for the weekly actual demand of the pharmaceutical new product (Losartan Potassium in units of 50 mg), which was collected one year after its launching and is presented in Table 4. Table 5 provides some descriptive statistics of the DPUT data for the new product, such as the sample size (n), minimum and maximum values, median, mean ($\bar{x}$), SD and the coefficients of variation (CV), skewness or asymmetry (CS) and kurtosis (CK). Note that the DPUT distribution of the new product, with data collected one year before its launching, has positive skewness and moderate kurtosis. Fig. 6(a) shows the histogram of these data with estimated gamma PDF; see details about the selection of this distribution in Section 4.5. From this figure, note that the gamma distribution reproduces the shape of the empirical distribution of the DPUT very well and it is consistent with results presented in Table 6. Thus, for the standard methodology, we use a gamma DPUT distribution and a TRI LT distribution, and then we approximate the LTD distribution according to the procedure proposed by [33]. In addition, by considering the costs C_h, C_o and C_p, the optimal values for the (Q, r) model of the product, obtained from the expected total cost per year given in (14), are $Q^\star = 983,435$ units and $r^\star = 138,608$ units, whereas the optimum total cost is $247,091.80.

4.4. The equivalent product methodology

As usual in practice when data for new products are unavailable, data from an equivalent product can be assumed. We use data of weekly actual demand of Acetyl Salicylic Acid (in units of 100 mg), which were collected one year before the launching of the new product and are presented in Table 4. Table 5 provides some

Table 3
Summary of the best fit degree, polynomial coefficients, KL, Q and r for the indicated scenario.

Scenario	1	2	3	4	5	6	7	8	9
Degree	11	11	10	8	10	10	12	12	9
	Polynomial coefficients								
α_0	0.0723	−0.4400	0.5177	0.4435	−0.7338	2.4653	37.8632	90.4228	−41.8525
α_1	−0.0194	0.0975	−0.0801	−0.0547	0.1460	−0.2226	−4.7841	−9.4490	2.8471
α_2	0.0022	−0.0093	0.0053	0.0029	−0.0106	0.0087	0.2674	0.4382	−0.0842
α_3	−0.0001	0.0005	−0.0002	−8.35E−05	0.0004	−0.0002	−0.00871	−0.0119	0.0014
α_4	6.03E−06	−1.72E−05	4.66E−06	1.47E−06	−9.51E−06	2.61E−06	0.0002	0.0002	−1.51E−05
α_5	−1.63E−07	3.94E−07	−7.17E−08	−1.60E−08	1.42E−07	−2.31E−08	−2.63E−06	−2.53E−06	1.04E−07
α_6	2.94E−09	−6.15E−09	7.32E−10	1.03E−10	−1.41E−09	1.32E−10	2.60E−08	2.11E−08	−4.73E−10
α_7	−3.55E−11	6.59E−11	−4.92E−12	−3.67E−13	9.10E−12	−4.75E−13	−1.77E−10	−1.22E−10	1.36E−12
α_8	2.81E−13	−4.75E−13	2.08E−14	5.47E−16	−3.71E−14	9.78E−16	8.04E−13	4.73E−13	−2.23E−15
α_9	−1.40E−15	2.20E−15	−5.02E−17	–	8.68E−17	−9.03E−19	−2.26E−15	−1.14E−15	1.60E−18
α_{10}	4.00E−18	−5.91E−18	5.27E−20	–	−8.85E−20	6.81E−23	3.16E−18	1.37E−18	–
α_{11}	−4.94E−21	6.99E−21	–	–	–	–	0	0	–
α_{12}	–	–	–	–	–	–	−3.94E−24	−1.28E−24	–
Q	260.4	281.1	284.8	278.2	304.6	284.8	283.4	285.0	403.6
r	94.7	108.7	112.0	105.8	121.8	112.1	110.7	112.4	170.0
	Polynomial approximation								
KL	0.0232	0.0314	0.0282	0.0202	0.0237	0.0217	0.0293	0.0281	0.0366
	MTE approximation								
KL	0.1400	0.0509	0.0127	0.0375	0.0231	0.0203	0.0323	0.0475	0.0132

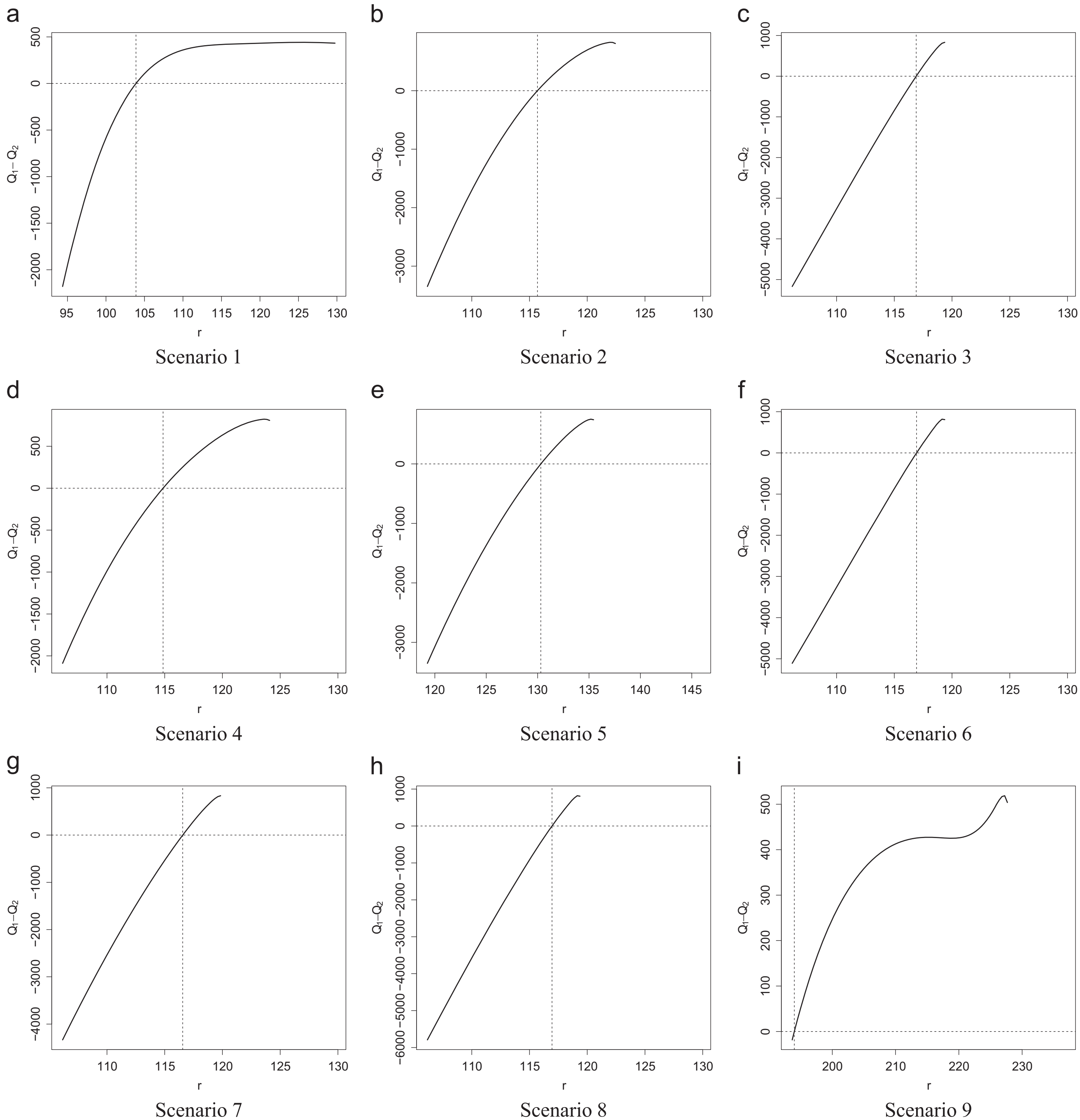

Fig. 3. Optimal value of r for the (Q, r) inventory model in the indicated scenario. (a) Scenario 1. (b) Scenario 2. (c) Scenario 3. (d) Scenario 4. (e) Scenario 5. (f) Scenario 6. (g) Scenario 7. (h) Scenario 8. (i) Scenario 9.

descriptive statistics of the DPUT data for the equivalent product. Note that the DPUT distribution of the equivalent product is also positively skewed and has a high kurtosis. Fig. 6(b) shows the histogram of the equivalent product DPUT data with estimated gamma PDF. From this figure, note that the gamma distribution reproduces the shape of the empirical distribution of the DPUT fairly and it has the best fit to DPUT data; see details about the selection of this distribution in Section 4.5. For the equivalent product methodology, we proceed analogously as in the standard methodology, obtaining the optimal values for the (Q, r) model to be $Q^\star = 1,064,812$ units, $r^\star = 205,733$ units and an optimum total cost of $279,537.70.

4.5. Fitting DPUT distributions and summary of results

For the standard and equivalent methodologies, Birnbaum–Saunders [25], exponential, gamma, Gaussian, inverse Gaussian, lognormal and Weibull distributions [29] are fitted to DPUT data; see [17] and [18] for more details about these distributions. Based

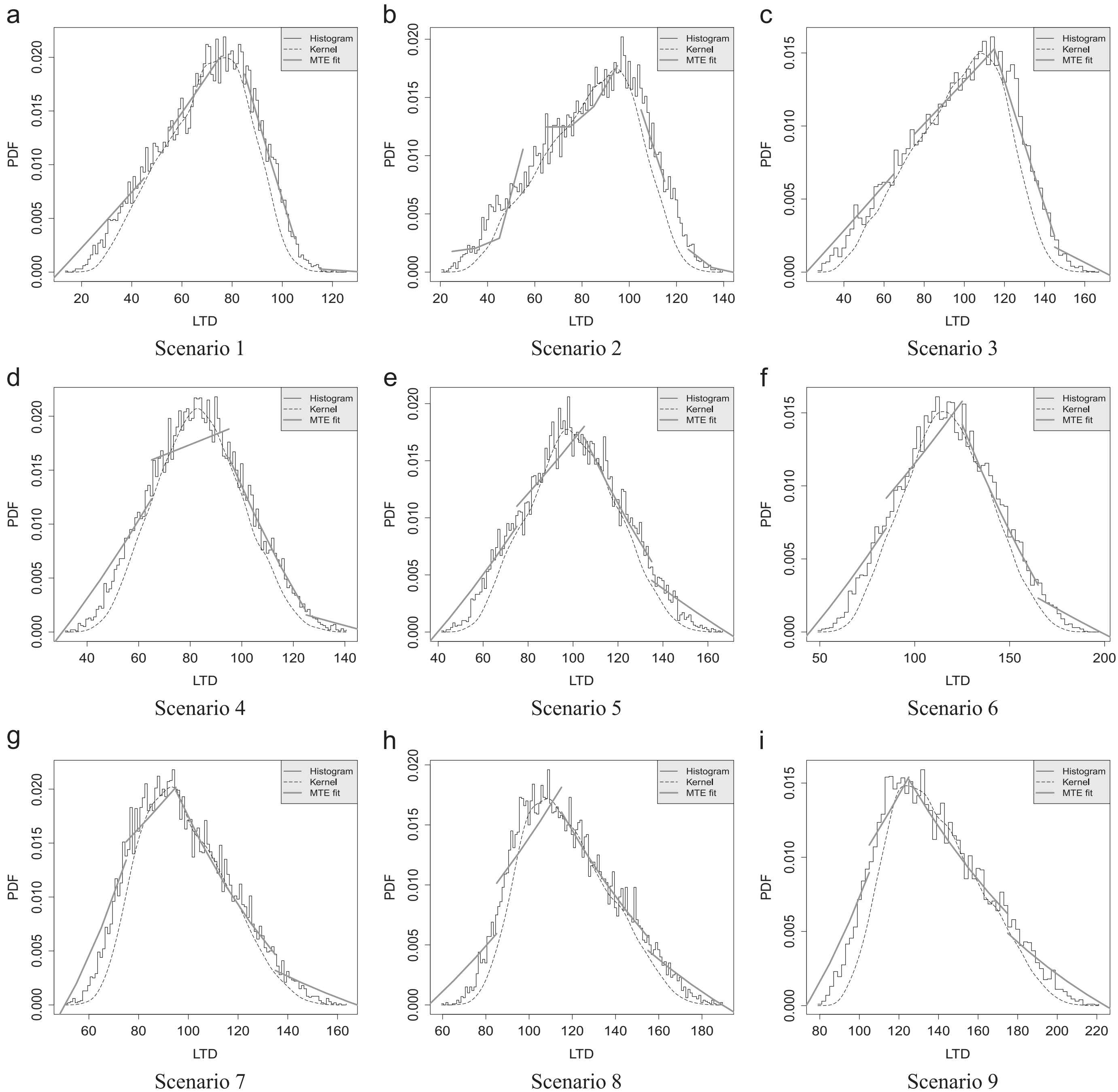

Fig. 4. Histograms and MTE fit for the PDF in the indicated scenario. (a) Scenario 1. (b) Scenario 2. (c) Scenario 3. (d) Scenario 4. (e) Scenario 5. (f) Scenario 6. (g) Scenario 7. (h) Scenario 8. (i) Scenario 9.

on the GOF Kolmogorov–Smirnov (KS) test, results presented in Table 6 suggest that the gamma distribution fits better the data in both standard and equivalent methodologies. We ratify visually this good fit of the gamma distribution to DPUT data from the histograms with estimated gamma PDF in Fig. 6 and probability–probability (PP) plots with 95% acceptance bands in Fig. 7; for more details about these graphical plots, see [2].

Based on each methodology, it is possible to calculate values for KL divergence, weeks on hand, turnover, lot size, expected shortage, fill rate and inventory annual total cost, as shown in Table 7. From this table, note that the TRI methodology produces values of the total cost, lot size and KL divergence that are closer to the standard methodology than to the equivalent methodology.

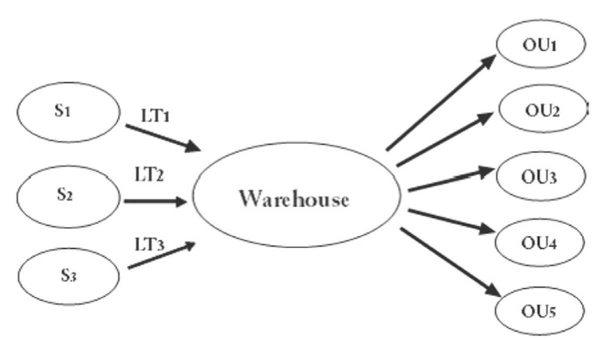

Fig. 5. Supply system in pharmacy units of primary health care centers in Chile, where S_i = supplier i, LT_i = LT of S_i for $i = 1, 2, 3$, and OU_j = output unit j, for $j = 1, \ldots, 5$.

Table 4
DPUT data from new (used with the standard methodology) and equivalent products.

Week	DPUT		Week	DPUT		Week	DPUT		Week	DPUT	
	Standard	Equiv		Standard	Equiv		Standard	Equiv		Standard	Equiv
1	10,000	10,000	14	20,000	14,000	27	0	14,000	40	15,300	10,000
2	20,000	45,000	15	15,000	21,000	28	3000	7000	41	39,000	20,000
3	0	0	16	0	10,000	29	10,000	7000	42	7000	0
4	10,000	0	17	15,000	10,000	30	5000	1200	43	30,000	35,000
5	0	0	18	0	11,000	31	0	15,000	44	10,000	0
6	8000	0	19	46,000	14,000	32	24,000	14,000	45	24,000	0
7	15,000	15,000	20	0	14,000	33	12,000	10,000	46	25,000	20,000
8	0	0	21	0	14,000	34	19,000	14,000	47	10,000	10,000
9	30,000	9000	22	20,000	11,000	35	6000	7000	48	20,000	10,000
10	20,000	20,000	23	25,000	21,000	36	0	24,000	49	10,000	10,000
11	12,000	12,000	24	9000	14,000	37	0	10,000	50	20,000	10,000
12	15,000	14,000	25	23,000	14,000	38	0	6000	51	15,000	10,000
13	14,000	11,000	26	0	10,000	39	24,000	16,000	52	10,000	19,000

Table 5
Descriptive statistics for the indicated DPUT data set.

Data set	Zeros	n	Minimum	Median	$\bar{x}$	SD	CV	CS	CK	Maximum
Standard	Yes	52	0	11,000	12,794.23	10,846.60	0.85	0.70	3.26	46,000
Equivalent	Yes	52	0	10,500	11,792.31	8490.36	0.72	1.25	6.40	45,000
Standard	No	39	3000	15,000	17,058.97	9123.07	0.53	1.03	4.15	46,000
Equivalent	No	44	1200	13,000	13,936.36	7410.963	0.53	2.09	8.96	45,000

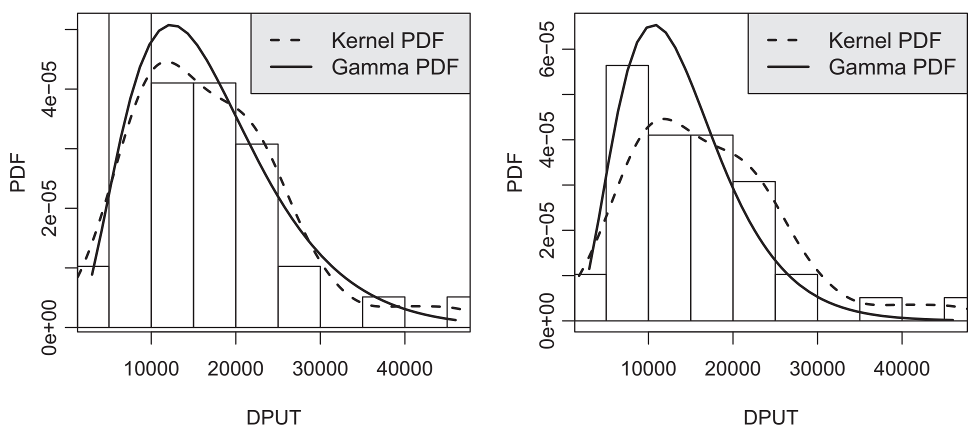

Fig. 6. Histograms with estimated PDFs from the gamma distribution and kernel method for standard (left) and equivalent (right) product DPUT data.

Table 6
KS p-values for the indicated methodology and distribution with DPUT data.

Distribution	KS p-value (standard)	KS p-value (equivalent)
Birnbaum–Saunders	0.649	0.006
Exponential	0.003	<0.001
Gamma	0.753	0.113
Gaussian	0.424	0.024
Inverse Gaussian	0.609	0.005
Lognormal	0.694	0.036
TRI	0.150	<0.001
Weibull	0.823	0.086

However, the value of fill rate for the TRI methodology is similar to the value of the standard and equivalent methodologies.

5. Conclusions and future research

We proposed a methodology to deal with the inventory management of new products by using triangular distributions for both demand per unit time and lead-time. Inventory shortage and total cost expressions for the (Q, r) model were provided and computationally implemented for triangularly distributed demand per unit time and lead-time, based on polynomial and mixture of truncated exponential approximations for the probability density function of the demand during lead-time. We evaluated our methodology under nine different scenarios in order to assess the robustness of the computational procedure. Managers, however, can easily alter the parameters of the original demand and lead-time distributions to assess any other particular case. New product inventory management represents an appropriate situation for applying the results derived in this paper, because demand data tend to be difficult or expensive to collect. The main advantage of assuming a triangular distribution is its ease of involving managers in the analytical process by capturing their subjective estimates in terms of the minimum, most likely and maximum values. This is particularly important when decisions regarding the first lot size and reorder point must be made. The idea is to build a consensus basis regarding minimal, most likely and maximal demand forecast for a given new product and to deploy its impacts on the inventory replenishment model. This non-learning based

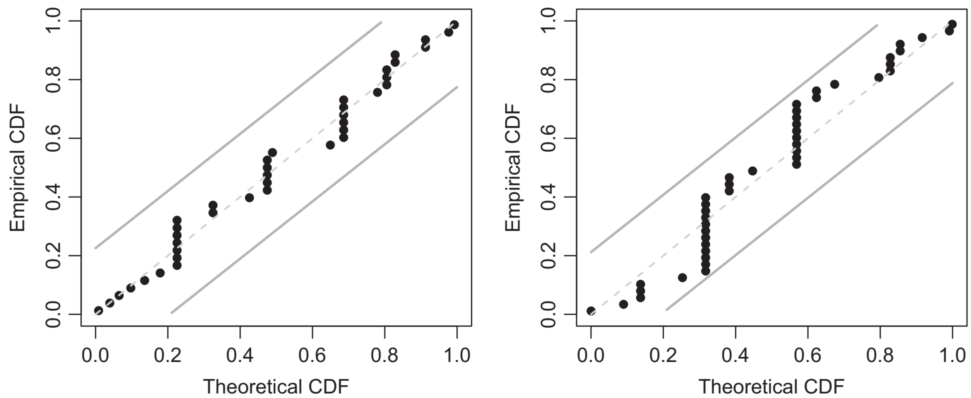

Fig. 7. PP plots with 95% acceptance bands for standard (left) and equivalent (right) product DPUT data.

Table 7
Summary of results for the indicated methodology.

Methodology	DPUT distribution	KL	Weeks on hand	Turnover (per year)	Lot size	Expected shortage	Fill rate	Total cost
Proposed	TRI	0.0288	88.5	0.59	885,187	23,726.1	0.9732	$189343.90
Standard	Gamma	0.1225	76.9	0.68	983,435	2213.2	0.9977	$247091.80
Equivalent	Gamma	0.1302	88.9	0.58	1,064,812	6866.5	0.9935	$279537.70

approach could be used as a first decision round on lot sizes and reorder points, whereas the learning process related to the distribution is still ongoing. The polynomial approximation to the triangularity assumption proved to be fairly robust, with extremely small Kullback–Leibler divergence values.

Future research should expand and generalize the solutions to the (Q, r) model, thus making their assumptions more flexible for handling other situations, such as different cycle service level measures. More generic solutions for triangularly distributed demand and lead-time could be used to support managerial initiatives to supply chain planning at other stages of the product life cycle.

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