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Breakdown of weak-turbulence and nonlinear wave condensation

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ABSTRACT

The formation of a large-scale coherent structure (a condensate) as a result of the long time evolution of the initial value problem of a classical partial differential nonlinear wave equation is considered. We consider the nonintegrable and unforced defocusing NonLinear Schrödinger (NLS) equation as a representative model. In spite of the formal reversibility of the NLS equation, the nonlinear wave exhibits an irreversible evolution towards a thermodynamic equilibrium state. The equilibrium state is characterized by a homogeneous solution (condensate), with small-scale fluctuations superposed (uncondensed particles), which store the information necessary for "time reversal". We analyze the evolution of the cumulants of the random wave as originally formulated by D.J. Benney and P.G. Saffman [D.J. Benney, P.G. Saffman, Proc. Roy. Soc. London A 289 (1966) 301] and A.C. Newell [A.C. Newell, Rev. Geophys. 6 (1968) 1]. This allows us to provide a self-consistent weak-turbulence theory of the condensation process, in which the nonequilibrium formation of the condensate is a natural consequence of the spontaneous regeneration of a non-vanishing first-order cumulant in the hierarchy of the cumulants' equations. More precisely, we show that in the presence of a small condensate amplitude, all relevant statistical information is contained in the off-diagonal second order cumulant, as described by the usual weak-turbulence theory. Conversely, in the presence of a high-amplitude condensate, the diagonal second-order cumulants no longer vanish in the long time limit, which signals a breakdown of the weak-turbulence theory. However, we show that an asymptotic closure of the hierarchy of the cumulants' equations is still possible provided one considers the Bogoliubov's basis rather than the standard Fourier's (free particle) basis. The nonequilibrium dynamics turns out to be governed by the Bogoliubov's off-diagonal second order cumulant, while the corresponding diagonal cumulants, as well as the higher order cumulants, are shown to vanish asymptotically. The numerical discretization of the NLS equation implicitly introduces an ultraviolet frequency cut-off. The simulations are in quantitative agreement with the weak turbulence theory without adjustable parameters, despite the fact that the theory is expected to breakdown nearby the transition to condensation. The fraction of condensed particles vs energy is characterized by two distinct regimes: For small energies ($H \ll H_c$) the Bogoliubov's regime is established, whereas for $H \lesssim H_c$ the small-amplitude condensate regime is described by the weak-turbulence theory. In both regimes we derive coupled kinetic equations that describe the coupled evolution of the condensate amplitude and the incoherent field component. The influence of finite size effects and of the dimensionality of the system are also considered. It is shown that, beyond the thermodynamic limit, wave condensation is reestablished in two spatial dimensions, in complete analogy with uniform and ideal 2D Bose gases.

1. Introduction

In the present work we study the statistical initial value problem for the nonintegrable Hamiltonian defocusing NonLinear Schrödinger (NLS) equation or Gross–Pitaevskii equation in space dimensions higher or equal than 2. The random nonlinear wave is known to exhibit a thermalization process characterized by an irreversible evolution of the field towards an equilibrium state. In the defocusing regime of

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the NLS equation, the thermalization of the random field is characterized by a condensation process, *i.e.* by the spontaneous formation of a homogeneous solution in the long term evolution of the field [1,2]. The condensation process manifests itself as a phase transition of the equilibrium state for sufficiently low energy density [3–6]. In the framework of weak-turbulence theory [7,8], the nonequilibrium formation of the condensate may be regarded as an inverse cascade of wave action from small scales to large scales and a direct cascade of energy towards small scale fluctuations, *i.e.* from small to large wavenumbers k. We provide a self-consistent weak-turbulence theory of the condensation process, in which the spontaneous emergence of a non-vanishing average of the field ($\langle\psi\rangle \neq 0$) results from the natural asymptotic closure of the cumulant equations for the random field. The theory is confirmed by a direct numerical integration of the NLS equation, which implicitly introduces an ultraviolet frequency cut-off in the wave system, *i.e.* a finite number of degrees of freedom.

It is important to discuss the phenomenology of the condensation process into a more general perspective. The classical condensation process may be regarded as a self-organization process that occurs in a conservative and reversible wave system. Let us recall in this respect that, contrary to dissipative systems, a conservative Hamiltonian system cannot evolve towards a fully ordered state, because such an evolution would imply a loss of statistical information for the system that would violate its formal reversibility. However, in spite of its formal reversibility, a *nonintegrable* Hamiltonian system is expected to exhibit an irreversible evolution towards an equilibrium state, as a result of an irreversible process of diffusion in phase-space [9]. In this regard, an important achievement was accomplished when Zakharov and collaborators reported in Ref. [10] numerical simulations performed in the framework of the *focusing* nonintegrable NLS equation. This study revealed that the Hamiltonian system would evolve, as a general rule, towards the formation of a large-scale coherent localized structure, *i.e.*, a solitary-wave, immersed in a sea of small-scale turbulent fluctuations. The solitary wave then plays the role of a "statistical attractor" for the Hamiltonian system, while the small-scale fluctuations contain, in principle, all the information necessary for time reversal. It is important to note, the solitary-wave solution corresponds to the solution that minimizes the energy (Hamiltonian), so that the system tends to relax towards the state of minimum energy, while the small-scale fluctuations compensate for the difference between the conserved energy and the energy of the coherent structure.

As was initially suggested in Refs. [10], a rigorous theoretical description of the long term evolution of the system would require a thermodynamic approach. It is only recently that statistical *equilibrium* models have been elaborated in the framework of statistical mechanics [11–13]. As a remarkable result, whenever the Hamiltonian system is constrained by an additional integral of motion (*e.g.*, number of particles), the increase of entropy of small-scale turbulent fluctuations requires the formation of coherent structures [11,13], so that it is thermodynamically advantageous for the system to approach the ground state which minimizes the energy [10]. More precisely, it is shown that a statistical equilibrium is reached, in which the energy not contained in the coherent structure is equally distributed among the modes of the small-scale fluctuations.

Let us now refers back to the condensation process that we shall discuss in the present paper. It is indeed important to note that the spontaneous formation of a homogeneous solution corroborates the general rule discussed above [10,11,13], because the homogenous solution realizes the minimum of the energy (Hamiltonian) in the defocusing case. This analogy between the focusing and defocusing regimes reveals that the formation of a condensate may be viewed as a consequence of the fact that the system naturally tends to increase its disorder (entropy). A simple explanation of this counterintuitive result may be given by recalling that the total energy of the field has a kinetic contribution and a nonlinear contribution. The kinetic energy being proportional to the gradient of the field, it provides a measure of the amount of fluctuations in the system. On the other hand, the nonlinear energy reaches its minimum value for a homogeneous solution. This merely explains why it is advantageous for the field to generate a condensate, because this permits the field to increase its disorder. In other terms, an increase of entropy in the field requires the generation of a homogenous solution (a plane-wave in optics terminology). According to this physical picture, there is a direct correspondence between the mechanisms underlying the spontaneous generation of a solitary-wave in the focusing regime and the condensation process in the defocusing regime. In both cases, the system tends to reach the most disordered state characterized by the presence of small-scale fluctuations in the field, which requires the generation of a large-scale coherent structure. The fact that the defocusing NLS equation exhibits a condensation process has been accurately confirmed in the context of thermal Bose fields, where numerical simulations of a "projected" NLS equation have been performed for both uniform [3,4] and non-uniform (*i.e.* trapped) [14] configurations of the gas.

In a recent Letter [5] we formulated a thermodynamic description of this condensation process on the basis of the weak-turbulence theory. We showed that the thermodynamic properties of the condensation process are analogous to those of a Bose–Einstein transition, despite the fact that the considered wave system is inherently classical. More precisely, the theoretical analysis of the equilibrium properties of the condensation process were found to be in agreement with the numerical simulations of the three-dimensional (3D) NLS equation [5]. The present paper is aimed at providing a deeper insight into the weak-turbulence description of the classical wave condensation process, in which particular attention will be devoted to the kinetic dynamics of condensate formation.

The weak-turbulence theory is often referred to the so-called "random phase-approximation" approach [1,7,15,16]. However, this approach should be rather considered as a convenient way of interpreting the results of the more rigorous technique based on the analysis of the cumulants of the nonlinear field, as originally formulated in Refs. [17–19]. This approach has been recently reviewed in Ref. [8], and studied in more detail through the analysis of the probability distribution function of the random field in Refs. [20]. The weak-turbulence theory describes the long time behavior of statistically homogeneous random waves under quite general assumptions [17–19]. It consists of a BBGKY-type of hierarchy equations for the cumulants of the random field, which is naturally closed because of the dispersive properties of the waves and the large separation between linear and nonlinear time scales [8,17–20]. The asymptotic closure of the hierarchy provides a kinetic equation governing the evolution of the spectrum $n_k(t)$ of the random field, which actually refers to the off-diagonal second-order cumulant [$Q^{(2)+-}(\boldsymbol{k})$] [8,17–19]. The structure of the kinetic equation is analogous to that of the Boltzmann's equation for classical dilute gases [21]. In particular, it exhibits a H-theorem of entropy growth that describes an irreversible evolution of the field towards the Rayleigh–Jeans equilibrium distribution, $n_k^{eq} = T/(k^2 - \mu)$, where T and μ are called, by analogy with thermodynamics, the temperature and the chemical potential. Considering the system at equilibrium, it was shown in Ref. [5] that, in analogy with the standard Bose–Einstein condensation of an ideal quantum gas, wave condensation arises when the chemical potential reaches zero (from the negative side) for a non-vanishing critical temperature T_c. Consequently, the number of particles $N = \sum_{\boldsymbol{k}} n_{\boldsymbol{k}}^{eq}$ was decomposed into a condensed part and an incoherent part, $N = N_0 + \sum'_{\boldsymbol{k}} n_{\boldsymbol{k}}^{eq}(\mu = 0)$, where the sum $\sum'_{\boldsymbol{k}}$ excludes the fundamental mode $\boldsymbol{k} = 0$. Let us underline that this decomposition is introduced artificially in the theory and, moreover, it does not consider N_0 as a dynamical variable and thus limits the theoretical description to the purely stationary equilibrium regime. *Here we report a self-consistent weak-turbulence theory in which* *the nonequilibrium formation of the condensate, i.e., a non-vanishing average $\langle\psi\rangle \neq 0$, is shown to naturally result from the spontaneous regeneration of the first-order cumulant of the field $Q^{(1)}(t)$ in the hierarchy of cumulants' equations.*

Moreover, the standard weak turbulence approach of the condensation process is not satisfactory whenever the equilibrium state is characterized by a significant fraction of condensed particles N_0/N, *i.e.*, for small values of the temperature, or equivalently for small values of the total energy H of the field. Indeed, the nonlinear frequency renormalization indicates that a new dispersion relation that takes into account the nonlinear interaction should be introduced. It turns out that the Bogoliubov's dispersion relation is the relevant dispersion relation that should be considered. A heuristic kinetic theory of this highly condensed regime (large condensate fraction N_0/N) was reported in Refs. [1,5,6] by adapting the Bogoliubov's transformation to the classical wave problem. In particular, in a finite degree of freedom system, that is in a truncated non-linear Schödinger equation, it was shown in Ref. [5] that the nonlinear interaction changes the nature of the transition to condensation, *i.e.*, the transition was shown to become subcritical (discontinuous). However, such a subcritical behavior was not identified in the numerical simulations [5]. *Here we clarify this issue by showing that the transition to condensation of a classical wave system is actually a supercritical (continuous) transition.*

In summary, we show that the phenomenon of wave condensation may be described by means of two distinct approaches. When the equilibrium state is characterized by a small fraction of condensed particles N_0/N (*i.e.*, high energy H), all the relevant statistical information is in the off-diagonal second order cumulant $Q^{(2)+-}(\boldsymbol{k})$, so that the standard weak-turbulence closure is valid and the field relaxes to the usual Rayleigh–Jeans distribution. This approach describes well the critical energy for the transition to condensation (H_c), as well as the small condensate regime. The corresponding condensation curve that expresses the fraction of condensed particles n_0/n vs the energy H, is given in Eq. (50). Conversely, for an equilibrium state characterized by a significant fraction of condensed particles N_0/N (*i.e.*, small energy H), we show that the diagonal second-order cumulants $Q^{(2)++}(\boldsymbol{k})$ [and $Q^{(2)--}(\boldsymbol{k})$] do not longer decrease to zero as in the usual weak-turbulence approach. This signals a breakdown of the standard weak-turbulence theory. The key idea behind the analysis is that the "normal variables" are no longer the Fourier's modes, but instead the Bogoliubov's modes. It is shown that in the framework of the Bogoliubov's basis, an asymptotic closure of the hierarchy of the cumulants' equations is still possible. The dynamics results in being governed by the Bogoliubov's off-diagonal second order cumulant, $Q_B^{(2)+-}(\boldsymbol{k})$, while the corresponding diagonal cumulants $Q_B^{(2)++}(\boldsymbol{k})$, $Q_B^{(2)--}(\boldsymbol{k})$, as well as the higher-order cumulants, are shown to vanish in the long time limit. The analysis reveals that the equilibrium spectrum turns out to be the Rayleigh–Jeans distribution with the Bogoluibov's dispersion relation and a zero chemical potential. The corresponding expression of the condensation curve (n_0/n vs H) of this high-amplitude condensate regime is given by Eq. (52).

Let us remark that, as was pointed out in Ref. [5], wave condensation does not occur in two spatial dimensions (2D) in the thermodynamic limit. Nevertheless, we shall see that for situations of practical interest in which a finite number of particles N and a finite size of the system V are considered, wave condensation is reestablished in 2D. This corroborates the numerical observations reported, *e.g.*, in Ref. [22]. Because the system is not in the thermodynamic limit, the chemical potential μ does not vanish exactly, and the condensation curve is given by means of two parametrically coupled equations for the condensate amplitude and the energy [see Eqs. (56) and (57)]. Actually, the critical energy for the transition to condensation tends to zero in the thermodynamic limit, because of the logarithmic infrared divergence of the equilibrium distribution, in complete analogy with uniform and ideal 2D Bose gases.

We finally note that, although we provide a consistent wave turbulence theory with a condensate, which is in quantitative agreement with the numerical simulations at equilibrium, the non-equilibrium dynamics of the random field still leaves open many questions. As many times stressed by A. Newell and collaborators [8], weak turbulence still does not constitute a complete and fully-satisfactory theory. Indeed, there are always some spatial scales (large or small) at which the theory breaks down [8,23,24]. In the considered example of wave condensation, we show that the theory breaks down near $k = 0$ in the thermodynamic limit for dimensions $D < 4$, and nonlinear processes come into play, such as, *e.g.,* a new dispersion relation, or the three-wave resonant interactions. A complete picture of wave condensation should also include the interaction with *localized coherent structures*, such as, *e.g.*, vortices or solitary waves as the Jones–Roberts rarefaction pulses [25]. In this respect, a complete statistical description of large scale coherent structures and their interactions with weakly nonlinear turbulent fluctuations is still lacking and will be the subject of future investigations.

The paper is organized as follows. In Section 2 we introduce the NLS model equation and recall the usual weak turbulence description, its properties and the mechanism underlying the dynamical formation of a condensate. In Section 3 we follow the weak-turbulence approach based on the analysis of the cumulants of the random field [8,17–19]. This approach will be shown to be more appropriate for the description of the spontaneous formation of a condensate, which turns out to be associated to the emergence of a non-vanishing first-order cumulant of the field, *i.e.,* a non-vanishing average ($\langle\psi\rangle \neq 0$). Moreover, this approach naturally decomposes the random field into its coherent part (condensate) and its incoherent part, whereas such a decomposition is introduced in an artificial way in the usual treatment of wave condensation. In Section 4 we develop the weak-turbulence theory in the Bogoliubov's basis, which is relevant for the description of the turbulent regime in the presence of a high-amplitude condensate. In this treatment the standard result of vanishing second-order diagonal cumulants does not hold. Finally, in Section 5 we analyze the role of finite size effects and of the dimensionality of the system, and we compare the predictions of weak turbulence theory with direct numerical simulations of the NLS equation.

2. NLS model equation

Let us consider the normalized defocusing NLS equation for the complex wave-function ψ [26]:

$$\mathrm{i}\partial_t\psi = -\Delta\psi + a|\psi|^2\psi \tag{1}$$

where Δ refers to the Laplacian operator in spatial dimension D while the constant $a > 0$ allow us to trace the role of the nonlinearity in the theoretical analysis. This equation describes the evolution of defocusing interacting waves through the cubic nonlinear term. The dynamics conserves the number of particles (mass)

$$N = \int |\psi|^2 \mathrm{d}^D\boldsymbol{x},$$

the linear momentum

$$\boldsymbol{P} = \mathrm{Im} \int \psi^* \nabla \psi \, \mathrm{d}^D \boldsymbol{x},$$

and the total energy or hamiltonian

$$H = \int \left(|\nabla \psi|^2 + \frac{a}{2} |\psi|^4 \right) \mathrm{d}^D \boldsymbol{x}, \tag{2}$$

which has a kinetic contribution $E = \int |\nabla\psi|^2 \mathrm{d}^D\boldsymbol{x}$, and a nonlinear contribution $U = \int \frac{a}{2}|\psi|^4 \mathrm{d}^D\boldsymbol{x}$. Moreover, let us note the following inequalities, $H \geq \frac{a}{2}\int |\psi|^4 \, \mathrm{d}^D\boldsymbol{x} \geq \frac{a}{2} \frac{(\int |\psi|^2 \mathrm{d}^D\boldsymbol{x})^2}{\int \mathrm{d}^D\boldsymbol{x}} = \frac{a}{2V} N^2 \equiv H_0$, where V refers to the system volume in any space dimension. $\frac{H_0}{V} = \frac{a}{2}(\frac{N}{V})^2$ then represents the minimum of the energy density of the field, which corresponds to the energy density of the homogeneous solution.

In the following we shall briefly summarize some important properties of the kinetic theory of the NLS equation as well as the kinetic process underlying wave condensation.

2.1. Weak turbulence theory for the nonlinear Schrödinger equation

Let us consider a spatially homogeneous random initial condition $\psi(\boldsymbol{x}, t = 0)$ of zero mean [$\int \psi(\boldsymbol{x}, t = 0)\mathrm{d}^D\boldsymbol{x} = 0$]. The corresponding first-order cumulant of the field is thus assumed to vanish. The wave turbulence theory provides a long-time description for the temporal evolution of the spectrum of the field, [7,8],

$$\partial_t n_p(t) = 2\pi a^2 \int n_p n_{k_2} n_{k_3} n_{k_4} \left(n_p^{-1} + n_{k_2}^{-1} - n_{k_3}^{-1} - n_{k_4}^{-1} \right) \delta(p^2 + k_2^2 - {k_3}^2 - {k_4}^2) \delta^{(D)}(\boldsymbol{p} + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4) \mathrm{d}^D \boldsymbol{k}_{234}, \tag{3}$$

here and throughout the manuscript $\mathrm{d}^D\boldsymbol{k}_{234} = \mathrm{d}^D\boldsymbol{k}_2 \mathrm{d}^D\boldsymbol{k}_3 \mathrm{d}^D\boldsymbol{k}_4$. We note that, in this equation, it is implicitly assumed that the first-order cumulant remains zero at any time.

A derivation of this equation is easily obtained from Section 3 and the Appendix A.2 by setting to zero the first-order cumulant. In the following we summarize the main properties of the four-wave kinetic equation (3) and refer the reader to Refs. [7,8] for more details.

2.2. Conservation laws, entropy growth and equilibrium

As it occurs for the usual Boltzmann's equation relevant for diluted gases, the total "mass" $N = V \int n_{\boldsymbol{k}}(t)\mathrm{d}^D\boldsymbol{k}$ (wave action in the weak turbulence terminology), the total momentum $\boldsymbol{P} = V \int \boldsymbol{k}\, n_{\boldsymbol{k}}(t)\mathrm{d}^D\boldsymbol{k}$ and the kinetic energy $E = V \int k^2 n_{\boldsymbol{k}}(t)\mathrm{d}^D\boldsymbol{k}$ are conserved by the evolution (3). Moreover, this kinetic equation (3) satisfies a H-Theorem[1]: $\partial_t S > 0$ for $S = V \int \ln n_{\boldsymbol{k}} \mathrm{d}^D\boldsymbol{k}$. The equilibrium (no-flux) solution, usually named the Rayleigh–Jeans distribution,

$$n_{\boldsymbol{k}}^{eq} = \frac{T}{k^2 - \boldsymbol{k}\cdot\boldsymbol{v} - \mu} \tag{4}$$

vanishes exactly the right-hand side of (3).

This is a formal solution only, because it does not yield a converging expression for the energy, nor even for the total mass. In order to regularize the divergences of the integrals, one usually introduces an ultraviolet frequency cut-off, k_c. Such a cut-off arises naturally in the numerical simulations, due to the spatial discretization of the NLS equation (1) [5], and it is also known to appear in real physical systems through viscosity or diffusion effects at the microscopic scale [7]. Note that in the context of optical waves, k_c corresponds to the spatial frequency cut-off inherent to a multimode waveguide configuration of optical wave propagation [27], *i.e.*, spatial frequencies k higher than k_c are not guided by the multimode waveguide. In the following we shall assume that the initial condition is such that the total initial momentum vanishes, so that the equilibrium distribution is spherically symmetric and the $\boldsymbol{k}\cdot\boldsymbol{v}$ term in (4) is absent.

As for the original NLS equation in its wave turbulence regime ($E \gg U$), the kinetic equation conserves the total mass and the kinetic energy. In 3D one obtains [5]

$$N/V = 4\pi \int_0^{k_c} \frac{T}{k^2 - \mu} k^2 \mathrm{d}k = 4\pi T k_c \left(1 - \frac{\sqrt{-\mu}}{k_c} \arctan\left(\frac{k_c}{\sqrt{-\mu}} \right) \right), \tag{5}$$

$$E/V = 4\pi \int_0^{k_c} \frac{Tk^2}{k^2 - \mu} k^2 \mathrm{d}k = \frac{4\pi T k_c^3}{3} \left(1 + 3\frac{\mu}{k_c^2} + 3\left(\frac{-\mu}{k_c^2} \right)^{3/2} \arctan\left(\frac{k_c}{\sqrt{-\mu}} \right) \right). \tag{6}$$

These equations should be interpreted as follows. The initial non-equilibrium state of the field ψ is characterized by a mass N and an energy E. The wave spectrum then irreversibly relaxes to the equilibrium distribution (4), whose temperature T and chemical potential μ are determined by Eqs. (5) and (6): a given pair (N, E) then determines a unique pair (T, μ). An inspection of Eq. (5) reveals that μ tends to

[1] This definition of the entropy exhibits a divergence. An improved definition could be

$$S = V \int \mathrm{d}^D\boldsymbol{k} \ln\left(n_{\boldsymbol{k}}/n_{\boldsymbol{k}}^{eq}\right)$$

where $n_{\boldsymbol{k}}^{eq}$ refers to the Rayleigh–Jeans distribution. This definition provides a vanishing entropy for the equilibrium distribution (4).

0 for a non-vanishing temperature T (keeping N/V constant), or a finite density N/V (keeping T constant), as illustrated by the following expansion

$$N/V \approx 4\pi T k_c \left(1 - \frac{\pi\sqrt{-\mu}}{2k_c} + \cdots\right)$$

for $|\mu| \ll k_c^2$. By analogy with the Bose–Einstein transition in quantum systems, this reveals the existence of a condensation process [5].

This reasoning may easily be generalized to other dimensions. For an odd space dimension ($D > 1$), one obtains (here C_D is the volume of an unitary sphere in D space dimensions)

$$N/V \approx C_D T k_c^{D-2}\left(\frac{1}{D-2} + \frac{\pi}{2\sin(\frac{D\pi}{2})}\left(-\mu/k_c^2\right)^{\frac{D}{2}-1} + \frac{\mu}{(D-4)k_c^2} + \mathcal{O}(\mu/k_c^2)\right),$$

while for an even space dimension a logarithm dominant term arises. For instance, for $D = 2$ & $D = 4$ one respectively has:

$$N/V|_{2D} = \pi T\log(1 - k_c^2/\mu), \qquad N/V|_{4D} = 2\pi^2 T\left(k_c^2 + \mu\log(1 - k_c^2/\mu)\right). \tag{7}$$

It becomes apparent that for $D = 2$, μ reaches 0 for a vanishing temperature T (N/V constant), or for a diverging density N/V (T constant). In analogy with standard Bose–Einstein condensation, this indicates that condensation no longer takes place in 2D. This fact originates in the infrared divergence of the integral of the "mass", $N/V = C_D\pi\int_{k_0} k^{D-1}\frac{T}{k^2}\mathrm{d}k \sim k_0^{D-2}$ as $k_0 \to 0$, which exhibits a divergence for $D \le 2$. Actually, this result is rigorously correct in the thermodynamic limit (*i.e.*, $V \to \infty$, $N \to \infty$, keeping N/V constant). Nevertheless, we shall see below that, for situations of physical interest in which N and surface area are finite, wave condensation is re-established in two dimensions, a remarkable property confirmed by the numerical simulations.

The expressions (7) also reveal that $D = 4$ plays the role of a "threshold dimension", in the sense that $\frac{\partial N}{\partial\mu}|_{\mu=0^-} \to \infty$ for $D < 4$ and $\frac{\partial N}{\partial\mu}|_{\mu=0^-}$ finite for $D > 4$. This aspect indicates that the dimension $D = 4$ should play a particular role in the interpretation of wave condensation as a phase-transition, a feature that will be discussed in more detail hereafter.

2.3. Non-equilibrium spectra and locality

Besides thermal equilibrium stationary solutions, V. Zakharov identified an important class of non-equilibrium stationary solutions that are characterized by the existence of a non-vanishing flux of energy or particles [7,28]. Zakharov suggested that a Kolmogorov–like analysis provides two other possible power law behaviors

$$n_{\boldsymbol{k}}^Q = C_Q\frac{Q^{1/3}}{k^{D-2/3}} \tag{8}$$

$$n_{\boldsymbol{k}}^P = C_P\frac{P^{1/3}}{k^D} \tag{9}$$

where $Q(P)$ is the mass (energy) flux in wave-number space per unit time, C_P, & C_Q are numerical constants. Those solutions may be derived through a simple dimensional analysis by setting Q and P as constant. It is important to notice, however, that the Kolmogorov–Zakharov spectra are exact solutions of the kinetic equation [28], which constitutes an essential property of (3). Note that a linear stability analysis of the solution (8) ($n_k^Q \sim k^{-7/3}$) has been performed recently in three space dimensions [29], which reveals that this solution is stable.

Finally, let us comment on the locality of the interactions, which refers to the convergence of the integrals involved in the collision term (3). Consider the power law solution $n_{\boldsymbol{k}} \sim k^{-2z}$ then the collisional integral in (3) converges for $1 \le z \le D/2$, [7,8].

2.4. Breakdown of weak-turbulence

As already discussed above in the introduction, it has been stressed that weak turbulence is not a completely satisfactory theory, because it always exhibits a breakdown at either large or small scales [8,23,24]. Because the theory is based on an asymptotic (ordered) perturbation procedure, its self-consistency requires higher-order terms to be smaller than lower-order terms, a property that should be verified for every scale, *i.e.* every wave number k. It should be noted that such self-consistency is not verified for the stationary power-law solutions considered above, Eqs. (4), (8) and (9).

As discussed by Newell and collaborators [8,23,24], a well-ordered asymptotic expansion procedure is obtained provided the nonlinear frequency:

$$\omega_{NL}(k) \sim \frac{\partial_t n_k(t)}{n_k(t)} \sim 2\pi a^2\int n_{k_s}n_{k_{s'}}\delta(k^2 + k_2^2 - k_3{}^2 - k_4{}^2)\delta^{(D)}(\boldsymbol{k} + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\mathrm{d}^D\boldsymbol{k}_{234} \tag{10}$$

is much smaller than the linear frequency $\omega_k = k^2$. In (10) (s, s') take any of the couple of values (3, 4) or (2, 3) or (2, 4). Considering now a power-law spectrum $n_k = Ck^{-2z}$, one has

$$\omega_{NL}(k) \sim a^2C^2k^{-4z-2+2D} \ll \omega_k = k^2$$

which leads to the condition

$$a^2C^2k^{-4z-4+2D} \ll 1. \tag{11}$$

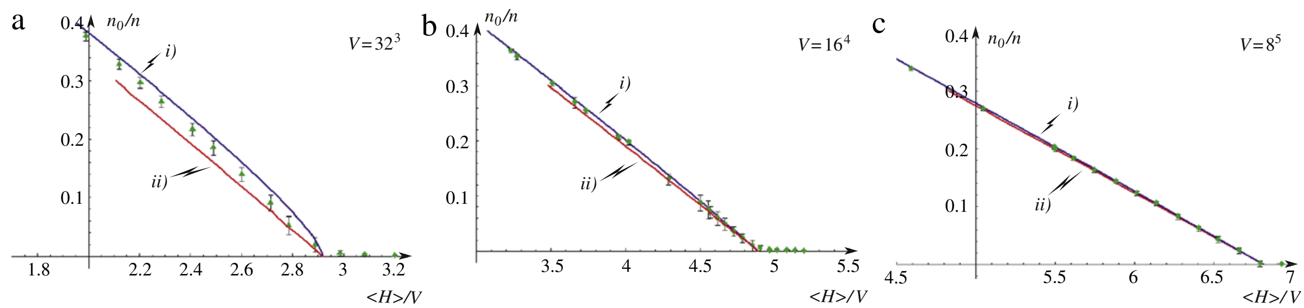

Fig. 1. The condensate fraction n_0/n as function of the energy density $\langle H\rangle/V$ for three (a), four (b) and five (b) space dimensions near the transition energy H_c. In all three numerical simulations the particle density is $N/V = 1/2$, $a = 1$, the domain is a box of volume $V = 32^3$ in three space dimensions, $V = 16^4$ in four and $V = 8^5$ in five space dimensions, in all cases the boundary conditions are periodic and the mesh size $dx = 1$ ($k_c = \pi$). The points (▲ & ♦) refer to the results of the numerics. The theoretical curves (i) and (ii) are plotted from Eqs. (52) and (50) respectively (see below for details). The transition energy $H_c \sim k_c^2$ depends explicitly on k_c, as shown below in Section 5. For the case of a cubic domain $H_c/V = an^2 + n\left(\sum'_{\boldsymbol{k}} 1\right)/\left(\sum'_{\boldsymbol{k}} \frac{1}{k^2}\right)$, where the $\sum'_{\boldsymbol{k}}$ denotes a sum in the discretized wavenumber space $k_i = (-\pi, \pi]$ except $\boldsymbol{k} = 0$, where $\Delta k = 2\pi/L$ is the wavenumber discretization. For the present cases (a) $H_c^{D=3}/V = 2.92$, (b), $H_c^{D=4}/V = 4.89$, and (c), $H_c^{D=5}/V = 6.81$. All plots show the agreement between the data and the theory near the transition energy H_c.

The explicit criterion to the wave number depends on the sign of $-4z - 4 + 2D$. If $D < 2(z+1)$ the weak turbulence equations are valid for

$$k \gg k_{NL} = (a\,C)^{1/(2z+2-D)}.$$

If k_{NL} is outside the frequency-bandwidth of interest delimited by the infrared and ultraviolet cut-offs: (k_{ir}, k_c) then the weak turbulence description is safe. But this is not the case in general. For instance, for the unforced NLS equation considered in this article, it is the Rayleigh–Jeans distribution which plays an important role. At the transition to condensation we may neglect the chemical potential [5], so that $z = 1$, $C = T$, which thus gives $k_{NL} = (a\,T)^{1/(4-D)}$ for $D < 4$. According to the weak-nonlinear limit $a \to 0$, the nonlinear scale $k_{NL} \to 0$ as $a \to 0$ so that the breakdown of weak turbulence arises at large wavelength scales, however, note that a finite size of the system provides an infrared cut-off, $k_{ir} \sim 1/L$, L^D being the system volume. Accordingly, for a finite system size, one has $k_{NL} \ll k_{ir}$ in the limit $a \to 0$ and the kinetic theory holds for any wavenumber k. A problem may arise in the thermodyamic limit $L \to \infty$ where a breakdown of the wave turbulence theory may occur in the small wavenumber region.

On the other hand if $D > 4$, the condition (11) reads

$$k \ll k_{NL} = (a\,T)^{-1/(D-4)},$$

so that the weak turbulence description is safe if k_{NL} is greater than the ultraviolet cut-off, $k_{NL} > k_c$.

Note that this criterion for the validity of the wave turbulence theory may also be obtained from the perturbation expansion procedure of Section 3 and Appendix A.1. It is easy to see that the following "small" expansion parameter

$$\omega_{NL}(k)/\omega_k = \left(aTk^{D-4}\right)^2 = \left(\frac{k}{k_{NL}}\right)^{2(D-4)} \tag{12}$$

comes from the asymptotic expansion. As before, this reveals that in the infrared regime, $k \to 0$, $\omega_{NL}(k)/\omega_k$ is always small for $D > 4$, which leads to well-ordered and convergent asymptotic series. However, for $D < 4$, $\omega_{NL}(k)/\omega_k$ is no longer as small as $k \to 0$.

As commented above through the analysis of Eq. (12), the dimension $D = 4$ then plays a particular role. This is actually not surprising. The thermodynamic properties of the Landau free energy (2) have been the subject of a detailed investigation in the theory of phase transitions, in which the partition function may be computed perturbatively (see for instance [30]). However, at the transition point where μ reaches zero, all integral terms of the perturbation expansion are known to diverge. Note that for the particular case $D = 4$, the integrals only exhibit a *logarithmic divergence*. As is well-known, the modern way to surround this problem is based on the Kadanoff–Wilson renormalization group theory of phase transitions and critical phenomena. Let us remark, however, that it seems that the dimension $D = 4$ does not play any particular role in the numerical simulations of NLS as we can see in the following Fig. 1 where the transition for wave condensation is plotted (for details of the plots see Section 5 later on).

2.5. *Finite time singularity of the 4-wave kinetic equation*

By means of numerical simulations, it was shown in Refs. [31,32] that Eq. (3) exhibits a finite-time singularity at $\boldsymbol{k} = 0$ in three spatial dimensions ($D = 3$). Although, it is not known whether any initial condition exhibits a self-similar singularity, the evolution of the linearized kinetic equation in three space dimensions around the equilibrium distribution $n^{eq}_{\boldsymbol{k}} = \frac{T}{k^2}$ evolves towards the low wavenumber scales [33]. Moreover, the linear evolution captures all the mass at $k = 0$ but in infinite time, naturally this fact does not contradict the possibility that the nonlinear evolution blows-up in finite time.

The self-similar solution has the form

$$n_{\boldsymbol{k}}(t) = \tau(t)^{-\nu}\,\phi\left(\frac{k^2}{\tau(t)}\right), \tag{13}$$

where τ is a function that only depends on time and ϕ was shown to satisfy an integro–differential equation. The parameter ν is a nonlinear eigenvalue which guarantees that the boundary condition $\phi(0)$ keeps a finite value, while $\phi(\zeta) \approx \zeta^{-\nu}$ as ζ tends to infinity.[2] The function $\tau(t)$ was shown to vanish in finite time as $\tau(t) = [2(\nu - 1)(t_* - t)]^{\frac{1}{2(\nu-1)}}$, where t_* depends on the initial condition, $n_{\boldsymbol{k}}(t = 0)$. The existence of this finite time singularity may be considered as a precursor for the spontaneous emergence of the condensate, although this aspect has not yet been rigorously proven.

Finally, as said in previous section, strictly speaking, the kinetic Eq. (3) is not 'uniformly valid' for all the wave-numbers $\boldsymbol{k}$, *i.e.*, its validity may become questionable for small wave-numbers for which dispersive effects may become of the same order as nonlinear effects. Indeed, by analogy with kinetic gas theory, one may define a "collision frequency" (or a nonlinear frequency) as $\omega_{NL}(t) = \dot{\tau}(t)/\tau(t) \sim 1/(t_* - t)$. For $t \lesssim t_*$, this frequency becomes much larger than the linear frequency $\omega_L(t) \sim \tau(t)$, which means that the kinetic description of the wave interaction by means of Eq. (3) breaks down before t_*. A similar argument follows from the usual assessment of a nonlinear scale k_{NL} in which wave turbulence theory breaks down [8]. The existence of such a characteristic spatial scale $1/k_{NL}$ reveals a breakdown of the assumption of spatial homogeneity of the field statistics.

2.6. Scenario for wave condensation

As discussed above, despite the formal reversibility of the NLS equation, the random nonlinear wave ruled by the NLS equation exhibits an irreversible evolution to equilibrium. The essential properties of this thermalization process are well described by the kinetic equation (3). The introduction of the ultraviolet cut-off, k_c, creates a deep analogy with a system of weakly interacting Bose gases: above a critical energy ($H_c \sim k_c^2$), the evolution is regular and the system reaches the Rayleigh–Jeans equilibrium distribution characterized by the couple of variables (μ, T). However if the initial energy is below the critical value ($H \lesssim H_c$), the equilibrium solution exhibits a divergence because $\mu \to 0$, which leads to a macroscopic population of the fundamental mode $\boldsymbol{k} = 0$. The kinetic equation (3) thus describes a flux of particles towards the small wavenumbers, which leads to a self-similar blow-up regime discussed here above in Section 2.5. Naturally, such a blow-up is fictitious because the weak turbulence expansion breaks down at large scales and it is also known to not hold at short time scales.

This indicates the emergence of isolated islands in the statistical properties of the field, so that the process of wave condensation does not occur uniformly in the physical space, but rather 'locally' within each island. The process of merging of such islands, as well as the corresponding merging process of vortices, occur in an intermediate stage of the field evolution. Finally, at some later stage a new state of the field emerges, which is characterized by an almost spatially homogeneous condensate interacting with statistically homogeneous turbulent fluctuations. It is this regime that we shall describe theoretically in the present paper.

Let us note that the discussion of $D > 4$ deserves here some special care. In principle the above scenario should also hold for $D > 4$. Although it is not known whether Eq. (3) exhibits a finite time singularity for any space dimensions, it was conjectured in [34], that the nonlinear eigenvalue ν should satisfy $D/2 - 1/3 < \nu < D/2$ which is possible for every dimension D. However, the breakdown of weak turbulence could drive the system to a complete different scenario in the large scale behavior. For instance intermittent events like creation and annihilation of vortex (manifolds of dimension $D - 2$ in a D space dimension) could not be stimulated by the system. This question deserves more study.

3. Weak turbulence theory with a small condensate amplitude

3.1. Cumulants' hierarchy

To provide a statistical description of the evolution of the random field ψ, we shall make use of the cumulants of the field rather than its moments [8,17–19] (see the Appendix A.1 for the definitions of the cumulants). Indeed, given the homogenous character of the statistics, the advantage of the cumulants with respect to the moments relies on the fact that the cumulants decrease to zero as the spatial coordinates involved in the average tend to infinity. For instance, for a field characterized by a non-zero mean, $\langle\psi\rangle \neq 0$, the second-order cumulant $R^{(2)+-}(\boldsymbol{x}_2) \equiv \langle\psi(\boldsymbol{x}, t)\psi^*(\boldsymbol{x} + \boldsymbol{x}_2, t)\rangle - \langle\psi\rangle^2$ tends to zero as $\boldsymbol{x}_2 \to \infty$. Let us consider the Fourier's amplitudes of the complex field ψ (s is a sign variable: $s = -$ refers to the complex conjugation, while $s = +$ leaves the quantity unchanged):

$$\psi^s(\boldsymbol{x}, t) = \int A^s_{\boldsymbol{k}}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}^D\boldsymbol{k}. \tag{14}$$

Note that the Fourier's amplitudes satisfy the relation $A^-_{\boldsymbol{k}} \equiv A^*_{-\boldsymbol{k}}$. The cumulants and the moments are related one to each other in the Fourier's space and the real physical space. In particular, under the assumption of space homogeneity, in Fourier's space they are related by the following relations [8]:

$$\left\langle A^{s_1}_{\boldsymbol{k}_1}\right\rangle = Q^{(1)s_1}(t)\delta^{(D)}(\boldsymbol{k}_1)$$

$$\left\langle A^{s_1}_{\boldsymbol{k}_1}A^{s_2}_{\boldsymbol{k}_2}\right\rangle = Q^{(2)s_1s_2}(\boldsymbol{k}_2)\delta^{(D)}(\boldsymbol{k}_1+\boldsymbol{k}_2) + Q^{(1)s_1}(t)Q^{(1)s_2}(t)\delta^{(D)}(\boldsymbol{k}_1)\delta^{(D)}(\boldsymbol{k}_2),$$

$$\begin{aligned}\left\langle A^{s_1}_{\boldsymbol{k}_1}A^{s_2}_{\boldsymbol{k}_2}A^{s_3}_{\boldsymbol{k}_3}\right\rangle = {} & Q^{(3)s_1s_2s_3}(\boldsymbol{k}_2,\boldsymbol{k}_3)\delta^{(D)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3) + \mathcal{P}_{123}Q^{(1)s_1}(t)Q^{(2)s_2s_3}(\boldsymbol{k}_3)\delta^{(D)}(\boldsymbol{k}_3)\delta^{(D)}(\boldsymbol{k}_2+\boldsymbol{k}_3)\\ &+ Q^{(1)s_1}(t)Q^{(1)s_2}(t)Q^{(1)s_3}(t)\delta^{(D)}(\boldsymbol{k}_1)\delta^{(D)}(\boldsymbol{k}_2)\delta^{(D)}(\boldsymbol{k}_3),\end{aligned}$$

[2] The numerics gives $\nu = 1.234\ldots$ in three space dimensions [32].

$$
\begin{aligned}
\left\langle A^{s_1}_{\boldsymbol{k}_1} A^{s_2}_{\boldsymbol{k}_2} A^{s_3}_{\boldsymbol{k}_3} A^{s_4}_{\boldsymbol{k}_4} \right\rangle &= Q^{(4)s_1s_2s_3s_4}(\boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4)\delta^{(D)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4) \\
&+ \mathcal{P}_{1234} Q^{(1)s_1}(t) Q^{(3)s_2s_3s_4}(\boldsymbol{k}_3, \boldsymbol{k}_4)\delta^{(D)}(\boldsymbol{k}_1)\delta^{(D)}(\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4) \\
&+ \mathcal{P}_{123} Q^{(2)s_1s_2}(\boldsymbol{k}_2) Q^{(2)s_3s_4}(\boldsymbol{k}_4)\delta^{(D)}(\boldsymbol{k}_1+\boldsymbol{k}_2)\delta^{(D)}(\boldsymbol{k}_3+\boldsymbol{k}_4) \\
&+ \mathcal{P}_{123} Q^{(1)s_1}(t) Q^{(1)s_2}(t) Q^{(2)s_3s_4}(\boldsymbol{k}_3, \boldsymbol{k}_4)\delta^{(D)}(\boldsymbol{k}_1)\delta^{(D)}(\boldsymbol{k}_2)\delta^{(D)}(\boldsymbol{k}_3+\boldsymbol{k}_4) \\
&+ \mathcal{P}_{123} Q^{(1)s_4}(t) Q^{(1)s_1}(t) Q^{(2)s_2s_3}(\boldsymbol{k}_2, \boldsymbol{k}_3)\delta^{(D)}(\boldsymbol{k}_4)\delta^{(D)}(\boldsymbol{k}_1)\delta^{(D)}(\boldsymbol{k}_2+\boldsymbol{k}_3) \\
&+ Q^{(1)s_1}(t) Q^{(1)s_2}(t) Q^{(1)s_3}(t) Q^{(1)s_4}(t)\delta^{(D)}(\boldsymbol{k}_1)\delta^{(D)}(\boldsymbol{k}_2)\delta^{(D)}(\boldsymbol{k}_3)\delta^{(D)}(\boldsymbol{k}_4),
\end{aligned}
$$

etc. (15)

where $\mathcal{P}_{i_1,\ldots,i_n}$ denotes the cyclic permutation of the sign $s_{i_1}\ldots s_{i_n}$ and wave-number $\boldsymbol{k}_{i_1}\ldots\boldsymbol{k}_{i_n}$ variables. Remark that the first-order cumulant denotes the average of the field, $Q^{(1)+}(t) = \langle\psi\rangle(t)$.

Eq. (1) may be written in Fourier's space by taking $\psi(\boldsymbol{x}, t) = \int A_{\boldsymbol{k}}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} d^D\boldsymbol{k}$

$$
i\partial_t A_{\boldsymbol{k}} = \omega_{\boldsymbol{k}} A_{\boldsymbol{k}} + a(2\pi)^D \int A_{\boldsymbol{k}_1} A_{\boldsymbol{k}_2} A^*_{\boldsymbol{k}_3} \delta^{(D)}(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}) d^D\boldsymbol{k}_{123}. \tag{16}
$$

By taking the average over the realizations, one readily obtains the first equation of the hierarchy governing the evolution of the first-order cumulant:

$$
\begin{aligned}
\frac{d}{dt} Q^{(1)l_1} &= -ial_1(2\pi)^D \int Q^{(3)-l_1l_1l_1}(\boldsymbol{k}_2, \boldsymbol{k}_3)\delta^{(D)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3) d^D\boldsymbol{k}_{123} \\
&- ial_1(2\pi)^D \int Q^{(1)-l_1} Q^{(2)l_1l_1}(\boldsymbol{k}) d^D\boldsymbol{k} + ial_1 Q^{(1)-l_1} Q^{(1)l_1} Q^{(1)l_1} \\
&- 2ial_1(2\pi)^D \left(\int Q^{(2)-l_1l_1}(\boldsymbol{k}) d^D\boldsymbol{k} + Q^{(1)-l_1} Q^{(1)l_1} \right) Q^{(1)l_1}.
\end{aligned} \tag{17}
$$

Note that $Q^{(1)-l_1} Q^{(1)l_1} = |Q^{(1)l_1}|^2 = |Q^{(1)-l_1}|^2$, so that the total number of particles takes the form

$$
N = V(2\pi)^D \left(\int Q^{(2)-l_1l_1}(\boldsymbol{k}) d^D\boldsymbol{k} + Q^{(1)-l_1} Q^{(1)l_1} \right). \tag{18}
$$

Accordingly, the last term of Eq. (17) is again a purely oscillation term, $-2ial_1 N/V Q^{(1)l_1}$, with the constant frequency $2aN/V$.

In a similar way, one may derive the equation governing the evolution of the second-order cumulant:

$$
\begin{aligned}
\frac{d}{dt} Q^{(2)l_1l_2}(\boldsymbol{p}_2) &= (-il_1\omega(\boldsymbol{p}_1) - il_2\omega(\boldsymbol{p}_2)) Q^{(2)l_1l_2}(\boldsymbol{p}_2) - 2ia(2\pi)^D \mathcal{P}_{1'2'} l_2 \left(\int Q^{(2)-l_2l_2}(\boldsymbol{k}) d^D\boldsymbol{k} + Q^{(1)-l_2} Q^{(1)l_2} \right) Q^{(2)l_1l_2}(\boldsymbol{p}_2) \\
&- ia(2\pi)^D \mathcal{P}_{1'2'} l_2 \left(\int Q^{(2)l_2l_2}(\boldsymbol{k}) d^D\boldsymbol{k} + Q^{(1)l_2} Q^{(1)l_2} \right) Q^{(2)l_1-l_2}(\boldsymbol{p}_2) \\
&- ia(2\pi)^D \mathcal{P}_{1'2'} l_2 \int Q^{(4)l_1-l_2l_2l_2}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\delta^{(D)}(\boldsymbol{p}_1+\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3) d^D\boldsymbol{k}_{123} \\
&- ia(2\pi)^D \mathcal{P}_{1'2'} l_2 \int Q^{(1)-l_2} Q^{(3)l_1l_2l_2}(\boldsymbol{k}_1, \boldsymbol{k}_2)\delta^{(D)}(\boldsymbol{p}_1+\boldsymbol{k}_1+\boldsymbol{k}_2) d^D\boldsymbol{k}_{12} \\
&- 2ia(2\pi)^D \mathcal{P}_{1'2'} l_2 \int Q^{(1)l_2} Q^{(3)l_1-l_2l_2}(\boldsymbol{k}_1, \boldsymbol{k}_2)\delta^{(D)}(\boldsymbol{p}_1+\boldsymbol{k}_1+\boldsymbol{k}_2) d^D\boldsymbol{k}_{12}
\end{aligned} \tag{19}
$$

where $\boldsymbol{p}_1+\boldsymbol{p}_2 = 0$ should be imposed. The second term in (19) is again a purely oscillation term, $-2iaN/V(l_1+l_2)Q^{(2)l_1l_2}(\boldsymbol{p}_2)$. The symbol $\mathcal{P}_{1'2'}$ means a permutation of the pair indices $\{l_1, \boldsymbol{p}_1\}$ and $\{l_2, \boldsymbol{p}_2\}$. In this equation, as well as in the following, we consider that the *l.h.s* depends only on the variable $\boldsymbol{p}$, while the $\boldsymbol{k}$'s variable are integrated in the *r.h.s* of the equation.

3.2. Multiscale expansion

It becomes apparent from these expressions that the equation for the third-order cumulant $Q^{(3)l_1l_2l_3}(\boldsymbol{p}_2, \boldsymbol{p}_3)$ will depend linearly on the fifth-order cumulant $Q^{(5)}$ and on products of the form $Q^{(1)}Q^{(4)}$, $Q^{(2)}Q^{(3)}$, $Q^{(1)}Q^{(1)}Q^{(3)}$, *etc*. In the same vein, the equation for the fourth-order cumulant $Q^{(4)}$ depends linearly on the sixth-order cumulant $Q^{(6)}$ and on products of the form $Q^{(1)}Q^{(5)}$, $Q^{(2)}Q^{(4)}$, $Q^{(1)}Q^{(1)}Q^{(4)}$, *etc*. A first conclusion that can be drawn from this simple observation is the following: If $Q^{(1)}(t=0) = 0,\ Q^{(3)}(t=0) = 0,\ \ldots,\ Q^{(2N+1)}(t=0) = 0, \ldots$, *i.e.*, if all odd-order cumulants are initially zero, then, they will remain zero at any time. *Conversely, if one of the odd cumulants is not zero initially, then $Q^{(1)}$ will be regenerated by the hierarchy of the equations and it is expected to asymptotically reach some stationary value.* In this sense wave condensation is a very robust phenomenon, which appears as a generic property of the NLS equation, and probably in more general class of wave systems.

Yet to that point we have an exact, but infinite hierarchy of cumulants' equations. The next step then consists in finding a closure of such hierarchy. This is possible in the limit of small amplitudes waves, under the weak assumption that the fields are initially uncorrelated at infinitely separated points. Then, the dispersive nature of the system allows for a fast decorrelation (phase-mixing) between different waves. Moreover, owing to the resonant interaction process, the nonlinear coupling term regenerates the higher-order cumulants in a special way. Such a regeneration process is given by the long-time cumulative effect in terms of the lower-order cumulants $Q^{(1)}$ and $Q^{(2)}$.

This time scale separation leads to a natural asymptotic closure of the hierarchy of cumulants' equations [8]. In substance, assuming that $A^s_{\boldsymbol{k}} \sim \epsilon \ll 1$, *i.e.*, $Q^{(n)} \sim \epsilon^n$, then one may perform a formal expansion of the n-th order cumulant in the form:

$$Q^{(n)l_1\ldots l_n} = \epsilon^n (Q_0^{(n)l_1\ldots l_n} + \epsilon Q_1^{(n)l_1\ldots l_n} + \epsilon^2 Q_2^{(n)l_1\ldots l_n} + \cdots).$$

Up to zero order, $\mathcal{O}(\epsilon^0)$, one has $\mathrm{d}Q_0^{(1)l_1}/\mathrm{d}t = 0$, while for higher-order cumulants one obtains (keeping in mind that $\boldsymbol{p}_1+\boldsymbol{p}_2+\cdots+\boldsymbol{p}_n = 0$)

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_0^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n) = -\mathrm{i}[l_1\omega(\boldsymbol{p}_1) + \cdots + l_n\omega(\boldsymbol{p}_n)] Q_0^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n).$$

Accordingly, we have $Q_0^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n) = \mathrm{e}^{-\mathrm{i}[l_1\omega(\boldsymbol{p}_1)+\cdots+l_n\omega(\boldsymbol{p}_n)]t} q_0^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n)$, with the exception that $Q_0^{(1)l_1} = q_0^{(1)l_1}$ and $Q_0^{(2)s-s}(\boldsymbol{p}) = q_0^{(2)s-s}(\boldsymbol{p})$, because $\omega(\boldsymbol{p}) = \omega(-\boldsymbol{p})$.

At the next order, $\mathcal{O}(\epsilon^2)$, one can show that $Q_1^{(n)}$ grows linearly in time. To avoid this secular growth one has to resort to a multiscale expansion technique. For this purpose, note that the n-th order cumulant at zero order in ϵ, $q_0^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n)$, depends on time in principle. However, we shall relax the definition of $q_0^{(n)}$ by assuming that $q_0^{(n)}$ depends only on long time scales as $\sim 1/\epsilon^2$ and/or $\sim 1/\epsilon^4$ *etc*. More precisely, by making use of the multiscale technique, we shall assume that the time derivative of $q_0^{(n)}$ exhibits a formal expansion in ϵ of the form:

$$\frac{\mathrm{d}}{\mathrm{d}t} q_0^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n) = \epsilon^2 F_2^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n) + \epsilon^4 F_4^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n) + \cdots \quad (20)$$

where the functions $F_r^{(n)}$ correspond to the $\sim 1/\epsilon^r$ time scale expansion of the n-th order cumulant $q_0^{(n)}$. According to this procedure, one can show that the second order expansion, $F_2^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n)$, only provides a nonlinear renormalization of the frequency. Indeed for the first order cumulant one gets

$$F_2^{(1)l_1} = -\mathrm{i}a(2\pi)^D l_1 \left[2\int Q_0^{(2)l_1-l_1}(\boldsymbol{k})\mathrm{d}^D\boldsymbol{k} + q_0^{(1)l_1} q_0^{(1)-l_1}\right] q_0^{(1)l_1},$$

so that $\frac{\mathrm{d}}{\mathrm{d}t}\left(q_0^{(1)l_1} q_0^{(1)-l_1}\right) = 0$, while for higher order cumulants, one has

$$F_2^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n) = -\mathrm{i}\,[l_1\widetilde{\omega}(\boldsymbol{p}_1) + \cdots + l_n\widetilde{\omega}(\boldsymbol{p}_n)]\, q_0^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n)$$

with

$$\widetilde{\omega}(\boldsymbol{p}) = 2a(2\pi)^D \left(\int Q_0^{(2)l_1-l_1}(\boldsymbol{k})\mathrm{d}^D\boldsymbol{k} + q_0^{(1)l_1} q_0^{(1)-l_1}\right). \quad (21)$$

The constant frequency in Eq. (21) does not depend on $\boldsymbol{p}$ for the simple case of the NLS equation (1) considered in this paper. However, applications of the above multiscale expansion technique to more involved equations would introduce a nontrivial renormalization of the frequency $\widetilde{\omega}(\boldsymbol{p})$. For the second order off-diagonal cumulant one then gets $\frac{\mathrm{d}}{\mathrm{d}t} q_0^{(2)-l_1 l_1} = 0$. Note that the condensate amplitude $n_0 = (2\pi)^D q_0^{(1)l_1} q_0^{(1)-l_1}$, is a quantity that does not depend explicitly on time up to this order. Up to the next order, however, we shall obtain a dynamical equation for $n_0(t)$ and for $n_{\boldsymbol{k}}(t)$, so that both $n_0(t)$ and $(2\pi)^D\int Q_0^{(2)l_1-l_1}(\boldsymbol{k})\mathrm{d}^D\boldsymbol{k}$ are no longer conserved, but the sum $(2\pi)^D\left(\int Q_0^{(2)l_1-l_1}(\boldsymbol{k})\mathrm{d}^D\boldsymbol{k} + q_0^{(1)l_1} q_0^{(1)-l_1}\right) \equiv N/V$ is still conserved by the dynamics. Note that the constant frequency given in Eq. (21) may be removed *ab-initio* from the NLS equation (1), by means of a special choice of a global uniform oscillation, *i.e.*, by writing $\mathrm{i}\partial_t\psi = -\Delta\psi + a(|\psi|^2 - N/V)\psi$ instead of Eq. (1).

3.3. Kinetic equations

At the fourth-order of the expansion, $\mathcal{O}(\epsilon^4)$, the asymptotic closure of the hierarchy of cumulants' equations remarkably gives a set of coupled kinetic equations (see the Appendix A.2 for the details). They describe the coupled evolutions of the condensate amplitude *i.e.*, the first-order cumulant $(2\pi)^D|Q_0^{(1)l_1}|^2 = (2\pi)^D|q_0^{(1)l_1}|^2 \equiv n_0(t) = N_0(t)/V$, and for the non-diagonal second-order cumulant $(2\pi)^D Q_0^{(2)+-}(\boldsymbol{k},t) = (2\pi)^D q_0^{(2)+-}(\boldsymbol{k},t) \equiv n_{\boldsymbol{k}}(t)$:

$$\partial_t n_0(t) = \alpha^2 n_0(t)\int n_{k_2}n_{k_3}n_{k_4}\left(n_{k_2}^{-1} - n_{k_3}^{-1} - n_{k_4}^{-1}\right)\delta(\omega_{k_2} - \omega_{k_3} - \omega_{k_4})\delta^{(D)}(\boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\mathrm{d}^D\boldsymbol{k}_{234}, \quad (22)$$

$$\begin{aligned}\partial_t n_{p_2}(t) \;=\; &\alpha^2\int n_{p_2}n_{k_2}n_{k_3}n_{k_4}\left(n_{p_2}^{-1} + n_{k_2}^{-1} - n_{k_3}^{-1} - n_{k_4}^{-1}\right)\delta(\omega_{p_2} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4})\delta^{(D)}(\boldsymbol{p}_2 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\,\mathrm{d}^D\boldsymbol{k}_{234}\\ &+\alpha^2 n_0(t)\int n_{p_2}n_{k_3}n_{k_4}\left(n_{p_2}^{-1} - n_{k_3}^{-1} - n_{k_4}^{-1}\right)\delta(\omega_{p_2} - \omega_{k_3} - \omega_{k_4})\delta^{(D)}(\boldsymbol{p}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\,\mathrm{d}^D\boldsymbol{k}_{34}\\ &+2\alpha^2 n_0(t)\int n_{p_2}n_{k_2}n_{k_4}\left(n_{p_2}^{-1} + n_{k_2}^{-1} - n_{k_4}^{-1}\right)\delta(\omega_{p_2} + \omega_{k_2} - \omega_{k_4})\delta^{(D)}(\boldsymbol{p}_2 + \boldsymbol{k}_2 - \boldsymbol{k}_4)\,\mathrm{d}^D\boldsymbol{k}_{24}\end{aligned} \quad (23)$$

where we define the constant $\alpha^2 = 4\pi a^2$.

The coupled set of kinetic equations (22) and (23) were obtained by extending the usual NLS-kinetic equation to singular distributions in Ref. [32]. Here, Eqs. (22) and (23) are shown to result from the natural asymptotic closure of the hierarchy of cumulants' equations, in which the average of the field $\langle\psi\rangle$ has not been implicitly assumed to vanish as it occurs in the 'random phase approximation' variant of weak-turbulence theory [7,16]. In the above derivation it is assumed that $N/V \ll 1$, or similarly $n_0(t) \sim n_k \sim \epsilon^2$. Let us remark that the diagonal part of the second-order cumulant, *i.e.* $Q^{(2)--}(\boldsymbol{k},t)$ and $Q^{(2)++}(\boldsymbol{k},t)$, as well as higher order cumulants exhibit an exponential decay in time (see Appendix A.2), as it is usually assumed in the 'random phase-approximation' approach of the weak-turbulence theory [7,16].

3.4. Equilibrium solutions

The structure of the kinetic equation (23) is analogous to the usual four wave kinetic equation (3), and thus exhibits similar properties. In the fourth-order expansion, $\mathcal{O}(\epsilon^4)$, Eq. (23) conserves the density of momentum

$$\frac{\boldsymbol{P}}{V}=\int \boldsymbol{k}\, n_{\boldsymbol{k}}(t)\mathrm{d}^D\boldsymbol{k},$$

the density of kinetic energy

$$\frac{E}{V}=\int \omega_{\boldsymbol{k}}\, n_{\boldsymbol{k}}\mathrm{d}^D\boldsymbol{k}=\int \boldsymbol{k}^2\, n_{\boldsymbol{k}}\mathrm{d}^D\boldsymbol{k} \tag{24}$$

and exhibits a H-theorem of entropy growth $\mathrm{d}\mathcal{S}/\mathrm{d}t\geq 0$, where $\mathcal{S}(t)=\int \ln(n_{\boldsymbol{k}})\mathrm{d}^D\boldsymbol{k}$ is the non-equilibrium entropy per unit volume.[3] The number of particles (mass) of the incoherent component $\int n_{\boldsymbol{k}}(t)\mathrm{d}^D\boldsymbol{k}$ is not conserved, because there is an exchange of particles between the condensate and the thermal component *via* the three-wave resonant interaction. Note however that the total number of particles $(N/V\equiv n_0+\int n_{\boldsymbol{k}}(t)\mathrm{d}^D\boldsymbol{k})$ is conserved by the coupled kinetic equations (22) and (23). The kinetic equations (22) and (23) then describe an irreversible evolution of the system towards an equilibrium state. Let us assume that the initial momentum of the field is zero. The equilibrium distribution, which vanishes the entropy production $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}[n_{\boldsymbol{k}}]$, and is

$$n_k^{eq}=\frac{T}{k^2}, \tag{25}$$

where T denotes the temperature by analogy with thermodynamics. Let us remark that n_k^{eq} vanishes exactly all the collision terms of the coupled kinetic equations (22) and (23). Important to note, the equilibrium value for the condensate amplitude n_0^{eq} can be calculated from the normalization condition (18),

$$n_0^{eq}=\frac{N}{V}-\int \frac{T}{k^2}\mathrm{d}^D\boldsymbol{k}. \tag{26}$$

The equilibrium spectrum (25) actually corresponds to the Rayleigh–Jeans distribution with zero chemical potential. Let us remark that $\mu=0$ is also justified by the fact that the mass of the incoherent wave component is not conserved due to its interaction with the condensate, which plays the role of a reservoir of particles. Also note that $\mu=0$ is consistent with the usual criterion for Bose–Einstein condensation in a uniform and ideal Bose gas [5,21]. This aspect has been extensively discussed in Ref. [5].

Let us now study the condensation curve relating the fraction of condensed particles N_0/N to the energy H. For this purpose one has to take into account the nonlinear energy contribution U to the total energy, $H=E+U$. Such a nonlinear contribution may be computed by expressing the hamiltonian in terms of the cumulants. Imposing that the diagonal second-, third- and fourth-orders cumulants vanish in the long time regime, one obtains [see Eq. (58) in the Appendix A.1],

$$\frac{\langle H\rangle}{V}=\int \omega_{\boldsymbol{k}}n_{\boldsymbol{k}}\mathrm{d}^D\boldsymbol{k}+a\left(\int n_{\boldsymbol{k}}\,\mathrm{d}^D\boldsymbol{k}\right)^2+2an_0\int n_{\boldsymbol{k}}\,\mathrm{d}^D\boldsymbol{k}+\frac{a}{2}n_0^2=\int \omega_{\boldsymbol{k}}n_{\boldsymbol{k}}\mathrm{d}^D\boldsymbol{k}+a\left(n^2-\frac{1}{2}n_0^2\right) \tag{27}$$

where we recall that $n_0\equiv N_0/V$ and $n\equiv N/V$. One may ask whether the nonlinear contributions of the energy are conserved by the dynamics. Because of the term $-\frac{a}{2}n_0^2$ in Eq. (27), the averaged energy is not conserved by the kinetic Eqs. (22) and (23) up to order $\mathcal{O}(\epsilon^6)$. One should note, however, that the term $a\left(n^2-\frac{1}{2}n_0^2\right)$ is of higher order and should be discarded. We shall discuss the importance of this term in the comparison with the numerical simulations in Section 5.

Finally, replacing the expression of the temperature in (26) into the expression of the energy (27), one obtains the following expression of the condensation curve, *i.e.,* a closed relation between the energy and the fraction of condensed particles:

$$\frac{\langle H\rangle}{V}=(n-n_0)\frac{\int \mathrm{d}^D\boldsymbol{k}}{\int \frac{1}{k^2}\mathrm{d}^D\boldsymbol{k}}+a\left(n^2-\frac{1}{2}n_0^2\right). \tag{28}$$

One may observe that there exists a critical value of the energy density, $\langle H_c\rangle/V=n\int \mathrm{d}^D\boldsymbol{k}/\int\frac{1}{k^2}\mathrm{d}^D\boldsymbol{k}+an^2$, above which no condensation occurs ($n_0=0$). Because the quotient $\int \mathrm{d}^D\boldsymbol{k}/\int\frac{1}{k^2}\mathrm{d}^D\boldsymbol{k}$ exhibits an ultraviolet divergence, one has $\langle H_c\rangle/V\to\infty$, *i.e.,* for a finite energy density all the particles result in being condensed, $n_0\to n$. As discussed above, to regularize the divergences of the integrals, one usually introduces an ultraviolet frequency cut-off, k_c. Then for a finite number of degrees of freedom, a finite value of the critical energy appears for condensation, and the condensation process turns out to depend critically on the energy density [5].

3.5. Nonequilibrium stationary solutions

Precisely, we investigate the existence of Zakharov's nonequilibrium stationary solutions to the coupled kinetic equations (22) and (23). For this purpose, let us assume that the spectrum exhibits a power-law dependence, $n_k=Ck^{-2z}$ [7,16]. Introducing this expression

[3] If one considers $\tilde{\mathcal{S}}(t)=\mathcal{S}(t)+\frac{1}{V}\ln n_0$ then the law of entropy growth is no longer satisfied. Although the sign of the first term in $\mathrm{d}\tilde{\mathcal{S}}/\mathrm{d}t=\mathrm{d}\mathcal{S}/\mathrm{d}t+\frac{1}{V}\frac{\partial_t n_0}{n_0}$ is positive, the sign of the second one, given by the *r.h.s.* of (22), is not well defined.

into the collision terms of Eqs. (22) and (23) and performing the Zakharov's transformation for Eq. (23), one obtains

$$\mathcal{C}oll_0 = \alpha^2 n_0 C^2 I^{(0)}_{z;D} \frac{1}{k_0^{2(2z+1-D)}}, \tag{29}$$

$$\mathcal{C}oll_3 = \alpha^2 C^3 I^{(3)}_{z;D} \frac{1}{k^{2(3z+D-1)}} + \alpha^2 n_0 C^2 I^{(2)}_{z;D} \frac{1}{k^{4z+2-D}}, \tag{30}$$

where $\mathcal{C}oll_0$ refers to the *r.h.s.* of (22) and $\mathcal{C}oll_3$ to the *r.h.s.* of (23). The integral $\mathcal{C}oll_0$ always diverges when the spectrum exhibits a power-law behavior. We thus need to introduce an infrared cut-off k_0 to prevent such a divergence. As an example, one obtains for $D = 3$,

$$I^{(0)}_{z;D=3} = \frac{-\sqrt{\pi} + 4^{z-1}\,\Gamma(2-z)\,\Gamma(z-1/2)}{2\sqrt{\pi}\ (z-1)^2}$$

where $\Gamma(\cdot)$ denotes the Gamma function. Note that the other integrals $I^{(2)}_{z;D}$, $I^{(3)}_{z;D}$ and their value and convergence only depend on z and on the space dimension D.

The explicit case of $D = 3$ is of interest. The first term in Eq. (30), $\alpha^2 C^3 I^{(3)}_{z;D}\, k^{-2(3z+D-1)}$, refers to the usual collision term of the four-wave kinetic equation of the NLS equation, *i.e.*, Eq. (3) [or Eq. (23) with $n_0 = 0$]. An extensive analysis of this term was given by Dyachenko et al. in Ref. [1]. The integrals for $I^{(3)}_{z;3}$ become a cumbersome expression involving special functions, however it could be easily computed numerically (see [34]) and it is shown to be formally convergent for $1 < z < 3/2$.

The collisional integral

$$I^{(2)}_{z;D=3} = \frac{1}{2\ (z-1)} \left(8 + \frac{4^z\,\sqrt{\pi}\,\Gamma(2-z)\ (\sec(\pi z) - 1)}{\Gamma(\frac{3}{2} - z)} \right)$$

is a convergent expression if $1/2 < z < 3$. Actually, it is possible to prove that, for the collision term involving $I^{(2)}_{z;D}$, the locality of the interaction is guaranteed for $1/2 < z < 3$ and $1/4 < z < 5/2$ for $D = 3$ and $D = 2$ respectively. Moreover, $I^{(2)}_{z;3}$ vanishes exactly for $z = 1\,\&\,z = 3/2$, so that in three space dimensions $D = 3$, $n_k = Ck^{-2z}$ is an exact solution of the right-hand side of (30) only if $z = 1$ or $z = 3/2$. These power-laws solutions refer, respectively, to the Rayleigh–Jeans equilibrium distribution and to the Zakharov–Kolmogorov (ZK) spectrum, which are solutions of the usual four-wave kinetic equation (3). Such a ZK spectrum then refers to a stationary nonequilibrium solution with a constant flux of energy from the large-scales towards the small-scales (*i.e.*, from small to large k). The flux of energy P associated to this nonequilibrium solution is given by the following expression [7]:

$$P = \left(C^3 \frac{\alpha^2}{3} \frac{\mathrm{d}I^{(3)}_{z;D}}{\mathrm{d}z} + n_0 C^2 \frac{\alpha^2}{2} \frac{\mathrm{d}I^{(2)}_{z;D}}{\mathrm{d}z} \right)_{z=3/2}. \tag{31}$$

As an example, one obtains in 3D, $\frac{\mathrm{d}I^{(2)}_{z;3}}{\mathrm{d}z}|_{z=3/2} = 8\pi(\pi - 4\log 2)$. From this equation it is easy to see that, for a given flux of energy P, *the constant C depends on the condensate amplitude* n_0.

Finally, let us remark that the integral $I^{(0)}_{z;3}$ is always positive for $z > 1$. It results from Eq. (29) that, for such a nonequilibrium ZK spectrum, the condensate amplitude increases with time, $\partial_t n_0 \geq 0$. One should note, however, that a stationary non-equilibrium solution requires n_0 to be constant. To maintain such a stationary non-equilibrium solution one should thus introduce an artificial sink at $k = 0$, so as to compensate for the increase of n_0. Then despite the fact that a KZ spectrum $n_k = Ck^{-2z}$ (with $z = 7/6$ or $z = 3/2$) is not, strictly speaking, a stationary solution of the isolated wave system, the previous discussion reveals that the nonequilibrium formation of the condensate may be regarded as an inverse cascade of wave action from small scales to large scales and a direct cascade of energy towards small scale fluctuations of the field.

3.6. The growth of a condensate

The coupled kinetic equations (22) and (23) do not provide a complete answer to the question regarding the formation of a condensate. Indeed, Eq. (22) seems to indicate that if the condensate amplitude vanishes initially [$n_0(t = 0) = 0$], it will vanish at any time. However this is not the case, because if $n_{\boldsymbol{k}}(t)$ exhibits the singular behavior given in Eq. (13), then the collision term in Eq. (22) becomes singular at $t = t_*$. Actually, at the singularity one has $dn_0/dt \equiv n_0\sigma/(t - t_*)$, with $\sigma = \frac{3/2-\nu}{2(\nu-1)}$ a positive constant (ν is the same as in Eq. (13)), so that the general solution of Eq. (22) reads

$$n_0(t) \sim (t - t_*)^{\sigma}. \tag{32}$$

Accordingly, the temporal evolution of the condensate amplitude exhibits a power-law behavior, even if it is assumed to vanish initially. However, let emphasize that this argument must be considered as purely qualitative. Indeed, the scaling properties of Eqs. (22) and (23) are the same than those of the standard four-wave kinetic equation (3), and both kinetic equations (3) and (22) are not uniformly valid in wave-number space. As discussed above in Section 2, the coupled kinetic equations (22) and (23) are thus expected to break down for the same k_{NL}. We shall come back to this point in the next section.

4. Weak-turbulence theory with a high amplitude condensate: The Bogoluibov's regime

In the presence of a small energy density, the weak-turbulence theory presented above only describes a transient evolution of the field, because in the long-time limit the condensate fraction at equilibrium may not be small, *i.e.*, $n_0/n \to 1$. In this regime of high-condensate amplitude, the diagonal terms of the second-order cumulant ($Q^{(2)++}, Q^{(2)--}$) no longer vanish, a peculiar feature which signals a breakdown of the usual weak-turbulence theory. Note that this breakdown seems to be of different origin than that discussed by Newell et al. in Refs. [8,23,24], in that it relies on the emergence of a condensate in the random field. Numerical evidence of this property will be illustrated in Section 5, Fig. 5. In the following we shall see that the diagonal second-order cumulants are relevant and have to be taken into account in the analysis.

4.1. Cumulants and hierarchy in the Bogoliubov's basis

The fact that the diagonal terms of the second-order cumulant do not vanish means that the function ψ should be recast in more appropriate "normal" variables. This becomes apparent from the linear part of the second-order cumulant in Eq. (19): the linear terms in $Q^{(2)l_1l_2}(\boldsymbol{k}_2)$ exhibit a non-diagonal coupling through the term $\mathrm{i}a(2\pi)^D\mathcal{P}_{1'2'}l_2Q^{(1)l_2}Q^{(1)l_2}Q^{(2)l_1-l_2}(\boldsymbol{p}_2)$.[4]

The novel "normal" variables can be conveniently defined by finding a linear transformation of every n-th order cumulant $Q^{(n)l_1l_2\cdots l_n}$ into a new set $Q_B^{(n)l_1l_2\cdots l_n}$. This transformation is actually the same than that used by Bogoliubov in the context of a weakly interacting quantum Bose gas [35]. Before proceeding, it is important to note that, whenever the condensate fraction n_0/n is close to unity, *i.e.* $n_0 \equiv (2\pi)^DQ^{(1)s}Q^{(1)-s} \lesssim n$, its value does not depend on time up to first order in ϵ^2. *This fact is of fundamental importance because* $(2\pi)^DQ^{(1)s}Q^{(1)-s}$ *enters explicitly into the transformation that we shall introduce below.*

The dynamics of $Q^{(1)s}$ is given by (17) and we shall assume that there exists a formal expansion of $Q^{(1)s}$ in the form

$$Q^{(1)s} = Q_0^{(1)s} + \epsilon^2 Q_2^{(1)s} + \epsilon^3 Q_3^{(1)s} + \cdots,$$

while for higher-order cumulants ($n > 2$)

$$Q^{(n)l_1\ldots l_n} = \epsilon^n\left(Q_0^{(n)l_1\ldots l_n} + \epsilon Q_1^{(n)l_1\ldots l_n} + \epsilon^2 Q_2^{(n)l_1\ldots l_n} + \cdots\right).$$

Under these assumptions one obtains, up to zero order in ϵ,

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_0^{(1)l_1} = -ial_1(2\pi)^D Q_0^{(1)-l_1}Q_0^{(1)l_1}Q_0^{(1)l_1}.$$

This equation exhibits a pure oscillatory temporal solution, because $\frac{\mathrm{d}}{\mathrm{d}t}\left(Q_0^{(1)-l_1}Q_0^{(1)l_1}\right) = 0$, which gives

$$Q_0^{(1)l_1}(t) = \left(Q_0^{(1)-l_1}Q_0^{(1)l_1}\right)^{1/2}\mathrm{e}^{-\mathrm{i}al_1(2\pi)^D\left(Q_0^{(1)-l_1}Q_0^{(1)l_1}\right)t}.$$

Keeping in mind that up to first order in ϵ the condensate amplitude is constant in time, we may now look for the required transformation. Rather than using the plane-wave basis [*i.e.*, the Fourier's amplitudes $A_{\boldsymbol{k}}^s$ given by Eq. (14)], we transform the wave-function into the quasi-particle basis of the Bogoliubov's theory. Let us define the amplitude $B_{\boldsymbol{k}}^s$ via the following linear transformation:

$$\begin{aligned}
&A_{\boldsymbol{k}}^s = u_k^s B_{\boldsymbol{k}}^s + v_k^s B_{\boldsymbol{k}}^{-s} \quad \text{or} \\
&B_{\boldsymbol{k}}^s = u_k^{-s} A_{\boldsymbol{k}}^s - v_k^s A_{\boldsymbol{k}}^{-s}, \quad \text{where} \\
&u_k^s = \frac{(2\pi)^{D/2}Q_0^{(1)s}}{\sqrt{n_0}}u_k \quad \& \quad v_k^s = \frac{(2\pi)^{D/2}Q_0^{(1)s}}{\sqrt{n_0}}L_k u_k.
\end{aligned} \tag{33}$$

Note that we have assumed that the function u_k^s and v_k^s do not depend explicitly on the direction of the wave-number $\boldsymbol{k}$, but only on its modulus $k = |\boldsymbol{k}|$. The functions u_k and L_k read

$$u_k = \frac{1}{\sqrt{1 - L_k^2}} \quad \& \quad L_k = \frac{-k^2 - an_0 + \omega_B(k)}{an_0},$$

[4] Another way to see the requirement of a different basis comes from the linear terms of Eq. (16). We are interested in the situation where the Fourier's amplitude associated to the zero wave-number is the largest one. It is thus desirable to split Eq. (16) into the zero mode and the non-zero modes. In order to clarify their roles in the dynamical equations for the Fourier's amplitudes, we shall consider a discrete version of Eq. (16). Let us take $\psi(\boldsymbol{x}, t) = \frac{1}{\sqrt{V}}\sum_{\boldsymbol{k}} A_{\boldsymbol{k}}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ then

$$\mathrm{i}\partial_t A_{\boldsymbol{k}}, =, \omega_{\boldsymbol{k}}A_{\boldsymbol{k}} + \frac{a}{V}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3} A_{\boldsymbol{k}_1}A_{\boldsymbol{k}_2}A_{\boldsymbol{k}_3}^*\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{k}_3+\boldsymbol{k}}$$

where $\delta_{\alpha,\alpha'}$ refers to the Kronecker-δ symbol (that takes the value 1 if $\alpha = \alpha'$, and 0 otherwise). The sum with the cubic term provides different kinds of contributions depending on the role of the zero mode $A_{\boldsymbol{k}=0}$. In particular, considering the case where two wave-numbers are identically zero, we have the following combinations: $\boldsymbol{k}_1 = \boldsymbol{k}_2 = 0$ and $\boldsymbol{k}_3 = -\boldsymbol{k}$, $\boldsymbol{k}_1 = \boldsymbol{k}_3 = 0$ and $\boldsymbol{k}_2 = \boldsymbol{k}$, and $\boldsymbol{k}_2 = \boldsymbol{k}_3 = 0$ and $\boldsymbol{k}_1 = \boldsymbol{k}$. This readily gives for the linear terms in $A_{\boldsymbol{k}}, A_{\boldsymbol{k}}^*$

$$\mathrm{i}\partial_t A_{\boldsymbol{k}} = \omega_{\boldsymbol{k}}A_{\boldsymbol{k}} + \frac{a}{V}(A_0^2A_{-\boldsymbol{k}}^* + 2|A_0|^2A_{\boldsymbol{k}}) + \textit{nonlinearities}.$$

It becomes apparent that the evolution is not diagonal, so that the normal modes are no longer the usual Fourier's basis ($A_{\boldsymbol{k}}, A_{\boldsymbol{k}}^*$).

with the Bogoliubov's dispersion relation

$$\omega_B(k) = \sqrt{k^4 + 2an_0k^2}, \tag{34}$$

where $n_0 \equiv (2\pi)^D Q_0^{(1)s} Q_0^{(1)-s}$ is a constant. This transformation is canonical, so that the Poisson brackets satisfy the following relations $\{A_{\boldsymbol{k}}, A^*_{\boldsymbol{k}}\} \equiv \{A^s_{\boldsymbol{k}}, A^{-s}_{-\boldsymbol{k}}\} = i = \{B_{\boldsymbol{k}}, B^*_{\boldsymbol{k}}\} \equiv \{B^s_{\boldsymbol{k}}, B^{-s}_{-\boldsymbol{k}}\}$.

From this linear transformation one may compute the cumulants in the Bogoliubov's basis in terms of the cumulants in the Fourier's basis (15):

$$\left\langle B^{l_1}_{\boldsymbol{p}_1}\right\rangle = Q_B^{(1)l_1}\delta^{(D)}(\boldsymbol{p}_1) \quad \text{with}$$

$$Q_B^{(1)l_1} = (u_{p_1}^{-l_1} Q^{(1)l_1} - v_{p_1}^{l_1} Q^{(1)-l_1}); \tag{35}$$

$$\left\langle B^{l_1}_{\boldsymbol{p}_1} B^{l_2}_{\boldsymbol{p}_2}\right\rangle = Q_B^{(2)l_1l_2}(\boldsymbol{p}_2)\delta^{(D)}(\boldsymbol{p}_1+\boldsymbol{p}_2) + Q_B^{(1)l_1}(t)Q_B^{(1)l_2}(t)\delta^{(D)}(\boldsymbol{p}_1)\delta^{(D)}(\boldsymbol{p}_2) \quad \text{with}$$

$$Q_B^{(2)l_1l_2}(\boldsymbol{p}_2) = \left(u_{p_1}^{-l_1}u_{p_2}^{-l_2}Q^{(2)l_1l_2}(\boldsymbol{p}_2) - u_{p_1}^{-l_1}v_{p_2}^{l_2}Q^{(2)l_1-l_2}(\boldsymbol{p}_2) - v_{p_1}^{l_1}u_{p_2}^{-l_2}Q^{(2)-l_1l_2}(\boldsymbol{p}_2) + v_{p_1}^{l_1}v_{p_2}^{l_2}Q^{(2)l_1l_2}(\boldsymbol{p}_2)\right); \quad etc. \tag{36}$$

Note that the first-order moment $\left\langle B^{l_1}_{\boldsymbol{p}_1}\right\rangle$ is ill-defined because $u_{p_1}^{-l_1}$ and $v_{p_1}^{l_1}$ diverge as $|\boldsymbol{p}_1| \to 0$. Let us inspect this divergence in the first-order cumulant in the Bogoliubov's basis:

$$Q_B^{(1)l_1} = \frac{\sqrt{n_0}}{(2\pi)^{D/2}}\frac{\sqrt{1-L_{p_1}}}{\sqrt{1+L_{p_1}}} + \epsilon^2(u_{p_1}^{-l_1}Q_2^{(1)l_1} - v_{p_1}^{l_1}Q_2^{(1)-l_1}), \tag{37}$$

with $1 + L_{p_1} \to 0$ if $|\boldsymbol{p}_1| \to 0$. This divergence will cause problems exclusively for the equation governing the evolution of the first-order cumulant, $Q_B^{(1)l_1}$. Indeed, we shall see that the equations for the higher-order cumulants are regular because such a divergence turns out to be canceled by the nonlinear terms. Furthermore, in the following we shall calculate the condensate fraction in the original basis, $Q^{(1)l_1}$; while the higher-order cumulants $Q_B^{(n)l_1\dots l_n}$ will be calculated in the Bogoliubov's basis.

One may express the original cumulants $Q^{(n)}$ as a function of the cumulants in the Bogoliubov's basis. Discarding the first-order cumulant, one has

$$Q^{(2)l_1l_2}(\boldsymbol{p}_2) = u^{l_1}_{\boldsymbol{p}_1}u^{l_2}_{\boldsymbol{p}_2}Q_B^{(2)l_1l_2}(\boldsymbol{p}_2) + u^{l_1}_{\boldsymbol{p}_1}v^{l_2}_{\boldsymbol{p}_2}Q_B^{(2)l_1-l_2}(\boldsymbol{p}_2) + v^{l_1}_{\boldsymbol{p}_1}u^{l_2}_{\boldsymbol{p}_2}Q_B^{(2)-l_1l_2}(\boldsymbol{p}_2) + v^{l_1}_{\boldsymbol{p}_1}v^{l_2}_{\boldsymbol{p}_2}Q_B^{(2)-l_1-l_2}(\boldsymbol{p}_2) \tag{38}$$

$$Q^{(3)l_1l_2l_3}(\boldsymbol{p}_2,\boldsymbol{p}_3) = u^{l_1}_{\boldsymbol{p}_1}u^{l_2}_{\boldsymbol{p}_2}u^{l_3}_{\boldsymbol{p}_3}Q_B^{(3)l_1l_2l_3}(\boldsymbol{p}_2,\boldsymbol{p}_3) + etc$$

where for $Q^{(2)l_1l_2}(\boldsymbol{p}_2)$ and $Q_B^{(2)l_1l_2}(\boldsymbol{p}_2)$ it should be imposed that $\boldsymbol{p}_1+\boldsymbol{p}_2 = 0$, for $Q^{(3)l_1l_2l_3}(\boldsymbol{p}_2,\boldsymbol{p}_3)$ and $Q^{(3)l_1l_2l_3}(\boldsymbol{p}_2,\boldsymbol{p}_3)$, $\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p}_3 = 0$, *etc*.

We shall now derive the equations for the cumulants in the Bogoliubov's basis. In practice, one may substitute the Bogoliubov's transformation (33) into the Fourier's transform of the NLS equation (16). This gives [5]:

$$\frac{\mathrm{d}B^s_{\boldsymbol{p}}}{\mathrm{d}t} + \mathrm{i}s(\omega(\boldsymbol{p}) - an_0)\left(\left(\left|u^s_{\boldsymbol{p}}\right|^2 + \left|v^s_{\boldsymbol{p}}\right|^2\right)B^s_{\boldsymbol{p}} + 2u^{-s}_{\boldsymbol{p}}v^s_{\boldsymbol{p}}B^{-s}_{\boldsymbol{p}}\right) = a(2\pi)^D\sum_{s_1s_2s_3}\int Lb^{ss_1s_2s_3}_{\boldsymbol{p}\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}B^{s_1}_{\boldsymbol{k}_1}B^{s_2}_{\boldsymbol{k}_2}B^{s_3}_{\boldsymbol{k}_3}\delta^{(D)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3-\boldsymbol{p})\mathrm{d}^D\boldsymbol{k}_{123} \tag{39}$$

where we define

$$\begin{aligned} Lb^{ss_1s_2s_3}_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3} = {}& is\frac{1}{24}u_ku_{k_1}u_{k_2}u_{k_3}\mathcal{P}_{123}\left[ss_1s_2s_3(L_k-1)(L_{k_1}-1)(L_{k_2}-1)(L_{k_3}-1) - ss_1(L_k-1)(L_{k_1}-1)(L_{k_2}+1)(L_{k_3}+1)\right.\\ &\left. +\ s_2s_3(L_k+1)(L_{k_1}+1)(L_{k_2}-1)(L_{k_3}-1) - (L_k+1)(L_{k_1}+1)(L_{k_2}+1)(L_{k_3}+1)\right]. \end{aligned}$$

Following the same procedure as that outlined in Section 3, one can derive an infinite hierarchy of cumulants' equations in the Bogoliubov's basis. Under the assumption of small fluctuations around the condensate one rescale $Q_B^{(n)} \to \epsilon^n Q_B^{(n)}$ for $n \ge 2$, and a straightforward calculation gives

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}Q_B^{(2)l_1l_2}(\boldsymbol{p}_2) + \mathrm{i}(l_1\omega_B(p_1) + l_2\omega_B(p_2))Q_B^{(2)l_1l_2}(\boldsymbol{p}_2)\\ &\quad = \epsilon^2 a(2\pi)^D\mathcal{P}_{1'2'}\sum_{s_1s_2s_3}\int Lb^{l_2s_1s_2s_3}_{\boldsymbol{p}_2\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}Q_B^{(4)l_1s_1s_2s_3}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\delta^{(D)}(\boldsymbol{p}_1+\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{123}\\ &\qquad + \epsilon a(2\pi)^D\mathcal{P}_{1'2'}\sum_{s_1s_2s_3}\int Lb^{l_2s_1s_2s_3}_{\boldsymbol{p}_2\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}\mathcal{P}_{123}\,Q_{B0}^{(1)s_1}(\boldsymbol{k}_1)Q_B^{(3)l_1s_2s_3}(\boldsymbol{k}_2,\boldsymbol{k}_3)\delta^{(D)}(\boldsymbol{k}_1)\delta^{(D)}(\boldsymbol{p}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{123}\\ &\qquad + 2\epsilon^2 a(2\pi)^D\mathcal{P}_{1'2'}\sum_{s_1s_2s_3}\int Lb^{l_2s_1s_2s_3}_{\boldsymbol{p}_2\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}\mathcal{P}_{123}\,Q_{B0}^{(1)s_1}(\boldsymbol{k}_1)Q_{B2}^{(1)s_2}(\boldsymbol{k}_2)Q_B^{(2)l_1s_3}(\boldsymbol{k}_3)\delta^{(D)}(\boldsymbol{k}_1)\delta^{(D)}(\boldsymbol{k}_2)\delta^{(D)}(\boldsymbol{p}_1+\boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{123}\\ &\qquad + \epsilon^2 a(2\pi)^D\mathcal{P}_{1'2'}\sum_{s_1s_2s_3}\int Lb^{l_2s_1s_2s_3}_{\boldsymbol{p}_2\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}\mathcal{P}_{123}\,Q_B^{(2)l_1s_1}(\boldsymbol{k}_1)Q_B^{(2)s_2s_3}(\boldsymbol{k}_3)\delta^{(D)}(\boldsymbol{p}_1+\boldsymbol{k}_1)\delta^{(D)}(\boldsymbol{k}_2+\boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{123} + \mathcal{O}(\epsilon^3) \end{aligned} \tag{40}$$

[5] Alternatively, one may derive the equations by calculating directly the time derivative $\frac{\mathrm{d}}{\mathrm{d}t}Q_B^{(2)l_1l_2}(\boldsymbol{p}_2) = \dots$ from Eq. (36) and making use of (19) to evaluate the time derivatives of the second-order cumulants in the original basis. In the equation for $\frac{\mathrm{d}}{\mathrm{d}t}Q_B^{(2)l_1l_2}(\boldsymbol{p}_2)$, one naturally gets from (36) terms involving the first-, second-, third-, and fourth-order cumulants in the original basis, and finally one should use the inverse of the transformation given in Eq. (38).

where $\boldsymbol{p}_1 + \boldsymbol{p}_2 = 0$ should be imposed and, from Eq. (37),

$$Q_{B0}^{(1)s_1} = \frac{\sqrt{n_0}}{(2\pi)^{D/2}} \frac{\sqrt{1-L_{k_1}}}{\sqrt{1+L_{k_1}}}, \quad \text{and} \quad Q_{B2}^{(1)s_1} = (u_{p_1}^{-s_1} Q_2^{(1)s_1} - v_{p_1}^{s_1} Q_2^{(1)-s_1}).$$

Note that $Q_{B0}^{(1)s_1}(\boldsymbol{k}_1)$ is always multiplied by a $\delta^{(D)}(\boldsymbol{k}_1)$, so that one recovers the divergence of $Q_{B0}^{(1)s_1}(\boldsymbol{k}_1)$ discussed above. This singularity is removed because of the pre-factor term $Lb_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}^{ss_1s_2s_3}$, which leads to a regular final expression. Following an analogous procedure, one can derive the equations governing the evolutions of all higher-order cumulants, $Q_B^{(n)}$ with $n > 2$.

4.2. Closure and kinetic equations

In a way similar as for the weak-turbulence in the presence of a small-condensate amplitude (Section 3), a natural closure of the hierarchy of the cumulants' equations in the Bogoliubov's basis may be achieved by making use of a multiscale perturbation scheme (see Appendix A.3). However, there is a major difference: the natural closure arises in this case only for the three-wave resonant interaction, instead of the four-wave resonant interaction considered in Section 3. This is due to the fact that in Section 3 the off-diagonal second-order cumulants and the condensate amplitude n_0 were of the same order, which thus led to kinetic equations mixing the three- and four-wave resonant processes. In the present case the condensate fraction is of order one, $n_0/n \sim \mathcal{O}(1)$, so that the new dynamical equations for the cumulants' are quadratic in the Bogoliubov's basis. In this way the three-wave interaction dominates the nonequilibrium kinetics. In other terms, the slow dynamical evolution is now obtained at second-order $\mathcal{O}(\epsilon^2)$ in the perturbation expansion procedure and the asymptotic closure only keeps terms involving the three-wave resonant process.

We refer the reader to the Appendix A.3 for the details of the derivation of the kinetic equation for the second-order cumulant in the Bogoliubov's basis. After a rather involved calculation, we obtained

$$\frac{\partial}{\partial t} n_B(\boldsymbol{p}_2) = 36\pi a^2 \epsilon^2 n_0 \sum_{s_2 s_3} \int \left| \tilde{Lb}_{\boldsymbol{p}_2\boldsymbol{k}_2\boldsymbol{k}_3}^{+s_2s_3} \right|^2 n_B(\boldsymbol{k}_2) n_B(\boldsymbol{k}_3) n_B(\boldsymbol{p}_2) \\ \times \left(\frac{1}{n_B(\boldsymbol{p}_2)} - \frac{s_2}{n_B(\boldsymbol{k}_2)} - \frac{s_3}{n_B(\boldsymbol{k}_3)} \right) \delta(s_2\omega_B(k_2) + s_3\omega_B(k_3) - \omega_B(p_2)) \delta^{(D)}(\boldsymbol{k}_2 + \boldsymbol{k}_3 - \boldsymbol{p}_2) \mathrm{d}^D \boldsymbol{k}_{23} \tag{41}$$

with $n_B(\boldsymbol{p}_2) = (2\pi)^D Q_{B0}^{(2)-ss}(\boldsymbol{p}_2)$ and

$$\tilde{Lb}_{\boldsymbol{k}\boldsymbol{k}_2\boldsymbol{k}_3}^{s_1s_2s_3} = \frac{is_1}{12} u_{k_1} u_{k_2} u_{k_3} [s_2 s_3 (L_{k_1}+1)(L_{k_2}-1)(L_{k_3}-1) - 3(L_{k_1}+1)(L_{k_2}+1)(L_{k_3}+1) - \mathcal{P}_{23} s_1 s_2 (L_{k_1}-1)(L_{k_2}-1)(L_{k_3}+1)]. \tag{42}$$

Note that, for simplicity, in the above derivation we have assumed reflexion symmetry: $Q_{B0}^{(2)-ss}(\boldsymbol{p}_2) = Q_{B0}^{(2)-ss}(-\boldsymbol{p}_2)$. A similar equation as (42) was first derived by Dyachenko et al. [1] under the assumption of a large and constant condensate fraction. In the theory developed here, it naturally appears that the condensate fraction remains constant up to order $\mathcal{O}(\epsilon^4)$, which means that n_0 is constant for times $t \ll \epsilon^{-4}$. In the long wavelength limit, *i.e.*, the limit where the system is non-dispersive [$\omega_B(k) = c_s k$ with $c_s = \sqrt{2an_0}$], Eq. (41) exhibits the usual Rayleigh–Jeans equilibrium distribution and a nonequilibrium ZK spectrum, as discussed in details by Dyachenko et al. [1].

Up to order ϵ^4 a similar analysis allows us to derive a set of coupled kinetic equations governing the coupled evolution of the condensate amplitude n_0 and the incoherent component $n_B(\boldsymbol{p}, t)$ [see Eq. (41)]. The equation for the condensate amplitude reads (see Appendix A.3)

$$\frac{\mathrm{d}}{\mathrm{d}t} n_0 = 6\pi a^2 \epsilon^4 n_0 \mathcal{P}_{123} \sum_{s_1 s_2 s_3} \int \tilde{Lv}_{\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}^{s_1s_2s_3} \tilde{Lb}_{\boldsymbol{k}_3-\boldsymbol{k}_1-\boldsymbol{k}_2}^{s_3-s_1-s_2} n_B(\boldsymbol{k}_1) n_B(\boldsymbol{k}_2) \delta(s_1\omega_B(k_1) + s_2\omega_B(k_2) + s_3\omega_B(k_3)) \delta^{(D)}(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3) \mathrm{d}^D \boldsymbol{k}_{123} \tag{43}$$

where a new scattering amplitude is introduced

$$\tilde{Lv}_{\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}^{s_1s_2s_3} = \mathrm{i}\frac{1}{12} u_{k_1} u_{k_2} u_{k_3} \mathcal{P}_{123} \left[s_1 (L_{k_1}-1) \left((L_{k_2}+1)(L_{k_3}+1) - s_2 s_3 (L_{k_2}-1)(L_{k_3}-1) \right) \right]. \tag{44}$$

Moreover, as previously discussed through Section 3, the total energy may be expressed in terms of the Bogoliubov's cumulants, in a way similar to Eq. (58) in the Appendix A.1. In the present case one obtains

$$\langle H \rangle / V = \int \omega_B(k) n_B(\boldsymbol{k}) \mathrm{d}\boldsymbol{k} + a\frac{n^2}{2} + \frac{a}{2}(n - n_0)^2. \tag{45}$$

The first term on the *r.h.s.* of Eq. (45) refers to the quadratic[6] contribution to the total energy, $E_B/V = \int \omega_B(k) n_B(\boldsymbol{k}) \mathrm{d}^D \boldsymbol{k}$ and it is formally conserved, up to $\mathcal{O}(\epsilon^5)$, by the set of coupled equations (41) and (43). On the other hand, taking into account the rescaling procedure outlined above, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t} n_0 + \epsilon^2 \frac{\mathrm{d}}{\mathrm{d}t} \int \frac{k^2 + an_0}{\omega_B(k)} n_B(\boldsymbol{p}) \mathrm{d}^D \boldsymbol{p} = 0.$$

The total number of particles is thus conserved up to $\mathcal{O}(\epsilon^4)$, and presumably is conserved up to order $\mathcal{O}(\epsilon^5)$.

Let us finally remark that, as for the wave turbulence theory in the presence of a small condensate amplitude (Section 3), the diagonal and higher-order cumulants are shown to vanish in the long time limit, $t \to \infty$, thus confirming that all the relevant statistical information is contained in the off-diagonal second-order cumulants in the Bogoliubov's basis.

[6] Let us note that, contrary to the kinetic energy E that has a purely linear origin, the kinetic energy E_B refers to the quadratic part of the total energy H expanded in the Bogoiubovs basis, which also includes nonlinear energy contributions.

4.3. Equilibrium distribution and condensation curve

We thus have shown that an asymptotic closure of the hierarchy of cumulants' equations is achieved in the framework of the Bogoliubov's normal variables. The diagonal terms of the second-order cumulants are shown to vanish asymptotically in the Bogoliubov's basis, $Q_B^{(2)++}(\boldsymbol{k})$, $Q_B^{(2)--}(\boldsymbol{k}) \to 0$, while the evolution of the off-diagonal cumulants exhibit an irreversible evolution to equilibrium. This may be shown by remarking that Eq. (41) exhibits a H-theorem of entropy growth, $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S} = \frac{\mathrm{d}}{\mathrm{d}t}\int \log[n_B((\boldsymbol{k}))]\mathrm{d}^D\boldsymbol{k} \geq 0$. Given the constraint on the conservation of the kinetic energy in Eq. (41), $E_B/V = \int \omega_B(k) n_B(\boldsymbol{k})\mathrm{d}^D\boldsymbol{k}$, the principle of maximum entropy provides the equilibrium distribution for the off-diagonal second-order cumulants,

$$(2\pi)^D q_{B0}^{(2)+-}(\boldsymbol{k}) = (2\pi)^D Q_{B0}^{(2)+-}(\boldsymbol{k}) = n_B(\boldsymbol{k}) = \frac{T}{\omega_B(k)}.$$

This equilibrium distribution refers to the Rayleigh–Jeans distribution, in which the quadratic dispersion relation $\omega(k) = k^2$ is substituted by the Bogoliubov's dispersion relation.

In order to compare the theory with the numerical simulations, let us calculate the second-order cumulants in the original Fourier's plane-wave basis. For this purpose, we shall make use of the transformation given in Eq. (38), which relates the second-order cumulants in the two basis. In the long time limit, they take the following expressions

$$(2\pi)^D Q^{(2)+-}(\boldsymbol{k}) \to T\frac{k^2 + an_0}{\omega_B(k)^2} \quad \& \quad (2\pi)^D |Q^{(2)++}(\boldsymbol{k})| = (2\pi)^D |Q^{(2)--}(\boldsymbol{k})| \to T\frac{an_0}{\omega_B(k)^2}, \tag{46}$$

where we have made use of the following identities: $|u_k|^2 + |v_k|^2 = (k^2 + an_0)/[2\omega_B(k)]$ and $2u_k v_k = n_0/\omega_B(k)$.

One thus obtains the following expressions for the densities of the particles and of the energy,

$$n = n_0 + \frac{T}{V}\int \frac{k^2 + an_0}{\omega_B(k)^2}\mathrm{d}^D\boldsymbol{k}, \tag{47}$$

$$\langle H \rangle /V = a\frac{n^2}{2} + \frac{a}{2}(n - n_0)^2 + \frac{T}{V}\int \mathrm{d}^D\boldsymbol{k}. \tag{48}$$

4.4. Breakdown of weak turbulence in the Bogoliubov regime

The kinetic equation (41) and the whole asymptotic theory has a certain domain of validity. Following the same procedure as in Section 2.4, one may evaluate the breakdown condition for (41):

$$\omega_B^{NL}(k) \sim \frac{\frac{\partial}{\partial t} n_B(k)}{n_B(k)} \sim a^2 n_0 \left|\tilde{Lb}(k)\right|^2 n_B(k)\frac{k^D}{\omega_B(k)},$$

where we made use of the following approximations $\omega_B(k) \sim \sqrt{2an_0}k$, $\tilde{Lb}(k) \sim (an_0)^{1/4}k^{-1/2}$. Considering the equilibrium distribution $n_B(k) = T/\omega_B(k)$ the breakdown criterion for the weak-turbulence in the Bogoliubov's regime reads:

$$\frac{\omega_B^{NL}(k)}{\omega_B(k)} \sim aTk^{D-4}. \tag{49}$$

Therefore, as for in the usual kinetic equation (3), weak turbulence theory in the Bogoliubov's regime breaks down in the long wavelength limit ($k \to 0$) if $D < 4$. It is surprising to note that the condensate fraction n_0 does not enter in the above breakdown criterion. This contrasts with the kinetic equation (23) of Section 3 derived in the limit of a small condensate amplitude. The kinetic equation (23) breaks down if $a^2T^2k^{2(D-4)} + a^2 n_0 T k^{D-6} \ll 1$ which thus extends the infrared divergence up to $D = 6$.

5. Numerical simulations of wave turbulence and wave condensation

To numerically integrate the NLS equation (1) one has to resort to a spatial discretization of the equation. The discretized field then exhibits a finite number of degrees of freedom, and the Rayleigh–Jeans distribution (25) may be reached without the trouble of the ultraviolet divergence discussed above. In the following we shall see that a good agreement is obtained between the numerical simulations and the wave turbulence theory reported in Sections 3 and 4, despite the fact that the theory has been developed for continuous integrals and an infinite number of modes.

5.1. Long time behavior of the nonlinear Schrödinger equation

Before entering into the details, let us briefly illustrate the condensation process by means of a direct numerical integration of the NLS equation (1). In our simulations we start with a nonequilibrium distribution $\psi(\boldsymbol{x}, t = 0) = \sum_k A_{\boldsymbol{k}}(t = 0)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})$, where the phases of the complex amplitudes $A_{\boldsymbol{k}}$ are distributed randomly [3,5]. In spite of the formal reversibility of the NLS equation (1), the nonlinear wave ψ exhibits an irreversible evolution towards thermal equilibrium. This is illustrated in Fig. 2(a)–(b), where the formation of the condensate is characterized by a process of entropy production, consistently with the H-theorem of entropy growth inherent to the weak-turbulence theory [see [7] and Eqs. (22) and (23)]. The condensate fraction n_0/n and the entropy $\mathcal{S}$ both saturate to a constant value at equilibrium ($n_0^{eq}/n \simeq 0.13$ for $\langle H \rangle /V = 5.6527$), as shown in Figs. 2(a) & (b). Let us remark that the kinetic contribution to the energy,

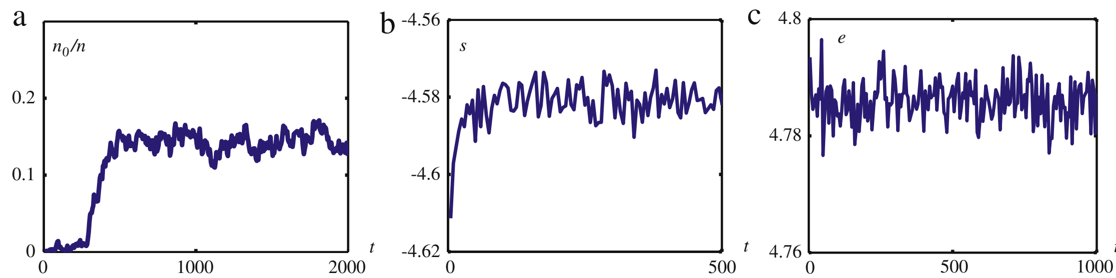

Fig. 2. Numerical simulations of the NLS equation (1) with periodic boundary conditions in a box $V = 32^3$ ($a = 1$, $D = 3$, and a mesh size $dx = 1$, $k_c = \pi$). The particle density is $n \equiv N/V = 1$ and the energy density is $H/V = 5.6527$. (a) The (non-averaged) condensate fraction n_0/n as a function of the dimensionless time. (b) The process of entropy growth saturates at equilibrium, $s \equiv \mathcal{S}/V = \frac{1}{V}\sum_{\mathbf{k}} \log|A_{\mathbf{k}}|^2$. (c) Temporal evolutions of the density of kinetic energy $e \equiv E/V = \frac{1}{V}\sum_{\mathbf{k}} k^2|A_{\mathbf{k}}|^2$.

$E = \int |\nabla\psi|^2 d^D\mathbf{x}$, keeps an almost constant value. This is consistent with the properties of the kinetic equations (22) and (23), which were shown to conserve the kinetic energy E. Note that in this numerical simulation the total energy [see Eq. (2)] is conserved up to 10^{-5}.

Let us remark that, for lower values of the total energy H, the condensation process is characterized by an increase of kinetic energy E and a reduction of nonlinear energy U, in agreement with the intuitive explanation of the condensation process given in the Introduction section. Note that in this case the kinetic energy E is no longer a conserved quantity. This does not contradict the kinetic theory, since we have shown that in the regime of a high-amplitude condensate the kinetic energy E_B has to be expressed in the Bogoliubov's basis, and the corresponding kinetic equation (41) and (43) conserves E_B.

5.2. Condensation curve in 3D

In order to compare the theory with the numerical simulations of the NLS equation, we shall replace the continuous integrals by discrete sums ($V\int d^Dk \to \sum_{\mathbf{k}}$), a feature that was discussed in detail in Ref. [5]. Then following the procedure outlined in Ref. [5], one may substitute the expression of the temperature in (26) into the expression of the energy (27). One obtains the following equation that expresses the condensed particles n_0 vs the energy density $\langle H\rangle/V$, *i.e.*, the condensation curve:

$$\frac{\langle H\rangle}{V} = (n - n_0)\frac{\sum'_{\mathbf{k}} 1}{\sum'_{\mathbf{k}} \frac{1}{k^2}} + a\left(n^2 - \frac{1}{2}n_0^2\right), \tag{50}$$

where $\sum'_{\mathbf{k}}$ denotes the sum over the whole frequency space excluding the mode $\mathbf{k} = 0$ ($n_0 \equiv N_0/V$, $n \equiv N/V$). As discussed above through Eq. (28), if the sum is extended over unbounded wave-number k's, the quotient $\sum'_{\mathbf{k}} 1/\sum'_{\mathbf{k}} \frac{1}{k^2}$ diverges. On the other hand, if one assumes the existence of an ultraviolet cut-off k_c, then Eq. (50) defines a critical energy

$$\frac{H_c}{V} = n\frac{\sum'_{\mathbf{k}} 1}{\sum'_{\mathbf{k}} \frac{1}{k^2}} + an^2, \tag{51}$$

above which the condensate amplitude is zero ($n_0 = 0$ for $H > H_c$), *i.e.*, no wave condensation occurs. Conversely, for an energy smaller than H_c, Eq. (50) reveals that the fraction of condensed particles increases linearly as $n_0 \sim (H_c - H)/V$.

The condensation curve (50) is plotted in Fig. 3, together with a comparison with the numerical simulations. Each point ($\diamond$) in Fig. 3 was obtained by averaging the condensate amplitude $n_0(t)$ over 3000 time units, once the equilibrium state was reached, *i.e.*, once the entropy growth saturates to a constant value consistently with the H-theorem ($\partial_t\mathcal{S} \simeq 0$). First of all we remark that a good agreement between the numerics and Eq. (50) is obtained for relatively small values of the condensate amplitude (typically $n_0/n < 0.3$). As shown in Section 4, for higher values of the condensate amplitude, the weak-turbulence theory must be developed in the Bogoliubov's basis. In this case the condensation curve follows from Eqs. (47) and (48):

$$\langle H\rangle/V = a\frac{n^2}{2} + \frac{a}{2}(n - n_0)^2 + (n - n_0)\frac{\sum'_{\mathbf{k}} 1}{\sum'_{\mathbf{k}} \frac{k^2 + an_0}{k^4 + 2an_0k^2}}. \tag{52}$$

Finally, let us remark in Fig. 3(c) that, when the particle density N/V becomes greater than the unity, a significant discrepancy appears between the numerical results and Eq. (50), a feature which is consistent with the assumption of small-amplitude waves that underlies the weak-turbulence theory presented in Section 3. Conversely, a quantitative agreement between Eq. (52) and the numerics is obtained regardless of the density N/V, a feature which is consistent with the weak-turbulence theory in the presence of a high condensate amplitude (see Section 4).

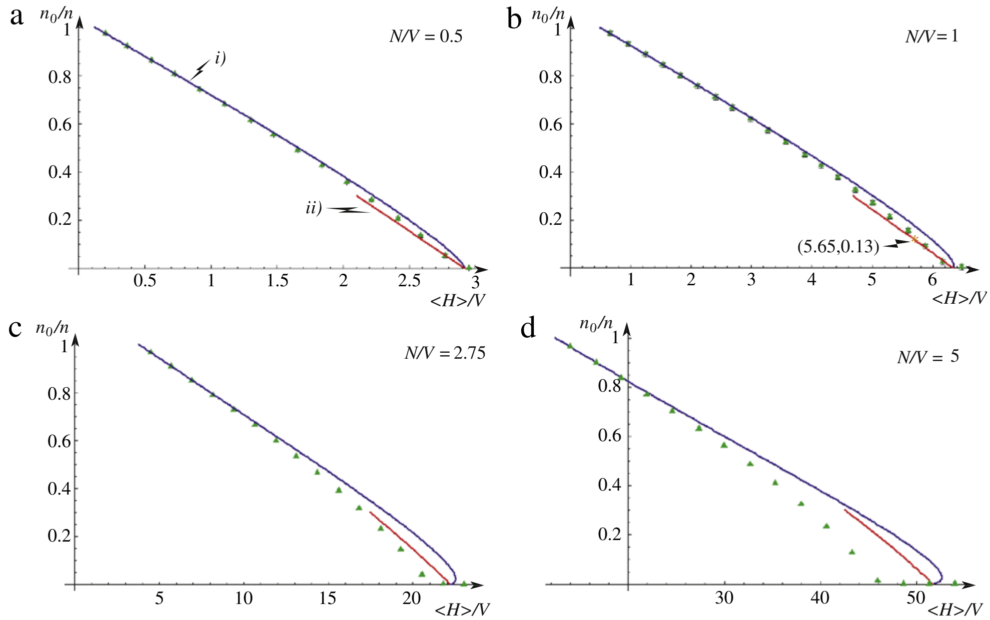

Fig. 3. The condensate fraction n_0/n as function of the energy density $\langle H\rangle/V$ for four different particle densities: (a) $N/V = 1/2$, (b) $N/V = 1$, (c) $N/V = 2.75$ and $N/V = 5$. The points (▲) refer to the results of the numerical simulation in a $V = 32^3$ box with periodic boundary conditions and a mesh size $dx = 1$ ($a = 1, D = 3, k_c = \pi$). The star point (*) in (b) refers to the numerical simulation reported in Fig. 2, that is $\langle H\rangle/V = 5.6527$ and $n_0/n = 0.13$. The theoretical curves (i) and (ii) (explicitly indicated in (a)) are plotted from Eqs. (52) and (50) respectively *i.e.*, from the weak-turbulence theory in the presence of a high and a small condensate amplitude. Let us remark a notable discrepancy between the simulations and the theory in (c) and (d), consistent with the fact that the assumption of small-amplitude waves is not satisfied for (c)–(d). Also note that a quantitative agreement between Eq. (52) and the numerics is obtained regardless of the density N/V, a feature which is consistent with the weak-turbulence theory in the presence of a high condensate amplitude (see Section 4).

5.3. Finite size effects and wave condensation in 2D

Let us comment the process of wave condensation in two spatial dimensions. It was shown in Ref. [5] that wave condensation does not take place in 2D in the thermodynamic limit. Indeed, in analogy with an ideal and uniform 2D Bose gas, it was shown that the chemical potential of the random field only reaches zero for a vanishing temperature in 2D. This result is rigorously correct in the thermodynamic limit, *i.e.*, $V \to \infty$, $N \to \infty$, keeping N/V constant (we recall that the 'volume' V refers to the system area in 2D). However, for situations of practical interest in which N and V are finite, wave condensation is reestablished in 2D, consistently with the numerical study reported in Ref. [22].

First of all let us note that, in order to derive the condensation curve given in Eqs. (50) and (52), no specific assumptions were made regarding the spatial dimensionality of the system. Then Eq. (50) is also valid for the two-dimensional case, as well as the critical energy for condensation (51), where $\sum_{\boldsymbol{k}}$ refers to the sum in the 2D wave-number space. Now one should consider that, in the thermodynamic limit ($V \to \infty$) the wave-number discretization ($dk = 2\pi/\sqrt{V}$) tends to zero, so that the discrete sums may be replaced by continuous integrals. Because of the *infrared* logarithmic divergence of the integral $\int d^2\boldsymbol{k}/k^2 \approx 2\pi \log(\sqrt{V}/2)$, the critical energy H_c/V tends to the its minimum value an^2, and wave condensation no longer take place in 2D.

However, in complete analogy with quantum Bose–Einstein condensation, for a finite volume V, wave condensation occurs for a non-vanishing value of the chemical potential, $\mu \neq 0$. In this respect, the relevant equilibrium distribution that should be considered reads:

$$n_{\boldsymbol{k}}^{eq} = \frac{T}{k^2 - \mu}. \tag{53}$$

Note that the equilibrium distribution given in Eq. (53) does not vanish the collision terms in the kinetic equations (22) and (23). Strictly speaking, the calculations presented here below should thus be considered as 'heuristic', although they are remarkably well validated by the numerical simulations (see Fig. 4).

According to Eq. (53), the number of particles and the energy satisfy the following equations

$$N = T \sum_{\boldsymbol{k}} \frac{1}{k^2 - \mu}, \tag{54}$$

$$H = T \sum_{\boldsymbol{k}}{}' \frac{k^2}{k^2 - \mu} + a\left(\frac{N^2}{V} - \frac{N_0^2}{2V}\right). \tag{55}$$

Let us now analyze the number of condensed particles in $\boldsymbol{k} = 0$ by decomposing the sum as follows, $N = N_0 + T\sum_{\boldsymbol{k}}' 1/[k^2 - \mu]$, where $N_0 = T/(-\mu)$. One may then substitute the expression of the temperature given in Eq. (54) into Eq. (55) to get the following coupled set

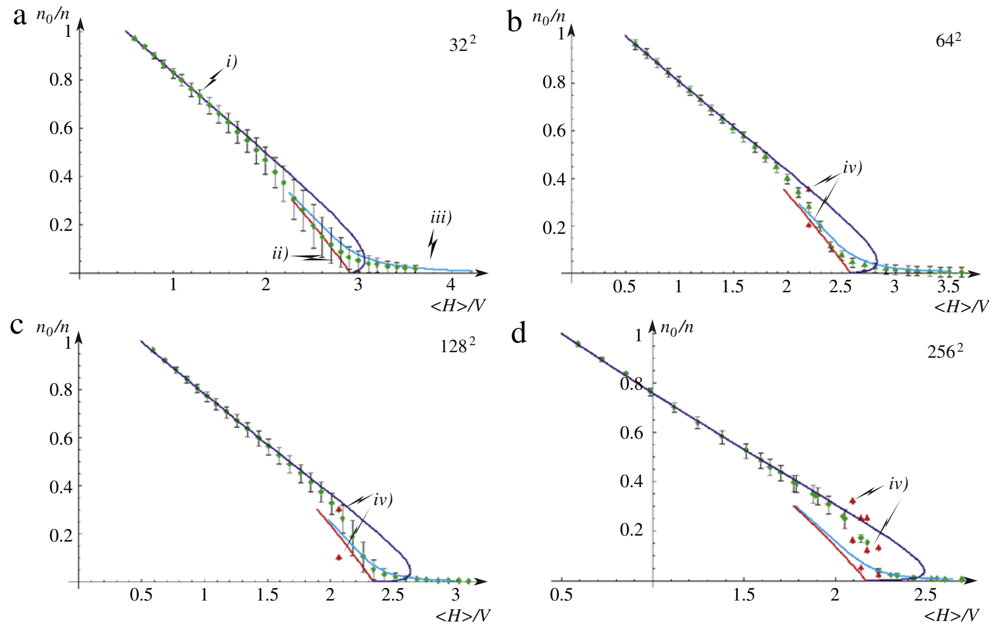

Fig. 4. The condensate fraction n_0/n vs $\langle H\rangle/V$ for four different system sizes: (a) $V = 32^2$, (b) $V = 64^2$, (c) $V = 128^2$, (d) $V = 256^2$. The points (♦) refer to the results of the numerical simulations realized with periodic boundary conditions ($N/V = 1$, $dx = 1$, $D = 2$, $a = 1$, $k_c = \pi$). The continuous lines refer to the condensation curves obtained theoretically, as explicitly indicated in (a). The curve (i) corresponds to Eq. (52), the curve (ii) corresponds to Eq. (50) and finally the curve (iii) is obtained through the parametric plot (with μ as a parameter) of Eqs. (56) and (57). In the presence of a small condensate amplitude (n_0/n typically less than 0.4), a good agreement is obtained with Eqs. (56) and (57) that takes into account finite size effects ($\mu \neq 0$). Conversely, in the presence of a high condensate amplitude, good agreement is obtained with Eq. (52) that takes into account the Bogoliubov's dispersion relation (see Section 4). At the frontier between the small- and the high-amplitude condensate regimes, the effects of a bistable-like behavior have been observed in the numerical simulations (see Section 5). Such 'bistable' behavior is represented by the triangles ▲, indicated by '(iv)' in ((b)–(c)–(d)). For instance, 'bistable' behavior has been observed in (d) for $\langle H\rangle/V = 2.0662$, in which the condensate amplitude is shown to exhibit fluctuations between $(n_0/n)_- \approx 0.1$ and $(n_0/n)_+ \approx 0.3$, as illustrated in Fig. 6. These upper and lower values of the condensate amplitudes are in qualitative agreement with those given by the corresponding condensation curves in the presence of a small- and a high-condensate amplitude, *i.e.*, Eqs. (52), (56) and (57) respectively. Also note that the critical energy for condensation H_c decreases as the system volume V increases [Eq. (51)], consistently with the fact that the 2D condensation process is suppressed in the thermodynamic limit.

of parametric equations for the condensate fraction and the energy density:

$$\frac{\langle H\rangle(\mu)}{V} = n\frac{\sum_{\boldsymbol{k}}' \frac{k^2}{k^2-\mu}}{\sum_{\boldsymbol{k}} \frac{1}{k^2-\mu}} + a\left(n^2 - \frac{1}{2}n_0^2\right), \tag{56}$$

$$\frac{n_0(\mu)}{n} = \frac{1}{-\mu}\frac{1}{\sum_{\boldsymbol{k}} \frac{1}{k^2-\mu}}. \tag{57}$$

We plotted in Fig. 4 the condensate fraction n_0/n (57) vs the energy density $\langle H\rangle/V$ (56), as a parametric function of μ. It reveals that a non-vanishing chemical potential makes the transition to condensation "smoother", with the appearance of a characteristic "tail" in the condensation curve. Such a "tail" progressively disappears as the system volume V increases, so that the condensation curve n_0/n vs. $\langle H\rangle/V$ tends to the expression derived in the thermodynamic limit, *i.e.*, Eq. (56) with $\mu = 0$ recovers Eq. (50). Let us remark that the theory is in quantitative agreement with the numerical simulations of the NLS equation (1), as illustrated in Fig. 4.

It results in that the critical behavior of the two-dimensional condensation curve looks similar to that of a genuine "phase transition". Note however that, strictly speaking, "phase transitions" only occur in the thermodynamic limit, so that such terminology is not appropriate for the two dimensional problem considered here. Nevertheless, if one considers the macroscopic occupation of the fundamental mode $\boldsymbol{k} = 0$ as the essential characteristic of condensation, one may say that wave condensation does occur in 2D.

To conclude, let us note that finite size effects may also play a significant role in 3D for very small box-volumes (typically $V \lesssim 8^3$). The corresponding expression of the condensation curve, n_0/n vs $\langle H\rangle/V$, is still given by the parametric plot of Eqs. (56) and (57), since these equations have been derived regardless of the dimensionality of the system.

5.4. Non-zero value of the diagonal second order cumulant $Q^{(2)s_1s_1}$

As discussed above through Section 4, in the regime of high-condensate amplitude the diagonal terms of the second-order cumulant ($Q^{(2)++}$, $Q^{(2)--}$) no longer vanish, a peculiar feature which signals a breakdown of the standard weak-turbulence theory. This aspect is illustrated in Fig. 5, in which we completed the results of Fig. 4(b) by adding the theoretical and the numerical calculations of the following quantity, $Q^{++} \equiv \left\langle \left| \sum_k' Q^{(2)++}(\boldsymbol{k}) \right| \right\rangle = \left\langle \left| \sum_k' A_k A_{-k} \right| \right\rangle$. In the numerics, the average $\langle\cdot\rangle$ refers to a time-averaging computed once the nonlinear field has reached an equilibrium state (*i.e.*, once the entropy growth saturates to a constant value, $\partial_t \mathcal{S} \simeq 0$). The results of Fig. 5 confirm that the diagonal second-order cumulants deviate from zero in the presence of a significant condensate amplitude n_0.

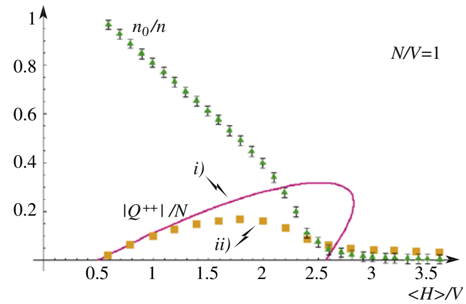

Fig. 5. Same as in Fig. 4(b), but with the additional plot of the second-order cumulant $Q^{++} \equiv \langle | \sum'_k Q^{(2)++}(\boldsymbol{k})| \rangle$. The continuous line (indicated by i) refers to the theory [Eq. (46)], the squares (indicated by ii) to the numerical simulations ($D = 2$, $N/V = 1$, $\mathrm{d}x = 1$, $V = 64^2$, $k_c = \pi$).

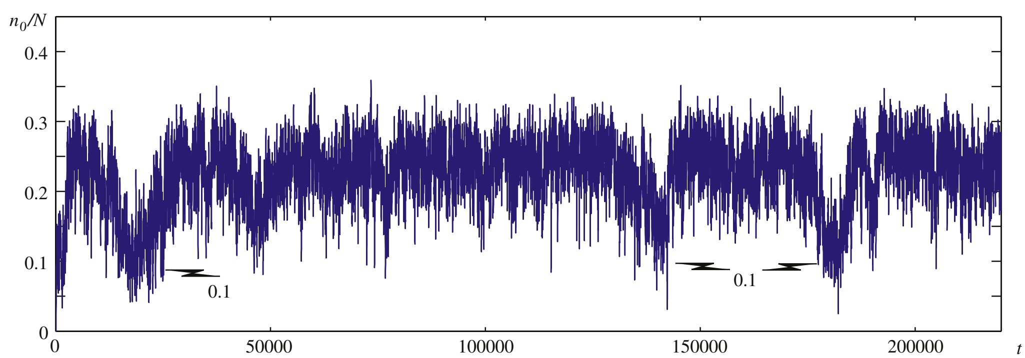

Fig. 6. Numerical simulations of the NLS equation showing the non-averaged temporal evolution of the condensate amplitude $n_0(t)$ for an energy in the vicinity of the crossover energy. The simulation has been realized in two spatial dimensions, for $\langle H \rangle / V = 2.0662$, $V = 256^2$, $N/V = 1$, $\mathrm{d}x = 1$ ($k_c = \pi$). The variance of the fluctuations of the energy density throughout the whole simulation ($t = 260\,000$ time units) are about $\simeq 10^{-4}$.

5.5. Bistable-like behavior

Let us underline that the expression of the condensation curve (52) coincides with the expression obtained in the framework of the heuristic approach reported in Ref. [5]. Provided one considers high values of the condensate amplitude n_0/n, Eq. (52) has been found in quantitative agreement with the numerical simulations, as remarkably illustrated in Figs. for $D = 3$ and in Figs. for $D = 2$. Conversely, for small values of the condensate amplitude, the condensation curve given by Eq. (52) exhibits a subcritical behavior in the vicinity of $H \simeq H_c$ (see Figs. 3 and 4). Note that such a subcritical behavior is more pronounced in two spatial dimensions than in three dimensions, as it would be appreciated by comparing Figs. 3 and 4. This would indicate that the Bogoliubov's expansion procedure (*i.e.*, the nonlinear interaction) would change the nature of the transition to condensation, which would become of the first-order (discontinuous). Actually, the weak-turbulence theory developed in the Bogoliubov's basis is not relevant for the description of a small condensate amplitude, as expressed by the assumptions underlying the derivation of the kinetic equation (41). In other terms, for small values of the condensate amplitude the diagonal parts of the second-order cumulant are still zero and the Bogoliubov's renormalization of the frequency is not yet realized. This signifies that the subcritical nature of the transition to condensation predicted by the theory is not physical, which corroborates the fact that it has not been identified in the numerical simulations.

In summary, the condensation curve at equilibrium is characterized by two regimes of interaction. For small energies, say for $H_0 \lesssim H \ll H_c$, with $H_0 = \frac{aN^2}{2V}$, the Bogoliubov's regime is established [Eq. (52)], whereas for $H_0 \gg H \lesssim H_c$ wave condensation may be described by the weak-turbulence theory in the presence of a small condensate amplitude [Eqs. (56) and (57)].

Let us remark that, at the frontier between the two regimes, a kind of bistable behavior is observed in the numerical simulations. Indeed, the temporal evolution of the condensate amplitude $n_0(t)$ may exhibit large fluctuations between the lower value $(n_0/n)_-$ predicted by the weak-turbulence theory [Eqs. (56) and (57)] and the upper value $(n_0/n)_+$ predicted by the Bogoliubov's theory [Eq. (52)]. A typical example of such large fluctuations is illustrated in Fig. 6, that shows the long time evolution of the non-averaged condensate amplitude for $H \lesssim H_c$. We can see in Fig. 6 that n_0/n spends a lot of time around $(n_0/n)_+ \simeq 0.3$, which corresponds to the Bogoliubov's condensation amplitude. However, it is remarkable that the condensate amplitude may spend more than 10 000 time units around $(n_0/n)_- \simeq 0.1$ ($10\,000 \lesssim t \lesssim 20\,000$), which corresponds to the value of the condensate amplitude given by the weak-turbulence theory [Eqs. (56) and (57)]. If the simulation were stopped at $t \simeq 30\,000$, one would have concluded that the lower amplitude state has been definitely selected by the system. However, at later times, the condensate amplitude fluctuates again between the corresponding lower and upper states (Fig. 6). In a loose sense, this seems to indicate that the condensate amplitude 'hesitates' between the upper and lower values predicted by the two regimes of condensation. Such behavior becomes more pronounced as the size of the system V increases, as illustrated in Fig. 4. Note that we also performed numerical simulations by slowing increasing/decreasing the energy H, and no clear hysteretic behavior has been identified in the simulations.

6. Conclusion

In summary, we analyzed the classical wave condensation process by means of the defocusing NLS equation. We extended the standard weak-turbulence theory by analyzing the asymptotic closure of the hierarchy of the cumulants' equations induced by the dispersive properties of the waves. This allowed us to formulate a self-consistent wave turbulence theory, in which the condensation process appears as a natural consequence of the emergence of a non-vanishing first-order cumulant. The condensation curve at equilibrium results in being characterized by two distinct regimes of interaction. For small energies, $H_0 \lesssim H \ll H_c$, the condensate amplitude follows the law Eq. (52) that has been established in the Bogoliubov's basis. Conversely, for $H \lesssim H_c$ the condensate amplitude has been calculated in the standard plane-wave basis and it has been shown to follow the law equations (56) and (57), or alternatively the law equation (50) in the thermodynamic limit. In both the high- and small-energy regimes, we derived coupled kinetic equations that govern the nonequilibrium evolution of the condensate amplitude and of the incoherent component.

We finally remark that, given the universal character of the NLS equation in nonlinear physics, the reported theory of wave condensation may find applicability in a large variety of fields, in particular in the context of optical waves, in which the NLS equation proved to be a relevant model in a variety of configurations [36,37]. The present study may also be relevant for an improved understanding of the nonequilibrium formation of a condensate in genuine thermal Bose gases. However, let us recall that the classical description of quantum condensation is only relevant in the limit where the modes are highly occupied, $n_k \gg 1$. In particular, this classical approach cannot describe a transfer of mass between scarcely occupied (high-energy) modes and the condensate [3,4,14,32,38].

Acknowledgements

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Appendix

A.1. Moments and cumulants

Let $\psi^s(\boldsymbol{x})$ be the wave field in the physical space, then the n-th order moment is defined by

$$M_N^{s_1,\dots s_N}(\boldsymbol{x},\boldsymbol{x}_2\cdots\boldsymbol{x}_N)=\left\langle\psi^{s_1}(\boldsymbol{x})\psi^{s_2}(\boldsymbol{x}+\boldsymbol{x}_2)\cdots\psi^{s_N}(\boldsymbol{x}+\boldsymbol{x}_N)\right\rangle.$$

From the definition of the Fourier's transform given in Eq. (16), one has

$$M_N^{s_1,\dots s_N}(\boldsymbol{x},\boldsymbol{x}_2\cdots\boldsymbol{x}_N)=\int \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1+\cdots+\boldsymbol{k}_N)\cdot\boldsymbol{x}}\left\langle A^{s_1}_{\boldsymbol{k}_1}\cdots A^{s_N}_{\boldsymbol{k}_N}\right\rangle \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{x}_2+\cdots+\boldsymbol{k}_N\cdot\boldsymbol{x}_N)}\mathrm{d}^D\boldsymbol{k}_1\dots\mathrm{d}^D\boldsymbol{k}_N.$$

Contrary to the moments, the cumulants decrease to zero as the relative distances in space coordinates go to infinity. In real physical space, the cumulants are related in a one to one way with the moments by the following relations:

$$\begin{aligned}
M_1^s &= \left\langle\psi^s(\boldsymbol{x})\right\rangle=\left\{\psi^s(\boldsymbol{x})\right\}=R^{(1)s},\\
M_2^{s_1,s_2}(\boldsymbol{x},\boldsymbol{x}_2) &= \left\langle\psi^{s_1}(\boldsymbol{x})\psi^{s_2}(\boldsymbol{x}+\boldsymbol{x}_2)\right\rangle=\left\{\psi^{s_1}(\boldsymbol{x})\psi^{s_2}(\boldsymbol{x}+\boldsymbol{x}_2)\right\}+\left\{\psi^{s}(\boldsymbol{x})\right\}\left\{\psi^{s_2}(\boldsymbol{x}+\boldsymbol{x}_2)\right\}\\
&= R^{(2)s_1s_2}(\boldsymbol{x}_2)+R^{(1)s_1}R^{(1)s_2},\\
M_3^{s_1,s_2,s_3}(\boldsymbol{x},\boldsymbol{x}_2,\boldsymbol{x}_3) &= \left\langle\psi^{s_1}(\boldsymbol{x})\psi^{s_2}(\boldsymbol{x}+\boldsymbol{x}_2)\psi^{s_3}(\boldsymbol{x}+\boldsymbol{x}_3)\right\rangle\\
&= R^{(3)s_1s_2s_3}(\boldsymbol{x}_2,\boldsymbol{x}_3)+R^{(1)s_1}R^{(2)s_2s_3}(\boldsymbol{x}_3-\boldsymbol{x}_2)+R^{(1)s_2}R^{(2)s_1s_3}(\boldsymbol{x}_3)+R^{(1)s_3}R^{(2)s_1s_2}(\boldsymbol{x}_2)+R^{(1)s_1}R^{(1)s_2}R^{(1)s_3},\\
M_4^{s_1,s_2,s_3,s_4}(\boldsymbol{x},\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4) &= R^{(4)s_1s_2s_3s_4}(\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4)+R^{(1)s_1}R^{(3)s_2s_3s_4}(\boldsymbol{x}_3-\boldsymbol{x}_2,\boldsymbol{x}_4-\boldsymbol{x}_2)\\
&\quad+\mathcal{P}_{234}R^{(1)s_2}R^{(3)s_1s_3s_4}(\boldsymbol{x}_3,\boldsymbol{x}_4)+\mathcal{P}_{234}R^{(2)s_1s_2}(\boldsymbol{x}_2)R^{(2)s_3,s_4}(\boldsymbol{x}_4-\boldsymbol{x}_3)\\
&\quad+R^{(1)s_1}\mathcal{P}_{234}R^{(1)s_2}R^{(2)s_3s_4}(\boldsymbol{x}_4-\boldsymbol{x}_3)+\mathcal{P}_{234}R^{(1)s_2}R^{(1)s_3}R^{(2)s_1s_4}(\boldsymbol{x}_4)+R^{(1)s_1}R^{(1)s_2}R^{(1)s_3}R^{(1)s_4},\\
&\quad\mathit{etc}.
\end{aligned}$$

One denotes by Qs the Fourier's transforms of the cumulants Rs,

$$R^{(N)s_1\dots s_N}(\boldsymbol{x}_2\cdots\boldsymbol{x}_N)=\int Q^{(N)s_1\cdots s_N}(\boldsymbol{k}_2\cdots\boldsymbol{k}_N)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{x}_2+\boldsymbol{k}_N\cdot\boldsymbol{x}_N)}\,\mathrm{d}^D\boldsymbol{k}_2\dots\mathrm{d}^D\boldsymbol{k}_N.$$

From this definition one readily obtains Eq. (15), which provide the relations between the moments and the cumulants in Fourier's space.

The average of the hamiltonian of the NLS equation (1) given in Eq. (2) may be expressed in terms of the cumulants,

$$\begin{aligned}
\frac{\langle H\rangle}{V} = \frac{(2\pi)^D}{2}\sum_s\Bigg[&\int\omega_{\boldsymbol{k}}Q^{(2)-ss}(\boldsymbol{k})\mathrm{d}^D\boldsymbol{k}+\frac{a}{2}(2\pi)^D\int Q^{(4)ss-s-s}(\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\mathrm{d}^D\boldsymbol{k}_{234}\\
&+2a(2\pi)^D\int Q^{(1)s}Q^{(3)s-s-s}(\boldsymbol{k}_2,\boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{23}+\frac{a}{2}(2\pi)^D\int Q^{(2)ss}(\boldsymbol{k}_1)Q^{(2)-s-s}(\boldsymbol{k}_2)\mathrm{d}^D\boldsymbol{k}_{12}\\
&+a(2\pi)^D\int Q^{(2)-ss}(\boldsymbol{k}_1)Q^{(2)-ss}(\boldsymbol{k}_2)\mathrm{d}^D\boldsymbol{k}_{12}+a(2\pi)^D\int Q^{(1)s}Q^{(1)s}Q^{(2)-s-s}(\boldsymbol{k}_1)\mathrm{d}^D\boldsymbol{k}_1\\
&+2a(2\pi)^D\int Q^{(1)s}Q^{(1)-s}Q^{(2)-ss}(\boldsymbol{k})\mathrm{d}^D\boldsymbol{k}+\frac{a}{2}(2\pi)^DQ^{(1)s}Q^{(1)s}Q^{(1)-s}Q^{(1)-s}\Bigg].
\end{aligned}\tag{58}$$

A.2. Multiscale expansion for the weak condensate regime

As discussed in Section 3.2, at the order $\mathcal{O}(\epsilon^2)$, it is necessary to introduce the functions $F_r^{(n)}$ to remove the secular growth of the terms $Q_1^{(n)}$. From the equation for the first order cumulant (17) one thus gets,

$$F_2^{(1)l_1}+\frac{\mathrm{d}}{\mathrm{d}t}q_2^{(1)l_1} = -\mathrm{i}al_1(2\pi)^D\int q_0^{(3)-l_1l_1l_1}(\boldsymbol{k}_2,\boldsymbol{k}_3)\mathrm{e}^{-\mathrm{i}l_1[-\omega(\boldsymbol{k}_1)+\omega(\boldsymbol{k}_2)+\omega(\boldsymbol{k}_3)]t}\delta^{(D)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{123}$$
$$-\mathrm{i}al_1(2\pi)^D\int q_0^{(1)-l_1}q_0^{(2)l_1l_1}(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}2l_1\omega(\boldsymbol{k})t}\mathrm{d}^D\boldsymbol{k}+\mathrm{i}al_1q_0^{(1)-l_1}q_0^{(1)l_1}q_0^{(1)l_1}$$
$$-2\mathrm{i}al_1(2\pi)^D\left(\int q_0^{(2)-l_1l_1}(\boldsymbol{k})\mathrm{d}^D\boldsymbol{k}+q_0^{(1)-l_1}q_0^{(1)l_1}\right)q_0^{(1)l_1}. \tag{59}$$

In a similar way, one obtains from the equation of the second-order cumulant (19):

$$F_2^{(2)l_1l_2}(\boldsymbol{p}_2)+\frac{\mathrm{d}}{\mathrm{d}t}q_2^{(2)l_1l_2}(\boldsymbol{p}_2) = -2\mathrm{i}a(2\pi)^D\mathcal{P}_{1'2'}l_2\left(\int q_0^{(2)-l_2l_2}(\boldsymbol{k})\mathrm{d}^D\boldsymbol{k}+q_0^{(1)-l_2}q_0^{(1)l_2}\right)q_0^{(2)l_1l_2}(\boldsymbol{p}_2)$$
$$-\mathrm{i}a(2\pi)^D\mathcal{P}_{1'2'}l_2\left(\int q_0^{(2)l_2l_2}(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}2l_2\omega(\boldsymbol{k})t}\mathrm{d}^D\boldsymbol{k}+q_0^{(1)l_2}q_0^{(1)l_2}\right)q_0^{(2)l_1-l_2}(\boldsymbol{p}_2)\mathrm{e}^{\mathrm{i}2l_2\omega(\boldsymbol{p}_2)t}$$
$$-\mathrm{i}a(2\pi)^D\mathcal{P}_{1'2'}l_2\int q_0^{(4)l_1-l_2l_2l_2}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\mathrm{e}^{-\mathrm{i}l_2[-\omega(\boldsymbol{p}_2)-\omega(\boldsymbol{k}_1)+\omega(\boldsymbol{k}_2)+\omega(\boldsymbol{k}_3)]t}\delta^{(D)}(\boldsymbol{p}_1+\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{123}$$
$$-\mathrm{i}a(2\pi)^D\mathcal{P}_{1'2'}l_2\int q_0^{(1)-l_2}q_0^{(3)l_1l_2l_2}(\boldsymbol{k}_1,\boldsymbol{k}_2)\mathrm{e}^{-\mathrm{i}l_2[-\omega(\boldsymbol{p}_2)+\omega(\boldsymbol{k}_1)+\omega(\boldsymbol{k}_2)]t}\delta^{(D)}(\boldsymbol{p}_1+\boldsymbol{k}_1+\boldsymbol{k}_2)\mathrm{d}^D\boldsymbol{k}_{12}$$
$$-2\mathrm{i}a(2\pi)^D\mathcal{P}_{1'2'}l_2\int q_0^{(1)l_2}q_0^{(3)l_1-l_2l_2}(\boldsymbol{k}_1,\boldsymbol{k}_2)\mathrm{e}^{-\mathrm{i}l_2[-\omega(\boldsymbol{p}_2)-\omega(\boldsymbol{k}_1)+\omega(\boldsymbol{k}_2)]t}\delta^{(D)}(\boldsymbol{p}_1+\boldsymbol{k}_1+\boldsymbol{k}_2)\mathrm{d}^D\boldsymbol{k}_{12} \tag{60}$$

where we did the change of variables $Q_2^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n)=\mathrm{e}^{-\mathrm{i}[l_1\omega(\boldsymbol{p}_1)+\cdots+l_n\omega(\boldsymbol{p}_n)]t}q_2^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n)$, with the exception that $Q_2^{(1)l_1}=q_2^{(1)l_1}$, and, as usual, $\boldsymbol{p}_1+\boldsymbol{p}_2+\cdots+\boldsymbol{p}_n=0$ through this and next Appendices. Following the same procedure, it is possible to find equations for all the higher-order cumulants. After an integration in time of Eqs. (59) and (60) it is easy to see that the elimination of the secular terms is achieved for the following expression of $F_2^{(1)l_1}$, and $F_2^{(2)l_1l_2}(\boldsymbol{p}_2)$,

$$F_2^{(1)l_1}=-\mathrm{i}al_1(2\pi)^D\left[2\int Q_0^{(2)l_1-l_1}(\boldsymbol{k})\mathrm{d}^D\boldsymbol{k}+q_0^{(1)l_1}q_0^{(1)-l_1}\right]q_0^{(1)l_1},$$

and

$$F_2^{(2)l_1l_2}(\boldsymbol{p}_2)=-2\mathrm{i}a(2\pi)^D\left[l_1\left(\int Q_0^{(2)l_1-l_1}(\boldsymbol{k})\mathrm{d}^D\boldsymbol{k}+q_0^{(1)l_1}q_0^{(1)-l_1}\right)+l_2\left(\int Q_0^{(2)l_2-l_2}(\boldsymbol{k})\mathrm{d}^D\boldsymbol{k}+q_0^{(1)l_2}q_0^{(1)-l_2}\right)\right]q_0^{(2)l_1l_2}(\boldsymbol{p}_2),$$

where $F_2^{(1)l_1}$ and $F_2^{(2)l_1l_2}(\boldsymbol{p}_2)$ refer to the long time cumulative effects of the oscillation frequency. The generalization to higher-order n-cumulant is straightforward and is given in Section 3.2.

Once we have removed the resonant terms to keep a well ordered expansion, one gets the fast time evolution for $q_2^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n)$, which is typically a fast oscillating function of time. We can now pursue the analysis to the next order. At order $\mathcal{O}(\epsilon^4)$, we are interested in the long time behavior of $q_0^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n)$, then following the standard method of weak-turbulence theory (see for example [8,18]) one has, for $n=1$:

$$F_4^{(1)l_1}\sim-\mathrm{i}al_1(2\pi)^D\int\int_0^t q_2^{(3)-l_1l_1l_1}(\boldsymbol{k}_2,\boldsymbol{k}_3;\tau)\mathrm{e}^{-\mathrm{i}l_1[-\omega(\boldsymbol{k}_1)+\omega(\boldsymbol{k}_2)+\omega(\boldsymbol{k}_3)]\tau}\mathrm{d}\tau\,\delta^{(D)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{123}+\text{“non-resonants terms”} \tag{61}$$

where the notation $F\sim A(t)$ indicates the long time resonant behavior of $A(t)$, *i.e.*, $F=\lim_{t\to\infty}\frac{1}{t}A(t)$. Obviously one has $\lim_{t\to\infty}\frac{1}{t}(\text{“non resonants terms”})\to 0$, so that we shall not write them explicitly here. For $n=2$ one has

$$F_4^{(2)l_1l_2}(\boldsymbol{p}_2)\sim-\mathrm{i}a(2\pi)^D\mathcal{P}_{1'2'}l_2\int\int_0^t q_2^{(4)l_1-l_2l_2l_2}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3;\tau)\mathrm{e}^{-\mathrm{i}l_2[-\omega(\boldsymbol{p}_2)-\omega(\boldsymbol{k}_1)+\omega(\boldsymbol{k}_2)+\omega(\boldsymbol{k}_3)]\tau}\mathrm{d}\tau\,\delta^{(D)}(\boldsymbol{p}_1+\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{123}$$
$$-\mathrm{i}a(2\pi)^D\mathcal{P}_{1'2'}l_2\int\int_0^t q_0^{(1)-l_2}q_2^{(3)l_1l_2l_2}(\boldsymbol{k}_1,\boldsymbol{k}_2;\tau)\mathrm{e}^{-\mathrm{i}l_2[-\omega(\boldsymbol{p}_2)+\omega(\boldsymbol{k}_1)+\omega(\boldsymbol{k}_2)]\tau}\mathrm{d}\tau\,\delta^{(D)}(\boldsymbol{p}_1+\boldsymbol{k}_1+\boldsymbol{k}_2)\mathrm{d}^D\boldsymbol{k}_{12}$$
$$-2\mathrm{i}a(2\pi)^D\mathcal{P}_{1'2'}l_2\int\int_0^t q_0^{(1)l_2}q_2^{(3)l_1-l_2l_2}(\boldsymbol{k}_1,\boldsymbol{k}_2;\tau)\mathrm{e}^{-\mathrm{i}l_2[-\omega(\boldsymbol{p}_2)-\omega(\boldsymbol{k}_1)+\omega(\boldsymbol{k}_2)]\tau}\mathrm{d}\tau\,\delta^{(D)}(\boldsymbol{p}_1+\boldsymbol{k}_1+\boldsymbol{k}_2)\mathrm{d}^D\boldsymbol{k}_{12}$$
$$+\text{“non-resonants terms”}. \tag{62}$$

As already noticed, at order $\mathcal{O}(\epsilon^2)$ of the expansion, one obtains the evolution for $q_2^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n)$, in particular one has that

$$q_2^{(3)l_1l_2l_3}(\boldsymbol{p}_2,\boldsymbol{p}_3;t)=-2\mathrm{i}a(2\pi)^D\,\mathcal{P}_{1'2'3'}l_3q_0^{(2)-l_3l_1}(\boldsymbol{p}_1)q_0^{(2)l_3l_2}(\boldsymbol{p}_2)q_0^{(1)l_3}\int_0^t\mathrm{e}^{-\mathrm{i}l_3[-\omega(\boldsymbol{p}_1)+\omega(\boldsymbol{p}_2)-\omega(\boldsymbol{p}_3)]\tau}\mathrm{d}\tau$$

$$
\begin{aligned}
& - 2\mathrm{i}a(2\pi)^D\,\mathcal{P}_{1'2'3'}l_3 q_0^{(2)l_3l_1}(\boldsymbol{p}_1)q_0^{(2)l_3l_2}(\boldsymbol{p}_2)q_0^{(1)-l_3}\int_0^t \mathrm{e}^{-\mathrm{i}l_3[\omega(\boldsymbol{p}_1)+\omega(\boldsymbol{p}_2)-\omega(\boldsymbol{p}_3)]\tau}\mathrm{d}\tau \\
& - 2\mathrm{i}a(2\pi)^D\,\mathcal{P}_{1'2'3'}l_3 q_0^{(2)l_3l_1}(\boldsymbol{p}_1)q_0^{(2)-l_3l_2}(\boldsymbol{p}_2)q_0^{(1)l_3}\int_0^t \mathrm{e}^{-\mathrm{i}l_3[\omega(\boldsymbol{p}_1)-\omega(\boldsymbol{p}_2)-\omega(\boldsymbol{p}_3)]\tau}\mathrm{d}\tau \\
& + \text{“non-resonants terms”}
\end{aligned}
\tag{63}
$$

and

$$
\begin{aligned}
q_2^{(4)l_1l_2l_3l_4}(\boldsymbol{p}_2,\boldsymbol{p}_3,\boldsymbol{p}_4;t) = & -2\mathrm{i}a(2\pi)^D\,\mathcal{P}_{1'2'3'4'}l_4 q_0^{(2)-l_4l_1}(\boldsymbol{p}_1)q_0^{(2)l_4l_2}(\boldsymbol{p}_2)q_0^{(2)l_4l_3}(\boldsymbol{p}_3)\int_0^t \mathrm{e}^{-\mathrm{i}l_4[-\omega(\boldsymbol{p}_1)+\omega(\boldsymbol{p}_2)+\omega(\boldsymbol{p}_3)-\omega(\boldsymbol{p}_4)]\tau}\mathrm{d}\tau \\
& - 2\mathrm{i}a(2\pi)^D\,\mathcal{P}_{1'2'3'4'}l_4 q_0^{(2)l_4l_1}(\boldsymbol{p}_1)q_0^{(2)l_4l_2}(\boldsymbol{p}_2)q_0^{(2)-l_4l_3}(\boldsymbol{p}_3)\int_0^t \mathrm{e}^{-\mathrm{i}l_4[\omega(\boldsymbol{p}_1)+\omega(\boldsymbol{p}_2)-\omega(\boldsymbol{p}_3)-\omega(\boldsymbol{p}_4)]\tau}\mathrm{d}\tau \\
& - 2\mathrm{i}a(2\pi)^D\,\mathcal{P}_{1'2'3'4'}l_4 q_0^{(2)l_4l_1}(\boldsymbol{p}_1)q_0^{(2)-l_4l_2}(\boldsymbol{p}_2)q_0^{(2)l_4l_3}(\boldsymbol{p}_3)\int_0^t \mathrm{e}^{-\mathrm{i}l_4[\omega(\boldsymbol{p}_1)-\omega(\boldsymbol{p}_2)+\omega(\boldsymbol{p}_3)-\omega(\boldsymbol{p}_4)]\tau}\mathrm{d}\tau \\
& + \text{“non-resonants terms”}\,.
\end{aligned}
\tag{64}
$$

Replacing (63) and (64) into (61) and (62), and taking the limit of long times by making use of the “formal” relation (“formal”, because it has a sense only once integration has been performed, for more details see [18]):

$$
\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_0^\tau \mathrm{e}^{(-\mathrm{i}\tilde\tau\Omega_i)}\mathrm{d}\tilde\tau\,\mathrm{e}^{(\mathrm{i}\tau\Omega_j)}\mathrm{d}\tau = \left[\pi\delta(\Omega_i)+\mathrm{i}P\left(\frac{1}{\Omega_i}\right)\right]\delta_{i,j},
\tag{65}
$$

where P represents the Cauchy principal value, one readily obtains:

$$
\begin{aligned}
F_4^{(1)l_1} = & -\epsilon^4 2\pi a^2 q_0^{(1)l_1}\int\left[\delta(\omega(k_1)-\omega(k_2)-\omega(k_3)) - \mathrm{i}l_1 P\left(\frac{1}{\omega(k_1)-\omega(k_2)-\omega(k_3)}\right)\right]\delta^{(D)}(\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3) \\
& \times n_{\boldsymbol{k}_1}n_{\boldsymbol{k}_2}n_{\boldsymbol{k}_3}\left(\frac{1}{n_{\boldsymbol{k}_3}}+\frac{1}{n_{\boldsymbol{k}_2}}-\frac{1}{n_{\boldsymbol{k}_1}}\right)\mathrm{d}^D\boldsymbol{k}_{123}
\end{aligned}
\tag{66}
$$

and

$$
\begin{aligned}
F_4^{(2)-l_2l_2}(\boldsymbol{p}_2) = & \,4\pi a^2\epsilon^4\int n_0 n_{\boldsymbol{p}_2}n_{\boldsymbol{k}_2}n_{\boldsymbol{k}_3}\left(\frac{1}{n_{\boldsymbol{p}_2}}-\frac{1}{n_{\boldsymbol{k}_2}}-\frac{1}{n_{\boldsymbol{k}_3}}\right)\delta(\omega(p_2)-\omega(k_2)-\omega(k_3))\delta^{(D)}(\boldsymbol{p}_2-\boldsymbol{k}_2-\boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{23} \\
& + 8\pi a^2\epsilon^4\int n_0 n_{\boldsymbol{p}_2}n_{\boldsymbol{k}_1}n_{\boldsymbol{k}_2}\left(\frac{1}{n_{\boldsymbol{p}_2}}+\frac{1}{n_{\boldsymbol{k}_1}}-\frac{1}{n_{\boldsymbol{k}_2}}\right)\delta(\omega(p_2)+\omega(k_1)-\omega(k_2))\delta^{(D)}(\boldsymbol{p}_2+\boldsymbol{k}_1-\boldsymbol{k}_2)\mathrm{d}^D\boldsymbol{k}_{12} \\
& + 4\pi a^2\epsilon^4\int n_{\boldsymbol{p}_2}n_{\boldsymbol{k}_1}n_{\boldsymbol{k}_2}n_{\boldsymbol{k}_3}\left(\frac{1}{n_{\boldsymbol{p}_2}}+\frac{1}{n_{\boldsymbol{k}_1}}-\frac{1}{n_{\boldsymbol{k}_2}}-\frac{1}{n_{\boldsymbol{k}_3}}\right) \\
& \times\delta(\omega(p_2)+\omega(k_1)-\omega(k_2)-\omega(k_3))\delta^{(D)}(\boldsymbol{p}_2+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{123}.
\end{aligned}
\tag{67}
$$

These relations provide the slow, up to order $\mathcal{O}(\epsilon^4)$, dynamical evolution of $n_0(t) = (2\pi)^D q_0^{(1)l_1}q_0^{(1)-l_1}$ and $n_{\boldsymbol{k}}(t)$, which corresponds to the set of coupled equations (22) and (23). Notice, however, that $\frac{\mathrm{d}}{\mathrm{d}t}n_0(t) = (2\pi)^D\left(q_0^{(1)l_1}F_0^{(1)-l_1}+F_0^{(1)l_1}q_0^{(1)-l_1}\right)$ so the principal value term in (66) vanishes in the final equation (22).

Furthermore, for the diagonal cumulants, $l_1 = l_2$, one gets the linear evolution:

$$
F_4^{(2)l_2l_2}(\boldsymbol{p}_2) = 2\left[il_2\Gamma^I(\boldsymbol{p}_2)+\Gamma^R(\boldsymbol{p}_2)\right]q_0^{(2)l_2l_2}(\boldsymbol{p}_2),
\tag{68}
$$

where

$$
\begin{aligned}
\Gamma^I(\boldsymbol{p}_2) = & \,2a^2 P\int\frac{(n_{\boldsymbol{k}_2}n_{\boldsymbol{k}_3}-n_{\boldsymbol{k}_1}n_{\boldsymbol{k}_2}-n_{\boldsymbol{k}_3}n_{\boldsymbol{k}_1})}{\omega(p_2)+\omega(k_1)-\omega(k_2)-\omega(k_3)}\delta^{(D)}(\boldsymbol{p}_2+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{123} \\
& - 2a^2 P\int\frac{n_0(n_{\boldsymbol{k}_2}+n_{\boldsymbol{k}_3})}{\omega(p_2)-\omega(k_2)-\omega(k_3)}\delta^{(D)}(\boldsymbol{p}_2-\boldsymbol{k}_2-\boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{23} \\
& + 4a^2 P\int\frac{n_0(n_{\boldsymbol{k}_3}-n_{\boldsymbol{k}_2})}{\omega(p_2)+\omega(k_2)-\omega(k_3)}\delta^{(D)}(\boldsymbol{p}_2+\boldsymbol{k}_2-\boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{23},
\end{aligned}
\tag{69}
$$

and

$$
\begin{aligned}
\Gamma^R(\boldsymbol{p}_2) = & \,2\pi a^2\int(n_{\boldsymbol{k}_2}n_{\boldsymbol{k}_3}-n_{\boldsymbol{k}_1}n_{\boldsymbol{k}_2}-n_{\boldsymbol{k}_3}n_{\boldsymbol{k}_1})\delta(\omega(p_2)+\omega(k_1)-\omega(k_2)-\omega(k_3))\delta^{(D)}(\boldsymbol{p}_2+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{123} \\
& - 2\pi a^2\int n_0(n_{\boldsymbol{k}_2}+n_{\boldsymbol{k}_3})\delta^{(D)}(\boldsymbol{p}_2-\boldsymbol{k}_2-\boldsymbol{k}_3)\delta(\omega(p_2)-\omega(k_2)-\omega(k_3))\mathrm{d}^D\boldsymbol{k}_{23} \\
& + 4\pi a^2\int n_0(n_{\boldsymbol{k}_3}-n_{\boldsymbol{k}_2})\delta^{(D)}(\boldsymbol{p}_2+\boldsymbol{k}_2-\boldsymbol{k}_3)\delta(\omega(p_2)+\omega(k_2)-\omega(k_3))\mathrm{d}^D\boldsymbol{k}_{23}.
\end{aligned}
\tag{70}
$$

It is of particular interest to prove that $\Gamma^R(\boldsymbol{p})$ is negative for every $\boldsymbol{p}$. The expression (70) is at least negative for the equilibrium distribution $n_k^{eq} = T/\omega_k$ (25). Accordingly, the damping rate $\Gamma^R(\boldsymbol{p}_2)$ of the diagonal cumulants is negative, since

$$2\Gamma^R(\boldsymbol{p}_2) = -2\pi a^2 \frac{\omega_{p_2}}{T}(I_1(\boldsymbol{p}_2) + n_0 I_2(\boldsymbol{p}_2)) \le 0 \quad \text{with}$$
$$I_1(\boldsymbol{p}_2) = \int n_{k_1}^{eq} n_{k_2}^{eq} n_{k_3}^{eq} \delta(\omega_{p_2} + \omega_{k_1} - \omega_{k_2} - \omega_{k_3})\delta^{(D)}(\boldsymbol{p}_2 + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{123}$$
$$I_2(\boldsymbol{p}_2) = \int n_{k_2}^{eq} n_{k_3}^{eq} \delta(\omega_{p_2} - \omega_{k_2} - \omega_{k_3})\delta^{(D)}(\boldsymbol{p}_2 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{23} + 2\int n_{k_2}^{eq} n_{k_3}^{eq} \delta(\omega_{p_2} + \omega_{k_2} - \omega_{k_3})\delta^{(D)}(\boldsymbol{p}_2 + \boldsymbol{k}_2 - \boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{23}.$$

One can develop a similar analysis for higher-order (diagonal or non-diagonal) cumulants, which leads to the conclusion that they exhibit a temporal exponential decay at equilibrium. The weak turbulence theory then provides a natural closure in terms of the off-diagonal second-order cumulants.

A.3. Multi-scale expansion in the Bogoliubov's basis

To find a closure of the hierarchy of cumulants' equations in the presence of a large condensate amplitude, we follow a similar approach as that developed above for the small condensate case. One expands the cumulants in series as

$$Q_B^{(n)l_1\ldots l_n} = Q_{B0}^{(n)l_1\ldots l_n} + \epsilon Q_{B1}^{(n)l_1\ldots l_n} + \epsilon^2 Q_{B2}^{(n)l_1\ldots l_n} + \cdots$$

and the corresponding equations are obtained order by order. Up to zero order, $\mathcal{O}(\epsilon^0)$, one gets

$$Q_{B0}^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n) = \mathrm{e}^{-\mathrm{i}[l_1\omega_B(\boldsymbol{p}_1)+\cdots+l_n\omega_B(\boldsymbol{p}_n)]t} q_{B0}^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n),$$

where, as usual, $\boldsymbol{p}_1 + \boldsymbol{p}_2 + \cdots + \boldsymbol{p}_n = 0$. The condensate fraction being constant $\frac{\mathrm{d}}{\mathrm{d}t}\left(Q_0^{(1)l_1} Q_0^{(1)-l_1}\right) = 0$ up to this order, the Bogoliubov's transformation is well defined. For higher order cumulants one gets up to order one, $\mathcal{O}(\epsilon^1)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_{B1}^{(2)l_1 l_2}(\boldsymbol{p}_2) + (\mathrm{i}l_2\omega_B(p_2) + \mathrm{i}l_1\omega_B(p_1))Q_{B1}^{(2)l_1 l_2}(\boldsymbol{p}_2)$$
$$= (2\pi)^D \mathcal{P}_{1'2'} \sum_{s_1 s_2 s_3} \int Lb_{\boldsymbol{p}_2\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}^{l_2 s_1 s_2 s_3} \mathcal{P}_{123}\, Q_{B0}^{(1)s_1} Q_{B0}^{(3)l_1 s_2 s_3}(\boldsymbol{k}_2, \boldsymbol{k}_3)\delta^{(D)}(\boldsymbol{k}_1)\delta^{(D)}(\boldsymbol{p}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{123} \quad (71)$$

which may be readily integrated

$$q_{B1}^{(2)l_1 l_2}(\boldsymbol{p}_2) = C_1^{(2)l_1 l_2} + (2\pi)^D \mathcal{P}_{1'2'} \sum_{s_1 s_2 s_3} \int \mathcal{P}_{123} Lb_{\boldsymbol{p}_2\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}^{l_2 s_1 s_2 s_3} Q_{B0}^{(1)s_1} q_{B0}^{(3)l_1 s_2 s_3}(\boldsymbol{k}_2, \boldsymbol{k}_3)\mathrm{e}^{-\mathrm{i}\Omega_{k_2 k_3 p_2}^{s_2 s_3 -l_2} t}\delta^{(D)}(\boldsymbol{k}_1)\delta^{(D)}(\boldsymbol{p}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{123} \quad (72)$$

where we have denoted $Q_{B1}^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n) = \mathrm{e}^{-\mathrm{i}[l_1\omega_B(\boldsymbol{p}_1)+\cdots+l_n\omega_B(\boldsymbol{p}_n)]t} q_{B1}^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n)$, $C_1^{(2)l_1 l_2}$ being a constant of integration and $\Omega_{k_1 k_2 k_3}^{s_1 s_2 s_3} = s_1\omega_B(\boldsymbol{k}_1) + s_2\omega_B(\boldsymbol{k}_2) + s_3\omega_B(\boldsymbol{k}_3)$.

For the third order cumulant

$$q_{B1}^{(3)l_1 l_2 l_3}(\boldsymbol{p}_2, \boldsymbol{p}_3)$$
$$= C_1^{(3)l_1 l_2 l_3} + (2\pi)^D \mathcal{P}_{1'2'3'} \sum_{s_1 s_2 s_3} \int \mathcal{P}_{123} Lb_{\boldsymbol{p}_3\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}^{l_3 s_1 s_2 s_3} Q_{B0}^{(1)s_1} q_{B0}^{(4)l_1 l_2 s_2 s_3}(\boldsymbol{p}_2, \boldsymbol{k}_2, \boldsymbol{k}_3)\mathrm{e}^{-\mathrm{i}\Omega_{k_2 k_3 p_3}^{s_2 s_3 -l_3} t}\delta^{(D)}(\boldsymbol{k}_1)\delta^{(D)}(\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{k}_2 + \boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{123}$$
$$+ (2\pi)^D \mathcal{P}_{1'2'3'} \sum_{s_1 s_2 s_3} \int \mathcal{PT}_{123} Lb_{\boldsymbol{p}_3\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}^{l_3 s_1 s_2 s_3} q_{B0}^{(2)l_1 s_1}(\boldsymbol{k}_1) q_{B0}^{(2)l_2 s_2}(\boldsymbol{k}_2) Q_{B0}^{(1)s_3}\mathrm{e}^{-\mathrm{i}\Omega_{k_1 k_2 p_3}^{s_1 s_2 -l_3} t}\delta^{(D)}(\boldsymbol{k}_3)\delta^{(D)}(\boldsymbol{p}_1 + \boldsymbol{k}_1)\delta^{(D)}(\boldsymbol{p}_2 + \boldsymbol{k}_2)\mathrm{d}^D\boldsymbol{k}_{123}, \quad (73)$$

where $C_1^{(3)l_1 l_2 l_3}$ is a constant of integration and $\mathcal{PT}_{123}$ represents all the possible permutations of the indices 1, 2 & 3.

At the next order, $\mathcal{O}(\epsilon^2)$, it is necessary to introduce a multiscale expansion to correctly compute the secular growth

$$\frac{\mathrm{d}}{\mathrm{d}t} q_0^{(1)l_1} = \epsilon^2 F_2^{(1)l_1} + \epsilon^3 F_3^{(1)l_1} + \cdots$$
$$\frac{\mathrm{d}}{\mathrm{d}t} q_{B0}^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n) = \epsilon^2 F_{B2}^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n) + \epsilon^3 F_{B3}^{(n)l_1\cdots l_n}(\boldsymbol{p}_2\cdots\boldsymbol{p}_n) + \cdots$$

where we define $Q_i^{(1)l_1} = q_i^{(1)l_1}\exp(-\mathrm{i}l_1 a n_0 t)$. By using the relation (38), one can find the equation for the first-order cumulant in the normal basis (for the sake of simplicity we considered a spatially isotropic system):

$$F_2^{(1)l_1} + \frac{\mathrm{d}}{\mathrm{d}t} q_2^{(1)l_1} = -\mathrm{i}a l_1 (2\pi)^D \Bigg[q_0^{(1)l_1} \int |u_k|^2\, 2(1 + L_k + L_k^2) q_{B0}^{(2)-l_1 l_1}(\boldsymbol{k})\mathrm{d}^D + q_0^{(1)l_1} q_0^{(1)l_1} q_2^{(1)-l_1} + q_0^{(1)-l_1} q_0^{(1)l_1} q_2^{(1)l_1}$$
$$+ q_0^{(1)l_1} \int |u_k|^2\, (2L_k + L_k^2) q_{B0}^{(2)-l_1 -l_1}(\boldsymbol{k})\mathrm{e}^{2\mathrm{i}l_1\omega_B(k)t}\mathrm{d}^D\boldsymbol{k} + q_0^{(1)l_1} \int |u_k|^2\, (1 + 2L_k) q_{B0}^{(2)l_1 l_1}(\boldsymbol{k})\mathrm{e}^{-2\mathrm{i}l_1\omega_B(k)t}\mathrm{d}^D\boldsymbol{k} \Bigg]. \quad (74)$$

As a result, at any order in ϵ, we obtain equations of the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}q_j^{(1)l_1} = H_j^{l_1}(t) - \mathrm{i}al_1(2\pi)^D\left(q_0^{(1)l_1}q_0^{(1)l_1}q_j^{(1)-l_1} + q_0^{(1)-l_1}q_0^{(1)l_1}q_j^{(1)l_1}\right),$$

where $H_j^{l_1}(t)$ corresponds to the non-homogeneous part of this temporal ordinary differential equation. The general solution of this *o.d.e.* is

$$q_j^{(1)l_1} = -\mathrm{i}al_1(2\pi)^D q_0^{(1)l_1}\int_0^t\int_0^\tau (q_0^{(1)-l_1}H_j^{l_1}(\tau_1) + q_0^{(1)l_1}H_j^{-l_1}(\tau_1))\mathrm{d}\tau_1\mathrm{d}\tau$$
$$- \mathrm{i}al_1(2\pi)^D tq_0^{(1)l_1}\left(q_0^{(1)-l_1}C_j^{(1)l_1} + q_0^{(1)l_1}tC_j^{(1)-l_1}\right) + C_j^{(1)l_1} + \int_0^t H_j^{l_1}(\tau)\mathrm{d}\tau \quad (75)$$

where $C_j^{(1)l_1}$ is a constant of integration. In general we can write $H_j^{l_1}(t) = -F_j^{(1)l_1} + A_j^{(1)l_1} + h_j^{l_1}(t)$, where $A_j^{(1)l_1}$ represent the terms without a time dependence (at least that $\lim_{t\to\infty}\frac{1}{t}\int_0^t A_j^{(1)l_1}\mathrm{d}\tau \to Cte$) and $h_j^{l_1}(t)$ correspond to all other terms that typically exhibit a fast oscillatory time dependence (that is $\lim_{t\to\infty}\frac{1}{t}\int_0^t h_j^{l_1}\mathrm{d}\tau \to 0$). In order to remove the secular term we impose

$$F_j^{(1)l_1} = A_j^{(1)l_1} - \mathrm{i}al_1(2\pi)^D q_0^{(1)l_1}\mathcal{C}\left[\int_0^t\int_0^\tau (q_0^{(1)-l_1}h_j^{l_1}(\tau_1) + q_0^{(1)l_1}h_j^{-l_1}(\tau_1))\mathrm{d}\tau_1\mathrm{d}\tau\right]$$
$$- \mathrm{i}\frac{al_1}{2}(2\pi)^D q_0^{(1)l_1}\left(q_0^{(1)-l_1}(A_j^{l_1} - F_j^{(1)l_1}) + q_0^{(1)l_1}(A_j^{l_1} - F_j^{(1)l_1})\right)t - \mathrm{i}al_1(2\pi)^D q_0^{(1)l_1}(q_0^{(1)-l_1}C_j^{(1)l_1} + q_0^{(1)l_1}C_j^{(1)-l_1}) \quad (76)$$

where $\mathcal{C}[u(t)] = \lim_{t\to\infty}\frac{u(t)}{t}$ and corresponds to the resonant term of $u(t)$. Note that Eq. (76) seems to be ill-defined, because of the third term which is linearly proportional to time. However, it is easy to see from the same equation, that such a term is identically zero, because $q_0^{(1)-l_1}(A_j^{l_1} - F_j^{(1)l_1}) + q_0^{(1)l_1}(A_j^{-l_1} - F_j^{(1)-l_1}) = 0$. Moreover, Eq. (76) gives us the slow dynamical evolution for the condensate part n_0 at order j because $\frac{\mathrm{d}}{\mathrm{d}t}q_0^{(1)l_1} = \sum_j \epsilon^j F_j^{(1)l_1}$, thus

$$\frac{\mathrm{d}}{\mathrm{d}t}n_0 = (2\pi)^D\sum_j \epsilon^j\left(q_0^{(1)-l_1}F_j^{(1)l_1} + q_0^{(1)l_1}F_j^{(1)-l_1}\right) = (2\pi)^D\sum_j\epsilon^j\left(q_0^{(1)-l_1}A_j^{(1)l_1} + q_0^{(1)l_1}A_j^{(1)-l_1}\right).$$

In conclusion, if $(q_0^{(1)-l_1}A_j^{(1)l_1} + q_0^{(1)l_1}A_j^{(1)-l_1}) = 0$ one simply obtains a frequency renormalization. For instance, for $j = 2$ one obtains the frequency renormalization,

$$F_2^{(1)l_1} = -\mathrm{i}al_1(2\pi)^D q_0^{(1)l_1}\int |u_k|^2\, 2(1 + L_k + L_k^2)q_{B0}^{(2)-l_1l_1}(\boldsymbol{k})\,\mathrm{d}^D\boldsymbol{k} - a^2(2\pi)^D q_0^{(1)l_1}q_0^{(1)l_1}q_0^{(1)-l_1}(q_{B0}^{(2)l_1l_1}(0) - q_{B0}^{(2)-l_1-l_1}(0))$$
$$+ \mathrm{i}a^2(2\pi)^D l_1 q_0^{(1)l_1}q_0^{(1)l_1}q_0^{(1)-l_1}P\int\frac{(q_{B0}^{(2)l_1l_1}(\boldsymbol{k}) + q_{B0}^{(2)-l_1-l_1}(\boldsymbol{k}))}{2\omega_B(\boldsymbol{k})}\mathrm{d}^D\boldsymbol{k} - \mathrm{i}al_1(2\pi)^D q_0^{(1)l_1}(q_0^{(1)-l_1}C_2^{(1)l_1} + q_0^{(1)l_1}C_2^{(1)-l_1}). \quad (77)$$

Simultaneously one gets an expression for $q_2^{(1)l_1}(t)$ that has a fast oscillatory time dependence. Then the total condensate fraction $N_0/V = n_0 + \epsilon^2(2\pi)^D\left(q_0^{(1)l_1}q_2^{(1)-l_1} + q_0^{(1)-l_1}q_2^{(1)l_1}\right) + \cdots$ will exhibit small fluctuations around n_0. Nevertheless, they are not important because one can always chose $C_2^{(1)-l_1}$ in such a way that $\lim_{t\to\infty}q_2^{(1)l_1}(t) \to 0$, so that such oscillations will not produce a long-time cumulative effect. As we already said, at first order we have $\frac{\mathrm{d}}{\mathrm{d}t}n_0 = 0$, so that we can make use of the Bogolibov's transformation for the higher-order cumulants. At this point one obtains for the second-order cumulant the equation:

$$F_{B2}^{(2)l_1l_2}(\boldsymbol{p}_2)\mathrm{e}^{-\mathrm{i}(l_1\omega_B(p_1)+l_2\omega_B(p_2))t} + \frac{\mathrm{d}}{\mathrm{d}t}Q_{B2}^{(2)l_1l_2}(\boldsymbol{p}_2) + (\mathrm{i}l_1\omega_B(p_1) + \mathrm{i}l_2\omega_B(p_2))Q_{B2}^{(2)l_1l_2}(\boldsymbol{p}_2)$$
$$= (2\pi)^D\mathcal{P}_{1'2'}\sum_{s_1s_2s_3}\int Lb^{l_2s_1s_2s_3}_{\boldsymbol{p}_2\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}Q_{B0}^{(4)l_1s_1s_2s_3}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\delta^{(D)}(\boldsymbol{p}_1 + \boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{123}$$
$$+ (2\pi)^D\mathcal{P}_{1'2'}\sum_{s_1s_2s_3}\int Lb^{l_2s_1s_2s_3}_{\boldsymbol{p}_2\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}\mathcal{P}_{123}\,Q_{B0}^{(1)s_1}Q_{B1}^{(3)l_1s_2s_3}(\boldsymbol{k}_2,\boldsymbol{k}_3)\delta^{(D)}(\boldsymbol{k}_1)\delta^{(D)}(\boldsymbol{p}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{123}$$
$$+ 2(2\pi)^D\mathcal{P}_{1'2'}\sum_{s_1s_2s_3}\int Lb^{l_2s_1s_2s_3}_{\boldsymbol{p}_2\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}\mathcal{P}_{123}\,Q_{B0}^{(1)s_1}Q_{B2}^{(1)s_2}Q_{B0}^{(2)l_1s_3}(\boldsymbol{k}_3)\delta^{(D)}(\boldsymbol{k}_1)\delta^{(D)}(\boldsymbol{k}_2)\delta^{(D)}(\boldsymbol{p}_1 + \boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{123}$$
$$+ (2\pi)^D\mathcal{P}_{1'2'}\sum_{s_1s_2s_3}\int Lb^{l_2s_1s_2s_3}_{\boldsymbol{p}_2\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}\mathcal{P}_{123}\,Q_{B0}^{(2)l_1s_1}(\boldsymbol{k}_1)Q_{B0}^{(2)s_2s_3}(\boldsymbol{k}_3)\delta^{(D)}(\boldsymbol{p}_1 + \boldsymbol{k}_1)\delta^{(D)}(\boldsymbol{k}_2 + \boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{123}. \quad (78)$$

Following the method of wave turbulence theory [8,18], a straightforward calculation leads to a closed form for the slow dynamics of the second-order cumulant (here $\Omega^{s_1s_2s_3}_{k_1k_2k_3} \equiv s_1\omega_B(k_1) + s_2\omega_B(k_2) + s_3\omega_B(k_3)$)

$$F_{B2}^{(2)l_1l_2}(\boldsymbol{p}_2) = 3a(2\pi)^D\mathcal{P}_{1'2'}\sum_s\int Lb^{l_2l_2-ss}_{\boldsymbol{p}_2\boldsymbol{p}_2-\boldsymbol{k}\boldsymbol{k}}\,q_{B0}^{(2)-ss}(\boldsymbol{k})\mathrm{d}^D\boldsymbol{k}\,q_{B0}^{(2)l_1l_2}(\boldsymbol{p}_2)$$

$$+6a^2(2\pi)^D\mathcal{P}_{1'2'}\mathcal{P}_{123}\sum_{s_2s_3}\int\left|\sum_s Lb^{l_2ss_2s_3}_{\boldsymbol{p}_2\boldsymbol{k}\boldsymbol{k}_2\boldsymbol{k}_3}\right|^2\frac{|1-L_k|}{|1+L_k|}n_0q_{B0}^{(2)-s_2s_2}(\boldsymbol{k}_2)q_{B0}^{(2)-s_3s_3}(\boldsymbol{k}_3)$$
$$\times\left(\pi\delta(\Omega^{s_2s_3-l_2}_{k_2k_3p_2})+iP\left(\frac{1}{\Omega^{s_2s_3-l_2}_{k_2k_3p_2}}\right)\right)\delta_{l_1,-l_2}\delta^{(D)}(\boldsymbol{k}_2+\boldsymbol{k}_3-\boldsymbol{p}_2)\delta^{(D)}(\boldsymbol{k})\mathrm{d}^D\boldsymbol{k}_{23}$$
$$-6a^2(2\pi)^D\mathcal{P}_{1'2'}\mathcal{P}_{123}\sum_{s_2s_3}\int\left|\sum_s Lb^{l_2ss_2s_3}_{\boldsymbol{p}_2\boldsymbol{k}\boldsymbol{k}_2\boldsymbol{k}_3}\right|^2\frac{|1-L_k|}{|1+L_k|}\mathcal{P}_{23}\left(\frac{s_3}{l_2}\right)n_0q_{B0}^{(2)l_1l_2}(\boldsymbol{p}_2)q_{B0}^{(2)-s_2s_2}(\boldsymbol{k}_2)$$
$$\times\left(\pi\delta(\Omega^{-l_2s_2s_3}_{p_2k_2k_3})+iP\left(\frac{1}{\Omega^{-l_2s_2s_3}_{p_2k_2k_3}}\right)\right)\delta^{(D)}(\boldsymbol{k}_2+\boldsymbol{k}_3-\boldsymbol{p}_2)\delta^{(D)}(\boldsymbol{k})\mathrm{d}^D\boldsymbol{k}_{23}\qquad(79)$$

where $P\left(\frac{1}{x}\right)$ represents the Cauchy principal value for the integral. For the case $l_1=-l_2$ and using the relation $\sum_s Lb^{l_2ss_2s_3}_{\boldsymbol{p}_2\boldsymbol{k}\boldsymbol{k}_2\boldsymbol{k}_3}=\tilde{Lb}^{l_2s_2s_3}_{\boldsymbol{p}_2\boldsymbol{k}_2\boldsymbol{k}_3}\frac{(L_k+1)}{\sqrt{1-L_k^2}}$ one finally obtains the kinetic equation (41) governing the evolution of $n_B(\boldsymbol{p},t)$.

In order to derive the kinetic equation (43) for the evolution of the condensate amplitude n_0, we must pursue the expansion up to order ϵ^4. This is because at order ϵ^3 a suitable choice of the constant of integration $C_2^{(n)}$ gives $(q_0^{(1)-l_1}A_3^{(1)l_1}+q_0^{(1)l_1}A_3^{(1)-l_1})=0$. In this way one only gets a frequency renormalization at order ϵ^3,

$$F_3^{(1)l_1}=-\mathrm{i}al_1(2\pi)^Dq_0^{(1)l_1}(q_0^{(1)-l_1}C_3^{(1)l_1}+q_0^{(1)l_1}C_3^{(1)-l_1}).$$

At order ϵ^4, after a long, but similar calculation, one obtains

$$F_4^{(1)l_1}=6a^2(2\pi)^D\mathcal{P}_{123}\sum_{s_1s_2s_3}\int\tilde{L}^{l_1s_1s_2s_3}_{0\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}\tilde{Lb}^{s_3-s_1-s_2}_{\boldsymbol{k}_3-\boldsymbol{k}_1-\boldsymbol{k}_2}\sqrt{n_0}(q_{B0}^{(2)-s_1s_1}(\boldsymbol{k}_1)q_{B0}^{(2)-s_2s_2}(\boldsymbol{k}_2))$$
$$\times\left(\pi\delta(\Omega^{s_1s_2s_3}_{k_1k_2k_3})+iP\left(\frac{1}{\Omega^{s_1s_2s_3}_{k_1k_2k_3}}\right)\right)\delta^{(D)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\mathrm{d}^D\boldsymbol{k}_{123}-\mathrm{i}al_1(2\pi)^D\left(q_0^{(1)-l_1}C_4^{(1)l_1}+q_0^{(1)l_1}C_4^{(1)-l_1}\right)q_0^{(1)l_1},\qquad(80)$$

where

$$\tilde{L}^{ss_1s_2s_3}_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}=is\frac{q_0^{(1)l_1}}{24\sqrt{n_0}}u_{k_1}u_{k_2}u_{k_3}\mathcal{P}_{123}(-ss_1s_2s_3(L_{k_1}-1)(L_{k_2}-1)(L_{k_3}-1)$$
$$+ss_1(L_{k_1}-1)(L_{k_2}+1)(L_{k_3}+1)+s_2s_3(L_{k_1}+1)(L_{k_2}-1)(L_{k_3}-1)-(L_{k_1}+1)(L_{k_2}+1)(L_{k_3}+1)).\qquad(81)$$

Using the relation $\tilde{Lv}^{s_1s_2s_3}_{\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}\sqrt{n_0}=\tilde{L}^{l_1s_1s_2s_3}_{0\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}q_0^{(1)-l_1}+\tilde{L}^{-l_1s_1s_2s_3}_{0\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}q_0^{(1)l_1}$ one obtains $\frac{\mathrm{d}}{\mathrm{d}t}n_0=\epsilon^4(2\pi)^D(q_0^{(1)l_1}F_4^{(1)-l_1}+F_4^{(1)l_1}q_0^{(1)-l_1})$, which corresponds to the equation governing the evolution of the condensate amplitude given in Eq. (43).

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