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Citation: Physics of Plasmas **23**, 056301 (2016); doi: 10.1063/1.4943108
View online: https://doi.org/10.1063/1.4943108
View Table of Contents: http://aip.scitation.org/toc/php/23/5
Published by the American Institute of Physics

PHYSICS OF PLASMAS **23**, 056301 (2016)

A statistical model for relativistic quantum fluids interacting with an intense electromagnetic wave

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(Received 4 December 2015; accepted 17 December 2015; published online 7 March 2016)

A statistical model for relativistic quantum fluids interacting with an arbitrary amplitude circularly polarized electromagnetic wave is developed in two steps. First, the energy spectrum and the wave function for a quantum particle (Klein Gordon and Dirac) embedded in the electromagnetic wave are calculated by solving the appropriate eigenvalue problem. The energy spectrum is anisotropic in the momentum K and reflects the electromagnetic field through the renormalization of the rest mass m to $\mathcal{M}=\sqrt{m^2+q^2A^2}$. Based on this energy spectrum of this quantum particle plus field combination (QPF), a statistical mechanics model of the quantum fluid made up of these weakly interacting QPF is developed. Preliminary investigations of the formalism yield highly interesting results—a new scale for temperature, and fundamental modification of the dispersion relation of the electromagnetic wave. It is expected that this formulation could, inter alia, uniquely advance our understanding of laboratory as well as astrophysical systems where one encounters arbitrarily large electromagnetic fields. © *2016 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4943108]

I. INTRODUCTION

This paper is an attempt to develop a framework for dealing with extremely high energy density matter, in particular, an assembly of particles in the field of an electromagnetic (EM) wave of arbitrary strength. To investigate this rather extreme state of matter, we need a theory that is both quantum and relativistic.

Introduction of quantum effects in relativistic plasmas has followed several routes: models varying from fluids to kinetic, and from a simple quantum approach via Dirac equation to quantum field theoretical approach, have been proposed and studied.[1–10] Other notable mentions are a quantum electrodynamics model to describe a relativistic degenerate electron Fermi gas,[11] and studies of magnetized and unmagnetized relativistic quantum plasmas via kinetic covariant methods,[12,13] and from fluid theories.[8] Interestingly enough, the relatively simple Klein-Gordon (KG) equation has been often invoked to describe spinless electrons and positrons. KG based models have been constructed for both fluid-like[14–18] and kinetic systems.[19,20]

In most of the quantum plasma literature, the quantum equations (Schrodinger, Pauli, Klein-Gordon, and Dirac) are cast into fluidized variables via Madelung transforms, appropriately averaged to write down the fluid equations. The idea is that the standard quantum equations of motion can be translated into equivalent equations of "classical" particles whose dynamics is determined by "quantum forces" (like the gradient of the Bohm potential) in addition to the external forces.

We explore, here, an alternative route towards "fluidization." In this paper, we will study two of the most representative relativistic quantum systems: the assembly of KG and Dirac particles interacting with an electromagnetic field. We construct a quantum fluid by (1) directly solving the relevant quantum equations as eigenvalue problems: finding the microscopic energy spectrum and the corresponding eigenfunctions, and (2) constructing fluid variables (fluxes, currents, and energy momentum tensor) by taking quantum statistical averages. The calculated energy spectrum will be used to define the "Boltzmann" factor that decides the probability of a given micro state. The quantum fluid equations are, then, simply the standard conservation laws. The similarities between our formalism and an earlier one based on the Volkov solution of the Dirac equation[9] are mentioned in Sec. III B.

In order to build a conceptual framework for solving the eigenvalue problem describing the combined quantum particle + field (QPF) system, we take our electromagnetic field to be that of a circularly polarized EM wave (CPEM). Special properties of the CPEM provide tremendous simplification that lets us perform analytic calculations. The general methodology, however, can be applied to any form of the electromagnetic field; the system, however, might need considerable numerical input. Notice that similar attempts have been performed for the Dirac equation[21] in a simpler scheme.

The properties of the QPF system must depend on the details of the solution of, what may be called, the microscopic eigenvalue problem; the calculated energy spectrum will serve as the basis for constructing the statistical mechanics of the QPF.

The solution of the QPF (both for KG and Dirac particles) eigenvalue problem reveals expected but highly interesting features:

(1) The effective mass associated with the QPF quantum is the combination of the particle inertial mass (m) and the inertia contributed by the electromagnetic field $|qA|$.

Note: Paper UI3 3, Bull. Am. Phys. Soc. **60**, 353 (2015).
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Since this combination occurs repeatedly, we will call it the field-renormalized mass

$$M^2_{field-renormalized} \equiv \mathcal{M}^2 = m^2 + q^2A^2. \tag{1}$$

We will see later that change from m to $\mathcal{M}$ has profound consequences for setting the temperature scale in statistical averaging.

(2) The energy spectrum is always anisotropic in the particle momenta, i.e., the momenta parallel and perpendicular to the direction of the wave propagation appear with different weights. The extent of the anisotropy, just like the effective mass $\mathcal{M}$, depends on the wave intensity, measured by q^2A^2; the anisotropy is also a function of other wave characteristics like ω and k. Though a totally expected effect (since the propagating electromagnetic wave breaks the rotational symmetry by introducing a preferred direction), the anisotropy in the energy spectrum (implying an anisotropic energy momentum tensor) for the QPF system spells fundamental consequences.

(3) For the more complicated Dirac case, the interaction with the electromagnetic wave, in addition, splits the energy spectrum; the degeneracy between the positive and negative energy states and between the spin up and spin down states in either class is broken. Thus, the statistical mechanics of the QPF will, again, be profoundly different from that of the standard Dirac electron (without the field).

The content of the preceding statements will become more accessible and easier to appreciate as we proceed with the calculations. The general results obtained in this quantum plasma theory will have widespread applicability, particularly for processes in high energy plasma-laser interactions.[21,22]

Since we plan to solve single particle Klein-Gordon and Dirac equations in the presence of a CPEM, we begin with a brief description of CPEM.

II. THE CIRCULARLY POLARIZED ELECTROMAGNETIC WAVE

The contravariant components of electromagnetic four potential A^μ of a CPEM, propagating in the z-direction, may be written as

$$A^0 = 0 = A^z, \quad A^x = A\cos(\omega t - kz), \\ A^y = -A\sin(\omega t - kz), \tag{2}$$

where A is its constant amplitude, and the four wave vector $k^\mu = [\omega, 0, 0, k]$ in the lab frame. Notice that $k^\mu A_\mu = 0 = \mathbf{k}\cdot\mathbf{A}$. Why CPEMs are the most amenable to analytical calculations is underwritten by the fact that

$$A^\mu A_\mu = (A^0)^2 - A^iA_i = A^2 \tag{3}$$

is a constant. In this paper, the Minkowski signature tensor is taken to be $\eta^{\mu\nu} = \text{diag}[1, -1, -1, -1]$. The electric and magnetic fields associated with CPEM are readily calculated to be

$$\mathbf{E} = \omega A(\hat{e}_x \sin(\omega t - kz) + \hat{e}_y\cos(\omega t - kz)), \tag{4}$$

$$\mathbf{B} = kA(-\hat{e}_x\cos(\omega t - kz) + \hat{e}_y\sin(\omega t - kz)), \tag{5}$$

implying a constant Poynting flux along z

$$\mathbf{E}\times\mathbf{B} = k\omega A^2\hat{e}_z. \tag{6}$$

III. QPF—RELATIVISTIC QUANTUM PARTICLE IN A CPEM

We will now work out the solutions of the two standard quantum relativistic equations—the Klein Gordon (K-G) and the Dirac equation—in the presence of an arbitrary amplitude CPEM wave. The K-G equation is the description of a complex scaler field, while the Dirac equation describes a spin half particle. In the first part of the calculation, we will consider the CPEM to be externally specified (2).

A. Klein Gordon equation—Wave function and energy spectrum

Invoking the minimal coupling prescription, the K-G particle of mass m and charge q, interacting with the electromagnetic field, obeys ($\hbar = 1 = c$)

$$(p_\mu + qA_\mu)(p^\mu + qA^\mu)\Psi = (i\partial_\mu + qA_\mu)(i\partial^\mu + qA^\mu)\Psi = m^2\Psi, \tag{7}$$

where $p^\mu = [p^0 = E, \mathbf{p}]$ is the energy momentum four vector of the particle. For the CPEM of (2), Eq. (7) expands as

$$\partial_t^2\Psi - \nabla^2\Psi - 2iqA[\cos(\omega t - kz)\partial_x\Psi \\ - \sin(\omega t - kz)\partial_y\Psi] + \mathcal{M}^2\Psi = 0, \tag{8}$$

where $\mathcal{M} = \sqrt{m^2 + q^2A^2}$ is the field-renormalized effective mass as alluded to in Section I. In the laser plasma literature, $\mathcal{M}$ is often written as

$$\mathcal{M} = m\sqrt{1 + \frac{q^2A^2}{m^2}} = m\gamma_f, \tag{9}$$

with γ_f interpreted as the "relativistic mass increase" factor. This, however, can create a serious conceptual problem because γ_f is a Lorentz scalar, while the conventional "mass increase factor"—the standard kinematic γ—is the zeroth component of the four velocity. In order to accurately represent the origin of the field-renormalized mass and avoid confusion, we introduce a scalar Γ_f

$$\mathcal{M} = m\sqrt{1 + \frac{q^2A^2}{m^2}} \equiv m\Gamma_f, \tag{10}$$

where the suffix f will remind us about its origin and distinguish it from the lower case kinematic γ. We will reserve the upper case Γ to denote Lorentz scalars.

Out of the four space-time dimensions, the two dimensions orthogonal to the propagation direction (z) are ignorable. We can, then, define the independent Fourier mode

$$\Psi(x, y, z, t) = e^{iK_\perp(\cos\varphi x + \sin\varphi y)}\psi(t, z), \tag{11}$$

which obeys

$$\partial_t^2\psi - \partial_z^2\psi + 2qAK_\perp\cos(\omega t - kz)\psi + ({K_\perp}^2 + \mathcal{M}^2)\psi = 0, \tag{12}$$

where $K_x = K_\perp\cos\varphi$, $K_y = K_\perp\sin\varphi$, and the phase φ has been eliminated in a trivial redefinition of the variable z.

Before embarking on solving the spatial 3-D problem embodied in (12), let us orient ourselves by solving the simple 1D (spatial) problem assuming $K_\perp = 0$. The Klein Gordon equation, then, yields the field-modified energy eigenvalue

$$E^2 = K_z^2 + \mathcal{M}^2 \equiv K_z^2 + m^2{\Gamma_f}^2, \tag{13}$$

for the wavefunction

$$\psi(z,t) = \psi_0 e^{-iEt+iK_z z}. \tag{14}$$

If one was to insist on adhering to the strict definition of the Lorentz factor γ

$$\gamma^2 = \frac{E^2}{m^2} = \frac{K_z^2}{m^2} + {\Gamma_f}^2 = 1 + \frac{{K_z}^2 + q^2A^2}{m^2}, \tag{15}$$

the preceding expression mimics the kinematic γ of a particle whose perpendicular momentum is qA. We know, however, that qA cannot correspond to the kinematic momentum of the particle since we obtained the above result by putting the perpendicular kinematic momentum $K_\perp = 0$. When we solve the full eigenvalue problem (12), we will, of course, find $\gamma = E/m$ to be explicitly dependent on $K_\perp$.

For the spatial 3-D ($K_\perp \neq 0$) dynamics, we must solve Eq. (12), a hyperbolic partial differential equation in z and t. The only combination that appears in the coefficients is the wave phase $\zeta = \omega t - kz$. One can exploit this feature to find another ignorable mixed dimension, $\rho = kt - \omega z$, and extend the solution (14) to

$$\Psi(x,y,z,t) = e^{iK_\perp(\cos\varphi x + \sin\varphi y)} e^{iK_\rho\rho}\psi(\zeta), \tag{16}$$

which reduces the eigenvalue problem to a Mathieu equation in ζ

$$\frac{d^2\psi}{d\zeta^2} + (\mu + \lambda\cos\zeta)\psi = 0, \tag{17}$$

with

$$\mu = K_\rho^2 + \frac{\mathcal{M}^2 + K_\perp^2}{\omega^2 - k^2}, \quad \lambda = \frac{2qAK_\perp}{\omega^2 - k^2}. \tag{18}$$

That the dynamics of a Klein-Gordon particle under the CPEM can be reduced to a Mathieu equation was demonstrated long ago.[23,24] Our results (for example, Eq. (17)), however, are more general since our QPF is not restricted by the standard mass-shell constraint.

A similar equation with periodic coefficients, a Hill equation, will pertain for a Klein-Gordon particle in an arbitrary amplitude linearly polarized electromagnetic wave (see Appendix A).

1. The Mathieu equation

The Mathieu equation allows solutions of the Floquet (Bloch) form

$$\psi = \psi_0 e^{is\zeta} f(\zeta), \tag{19}$$

where $f(\zeta)$ is a periodic function of ζ. The index s is determined from the exact equation

$$\sin^2(\pi s) = \Delta_0(\mu,\lambda)\sin^2(\pi\sqrt{\mu}), \tag{20}$$

where $\Delta_0(\mu,\lambda)$ is an infinite determinant, a complicated function of μ and λ. In this paper, we will not dwell in detail on analyzing the exact solution (20); we refer the interested readers to a text book like Morse and Feschbach.[25] Before working with approximate solutions, we point out a few important conceptual features:

(1) For $\lambda = 0$ ($K_\perp = 0$), $\Delta_0(\mu,\lambda) \Rightarrow 1$, yielding the simplest solution $s^2 = \mu$ that will reproduce exactly the 1D energy eigenvalue condition (15).
(2) When λ is nonzero, there do exist regions in the $\lambda - \mu$ plane where s becomes complex (regions of instability), and the wave function becomes a growing (or decaying) function of the space-time combination $\zeta = \omega t - kz$, the phase of the electromagnetic wave. This class of solutions is likely to be extremely interesting, leading to, perhaps, totally new uncharted phenomena. Physically, these solutions are likely to correspond to the particle being trapped (a quantum particle always has a finite probability for tunneling out) in the trough of the high amplitude wave. The phenomenon is analogous to the allowed and forbidden energy bands for an electron moving in the periodic potential of crystalline solids. We plan to investigate these solutions in a future publication and limit ourselves, here, only to the stable solutions (real s). Naturally, the resulting theory is not complete.
(3) The apparent simplicity of (20) is a bit of an illusion due to the very complicated determinant $\Delta_0(\mu,\lambda)$. A simpler but approximate WKB (Wentzel–Kramers–Brillouin) solution is developed in Appendix B. The normalized wave function comes out to be

$$\Psi(x,y,z,t) \equiv \Psi = \left[\frac{(\mu+\lambda)}{16K(r)^2(\mu+\lambda\cos\zeta)}\right]^{1/4} \times \exp\left[i\mathbf{K}_\perp\cdot\mathbf{x}_\perp + iK_\rho(kt-\omega z) + iQ\right], \tag{21}$$

with the eikonal

$$Q = \int(\mu+\lambda\cos\zeta)^{1/2}d\zeta = 2\sqrt{\lambda+\mu}\,\mathbf{E}\left(\frac{\zeta}{2},\gamma\right), \tag{22}$$

where $\gamma = \sqrt{2\lambda/(\lambda+\mu)}$ is the argument of the elliptic functions: $K(\gamma)$ is the complete elliptic integral of the first kind and $\mathbf{E}(\zeta/2,\gamma)$ is the elliptic function of the second kind. The associated dispersion relation is

$$E^2 - K_z^2 = (\omega^2 - k^2)\left[-K_\rho^2 + \frac{(\lambda+\mu)\pi^2}{(2K(\gamma))^2}\right], \tag{23}$$

where E and K_z are, respectively, defined as the eigenvalues of the corresponding operators

$$E = i\int\left(\psi^*\frac{\partial\psi}{\partial t}\right)d\zeta,$$

$$K_z = -i\int\left(\psi^*\frac{\partial\psi}{\partial z}\right)d\zeta.$$

(4) Even the exact WKB form makes analytical advancement difficult. For the main development of this paper, we will make a leading order approximation to the system to illustrate, what we believe, is the fundamental message: The energy eigenvalue of a quantum particle in the background of a propagating electromagnetic wave is anisotropic, that is, the energy is no longer a function of only $K^2 = K_x^2 + K_y^2 + K_z^2$, since the direction of propagation breaks the rotational symmetry. In fact, to leading order, the energy eigenvalue is, explicitly, approximated as (see Appendix B)

$$E^2 = K_z^2 + \mathcal{M}^2 + (1-\alpha)K_\perp^2 \equiv m^2 + q^2A^2 + K_z^2 + (1-\alpha)K_\perp^2, \tag{24}$$

with

$$\alpha \approx \frac{3q^2A^2}{2(\mathcal{M}^2 + K_\perp^2)}, \tag{25}$$

in the small λ/μ limit ($\alpha \ll 1$). We notice that the contributions of the parallel and perpendicular kinematic momenta (with respect to the direction of wave propagation) to the eigen-energy carry different weights; the contribution of the perpendicular part is somewhat suppressed compared with the isotropic case. We do not have a simple looking formula when λ approaches μ ($|\lambda|/|\mu|$ is always less than unity) but even in this limit, the perpendicular contribution comes with a weight factor less than unity, in fact, the weight factor can become considerably less than unity. For the rest of this section, therefore, we will work with the "model" approximation ($a = 1-\alpha < 1$)

$$E^2 = K_z^2 + \mathcal{M}^2 + (1-\alpha)K_\perp^2 \equiv m^2 + q^2A^2 + K_z^2 + a^2K_\perp^2, \tag{26}$$

as it encompasses the essential qualitative feature of the dynamics of a quantum KG particle in an arbitrary electromagnetic wave.

It must be emphasized that a is, in general, a complicated function of the wave attributes (ω, k) and the wave amplitude measured by $|qA|$; a must approach unity as $A \to 0$.

2. The Klein Gordon current

For the K-G particle in an electromagnetic field, the probability current has an induced part in addition to the standard kinematic part

$$j^\mu = \frac{1}{2i\mathcal{M}}[\Psi^*\partial^\mu\Psi - (\partial^\mu\Psi^*)\Psi] + \frac{q}{\mathcal{M}}\Psi^*\Psi A^\mu, \tag{27}$$

where the current has to take into account of the effective mass.

As expected, the induced current is proportional to the vector potential. The quantum expectation value of this current (for the wave function normalized to unity) is readily calculated to be

$$J^\mu = \frac{1}{\mathcal{M}}[E, (\mathbf{K}_\perp + q\mathbf{A}_\perp), K_z]. \tag{28}$$

Equations (21), (23) and its simplified version (26), and (28) constitute the complete approximate analytical solution of the eigenvalue problem associated with a K-G particle in an arbitrary amplitude CPEM field.

B. A Dirac particle in an intense electromagnetic field of a CPEM

In the presence of an electromagnetic field, the Dirac equation reads

$$\gamma^\mu(i\partial_\mu + qA_\mu)\Psi = m\Psi, \tag{29}$$

where Ψ is the four component spinor

$$\Psi = \begin{pmatrix}\varphi_1\\ \chi_1\\ \varphi_2\\ \chi_2\end{pmatrix}, \tag{30}$$

and γ^μ are the 4×4 Dirac matrices. The vector potential corresponding to the arbitrary amplitude CPEM was given in (2), i.e., $\mathbf{A} = A(\hat{e}_x\cos\zeta - \hat{e}_y\sin\zeta)$, $\zeta = \omega t - kz$.

For the K-G system, the 3D (spatial) eigenvalue problem was reducible to an ordinary differential equation [the Mathieu equation (17)]. The equivalent Dirac electron problem is algebraically far more complicated (leading to four coupled generalized Hill equations) and will be dealt with in a later report. We will resort here to an analytically solvable, relatively simpler one spatial D (the wave function is a function of z and t, $\partial_1 = 0 = \partial_2$) system.

With the simplicity afforded for the CPEM and the 1D assumption, the following wave function:

$$\begin{aligned}\varphi_1 &= P_1e^{i\zeta/2 - i(Et-K_zz)}\\ \varphi_2 &= P_2e^{i\zeta/2 - i(Et-K_zz)}\\ \chi_1 &= Q_1e^{-i\zeta/2 - i(Et-K_zz)}\\ \chi_2 &= Q_2e^{-i\zeta/2 - i(Et-K_zz)}\end{aligned} \tag{31}$$

solves the Dirac equation. When (31) is substituted into (29), the system transforms to the following set of homogeneous equations in the amplitudes (P_1, P_2, Q_1, Q_2):

$$\begin{aligned}(W_- - m)P_1 - K_-P_2 &= qAQ_2,\\ (W_+ - m)Q_1 + K_+Q_2 &= qAP_2,\\ (W_- + m)P_2 - K_-P_1 &= qAQ_1,\\ (W_+ + m)Q_2 + K_+Q_1 &= qAP_1;\end{aligned} \tag{32}$$

the solvability condition for (32) yields the eigenvalue condition

$$(W_-^2 - K_-^2 - \mathcal{M}^2)(W_+^2 - K_+^2 - \mathcal{M}^2) = q^2A^2(k^2 - \omega^2), \quad (33)$$

where $W_\pm = E \pm \omega/2$ and $K_\pm = K_z \pm k/2$ are, respectively, the shifted energy and z-momentum of the particle; the shift is caused by the electromagnetic wave that affects different spin states in different ways. As per assumption, the Dirac particle associated with the wave function (31) has no kinematic momentum in the directions perpendicular to the wave propagation ($K_\perp = 0$). Notice that, just as in the K-G case, the particle mass has been replaced by the field renormalized mass $\mathcal{M} = \sqrt{m^2 + q^2A^2}$.

At this time, we remind the reader that, in the standard representation of the γ^μ matrices, employed in this paper, φ_1 and χ_1 (φ_2 and χ_2) are, respectively, the spin up and spin down components of the positive energy (negative energy) solution. Naturally, for the four component Dirac electron, the eigenvalue equation (29) is fourth order in E and has four roots.

In order to complete the explicit solution of the Dirac electron interacting with an arbitrary amplitude circularly polarized electromagnetic wave, we must solve the amplitudes assuming, for instance, the normalization $P_1^2 + P_2^2 + Q_1^2 + Q_2^2 = 1$. The reader is referred to Appendix C for details. We just display here a few salient steps. The Dirac current, written in the Gordon decomposed form (that explicitly reveals the spin current), reads

$$J_D^\mu = \bar{\Psi}\gamma^\mu\Psi = \frac{q}{m}A^\mu\bar{\Psi}\Psi + \frac{i}{2m}(\bar{\Psi}\partial^\mu\Psi - \partial^\mu\bar{\Psi}\Psi) + \frac{1}{2m}\partial_\nu(\bar{\Psi}\sigma^{\mu\nu}\Psi), \quad (34)$$

with

$$\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu); \quad (35)$$

the first term is the field induced current, the second is the K-G like kinematic current, and the third is the spin current. These are readily calculated in terms of the amplitudes (Ps and Qs) and we display here the zero and the perpendicular parts

$$J_D^0 = \hat{n}_0 = P_1^2 + P_2^2 + Q_1^2 + Q_2^2,$$
$$\mathbf{J}_D^\perp = \frac{q}{m}\mathbf{A}^\perp(P_1^2 + Q_1^2 - P_2^2 - Q_2^2) + \frac{\Lambda}{mA}\mathbf{A}^\perp, \quad (36)$$

with

$$\Lambda = \omega(P_2Q_1 - P_1Q_2) - k(P_1Q_1 - P_2Q_2), \quad (37)$$

where $\hat{n}_0$ stands for the probability density. Notice that all four components contribute symmetrically to the density, but very asymmetrically to the current. In fact, in the first term in $\mathbf{J}_D^\perp$, the negative energy contribution (from P_2, Q_2) is in exact opposition to the current from the positive energy components (P_1, Q_1) as they must since antiparticles must carry an opposite charge.

The Dirac solution (the eigenfunction, the energy eigenvalue, and the Dirac current), even when the wave function is independent of the perpendicular coordinates, is considerably more complicated than the corresponding K-G solution, particularly because of the additional spin degree of freedom. By the same token, it is endowed with much richer physics. For example, as the energy of the Dirac particle rises, the amplitude of the negative energy (positron) component becomes larger implying that the pair production is generic to Dirac equation. Although pair production is, normally, identified in the second quantization of Dirac theory, it is a phenomena that is inherently built into the Dirac equation itself; it has been explicitly shown in solutions[29] and in the resolution of the Klein paradox.[30,31]

The intent of this paper as well as our formalism and results are very different from a recent work[9] that, building on the Volkov solution[32,33] of the Dirac equation, studies the dynamics of one electron in an electromagnetic field in a classical plasma. Reference 9 solution is derived in what may be termed as the "non-interacting" limit: neither the electromagnetic dispersion ($\omega^2 - k^2 = \omega_p^2$) nor the electron dispersion (energy-momentum relationship, $E^2 - p^2 = m^2$) is changed due to the presence of the other. In contrast, the single particle part of our formalism focusses on precisely the interaction between the two constituents; the wave dispersion and the particle energy momentum relationship are profoundly affected by the other. The qualitative difference between the current and the Volkov solution can be appreciated by the fact that the expression for electron "energy" fully reflects the effects of the electromagnetic wave; it depends, non-trivially, on the amplitude, the frequency, and the wavenumber of the EM wave.

Additionally, the induced electron current strongly modifies the wave dispersion relation; the modifications include the changes due to the EM field-induced effective mass that lowers the "plasma frequency;" the effective mass $\mathcal{M}$ could be much bigger than m for large amplitude waves.

The guiding aim of the current work, however, is to develop a statistical model for a system of interacting electron waves and arbitrary strength electromagnetic field. To construct the correct Boltzmann factor for this highly interacting system, we need a detailed knowledge of the energy spectrum. A fundamental warranted physical effect, which the electron energy be anisotropic in the momenta parallel and perpendicular to the direction of propagation, is automatically recovered.

IV. CONSTRUCTING A QUANTUM PLASMA—STATISTICS MECHANICS

We have just derived analytical "solutions," in particular, the energy eigenvalues for the Klein-Gordon and the Dirac particles interacting with an arbitrary amplitude CPEM. These will, now, serve as defining the micro-state of the constituent particles of a relativistic K-G or Dirac fluid. By developing a straightforward statistical mechanics of these systems, we will derive the macroscopic characteristics of these fluids, the fluxes, the energy momentum tensor, the averaged induced current, etc. We will, then, discuss the profound changes emerging from the fact that the invoked microscopic state was obtained by calculating the exact dynamics of the particle embedded in an intense EM field. In a sense, the particle micro-state is the combined state of the particle plus the EM field or a particle renormalized state.

Once we have the energy spectrum of the quantum particles, we can build the appropriate statistical mechanics so that we can calculate the ensemble averages of physical quantities for a fluid or a gas of such particles.

A. Statistical averages of Klein-Gordon solution

The K-G particles are spin-less bosons and we should, in principle, use the B-E statistics to determine ensemble averages. However, for conceptual demonstration, an explicit analytic calculation may be more revealing. So, we will treat the K-G assembly as a relativistic Maxwell- Boltzmann gas, the high temperature limit of the B-E gas.

The covariant relativistic M-B distribution is

$$f = Ne^{-\frac{K_\mu U^\mu}{T}}, \tag{38}$$

where U^μ is the bulk four velocity of the K-G fluid, and $K^\mu = (E, aK_x, aK_y, K_z)$ is the microscopic particle energy-momentum four vector obeying $K^\mu K_\mu = \mathcal{M}^2$; the new invariant mass of the particle is the field renormalized mass $\mathcal{M}$.

We will calculate the averages in the fluid rest frame $U^\mu = [1, 0, 0, 0]$; the physical quantities for a moving fluid can be obtained by an appropriate Lorentz transformation. Thus, the only nonzero component of the flux is

$$\Gamma^\mu = \int \frac{d^3K}{E} K^\mu f_R, \tag{39}$$

with the rest frame distribution

$$f_R = Ne^{-\frac{E}{T}} = N \exp\left[-\frac{\sqrt{K_z^2 + \mathcal{M}^2 + a^2 K_\perp^2}}{T}\right]. \tag{40}$$

The density is (from now on, unless otherwise stated, all physical quantities pertain to the rest frame)

$$n_R \equiv \Gamma^0 = \frac{N\mathcal{M}^3}{a^2} \int d^3p \exp\left(-\zeta\sqrt{1+p^2}\right), \tag{41}$$

where $\zeta = \mathcal{M}/T$, and we have defined the adjusted momentum variables $p_z = K_z/\mathcal{M}$, $p_x = aK_x/\mathcal{M}$, $p_y = aK_y/\mathcal{M}$. In the p variables ($p_0 = E/\mathcal{M}$), $p^\mu p_\mu = 1$. On carrying out the integral, we can fix the normalization constant N in terms of n_R, yielding

$$N = \frac{a^2 n_R \zeta}{4\pi\mathcal{M}^3 K_2(\zeta)} \Rightarrow f_R = \frac{a^2 n_R \zeta}{4\pi\mathcal{M}^3 K_2(\zeta)} e^{-E/T}. \tag{42}$$

Next step is to evaluate the energy momentum tensor

$$T_R^{\mu\nu} = \int \frac{d^3K}{E} K^\mu K^\nu f_R. \tag{43}$$

Because f_R is a function of K_z^2 and $K_\perp^2$, only diagonal elements survive. The energy density

$$\begin{aligned}\mathcal{E} = T_R^{00} &= \int d^3K E f \\ &= \frac{n_R\zeta\mathcal{M}}{K_2(\zeta)} \int d^3p\sqrt{1+p^2}\exp\left(-\zeta\sqrt{1+p^2}\right) \\ &= \frac{n_R\zeta\mathcal{M}}{K_2(\zeta)} \frac{d^2}{d\zeta^2}\left(\frac{K_1(\zeta)}{\zeta}\right),\end{aligned} \tag{44}$$

the parallel pressure ($P_\parallel = P_z = T_R^{zz}$)

$$\begin{aligned}P_\parallel = \int \frac{d^3K}{E} K^z K^z f = \frac{n_R\zeta\mathcal{M}}{K_2(\zeta)} \int_0^\infty dp p^2 \\ \times \frac{p_z^2}{\sqrt{1+p^2}} \exp\left(-\zeta\sqrt{1+p^2}\right) = nT,\end{aligned} \tag{45}$$

and the perpendicular pressure ($P_\perp = T_R^{xx} = T_R^{yy}$)

$$P_\perp = \frac{n_R T}{a^2} = \frac{P_\parallel}{a^2}. \tag{46}$$

The differentiated parallel and perpendicular particle response to high intensity electromagnetic wave (propagating along z-axis) creates the pressure anisotropy; the effective perpendicular temperature (T/a^2) turns out to be greater than the parallel temperature T.

It is, perhaps, necessary to stress that, though the exact solution of the eigenvalue problem (to be presented in a later work) is considerable more involved than (26), the featured asymmetry between the parallel and perpendicular (and symmetry in the two perpendicular directions) is always maintained in this model problem.

1. General energy momentum tensor—Quantum KG in CPEM

By using standard methods (see, for instance, Ref. 27), we can construct a general (special relativistic, for this paper) energy momentum tensor $T^{\mu\nu}$ for the K-G fluid that, in the rest frame, reduces to $T_R^{\mu\nu} = diag[\mathcal{E}, P_\parallel, P_\perp, P_\perp]$ of Eqs. (44)–(46) of Section IV A. This tensor, capturing the qualitative essence of the system, must be of the form

$$T^{\mu\nu} = -\eta^{\mu\nu}P_\parallel + \beta^{\mu\nu}(P_\parallel - P_\perp) + (\epsilon + P_\parallel)U^\mu U^\nu, \tag{47}$$

where U^μ, as usual, is the fluid four velocity, $\eta^{\mu\nu} = (1, -1, -1, -1)$, and $\beta^{\mu\nu}$ is the tensor that weights the anisotropy $(P_\parallel - P_\perp)$ contribution. There are several constraints on $\beta^{\mu\nu}$. First, it must be constructed from the tensors peculiar to the system; for a plasma interacting with an intense electromagnetic wave, these are: the vector potential A^μ, the Faraday tensor $F^{\mu\nu}$, and the wave four vector k^μ. Second, in order to recover the known rest frame form, the diagonal terms must be $\beta^{00} = 0 = \beta^{33}$ and $\beta^{11} = -1 = \beta^{22}$.

It is straightforward to show that for the circularly polarized electromagnetic wave, the dimensionless tensor

$$\beta^{\mu\nu} = \frac{F^{\mu\kappa}F_\kappa{}^\nu}{F^{\alpha\beta}F_{\alpha\beta}} + \frac{k^\mu k^\nu}{k^\alpha k_\alpha} - \frac{A^\mu A^\nu}{A^\alpha A_\alpha} \tag{48}$$

fits the bill; in fact for this tensor, all nondiagonal terms are zero, i.e., $\beta^{\mu\nu} = 0$, for $\mu \neq \nu$. We will come back for discussing the general implications of the work after we derive the corresponding results for the relativistic Dirac equation since our eventual interest is to deal with a system of electrons.

2. Klein-Gordon current—Closure with Maxwell's equations

Till now, we have pretended that the CPEM was given and we were simply calculating the particle response: the eigenstate, fluxes, the induced current, etc. This situation might pertain in some situations. However, the more general problem would entail demonstrating the consistency between the induced currents and the EM field through the Maxwell equations. Such consistency, of course, will require a solubility constraint, the dispersion relation of the wave in the given charged fluid. The KG microscopic particle current (28), on averaging over distribution (40), yields

$$\langle J_0\rangle = \frac{n_R}{\mathcal{M}}, \tag{49}$$

$$\langle \mathbf{J}_\perp\rangle = \frac{q n_R K_1(\zeta)}{\mathcal{M} K_2(\zeta)}\mathbf{A}_\perp. \tag{50}$$

Thus, we find that only the induced current [proportional to $(\mathbf{A}_\perp)$] survives statistical averaging. The corresponding electrical current will be $q\langle \mathbf{J}_\perp\rangle$, which, when substituted into the nontrivial perpendicular components of the Maxwell equations, converts them to

$$\frac{\partial^2 \mathbf{A}_\perp}{\partial t^2} - \nabla^2 \mathbf{A}_\perp = -\frac{4\pi q^2 n_R K_1(\zeta)}{\mathcal{M} K_2(\zeta)}\mathbf{A}_\perp\,, \tag{51}$$

yielding the dispersion relation

$$\omega^2 - k^2 = \frac{{\omega_p}^2}{\Gamma_f \Gamma_{th}}, \tag{52}$$

where $\omega_p = \sqrt{4\pi q^2 n_R/m}$ is the invariant (rest frame) plasma frequency, $\Gamma_f = \mathcal{M}/m$ is the field-renormalized mass enhancement factor, and $\Gamma_{th} = K_2(\zeta)/K_1(\zeta) > 1$ (with $\zeta = \sqrt{(m^2+q^2A^2)}/T$) signifies the thermal enhancement of the particle inertia. Notice that, for this equation to be Lorentz invariant, the right hand side must be a Lorentz scalar since the left hand side $\omega^2 - k^2 = k^\mu k_\mu$ is a Lorentz scalar. Thus, both the gamma factors are numbers. Both of these factors tend to decrease the effective plasma frequency so that the plasma becomes more transparent when its temperature is high and when a high amplitude wave is incident. It is important, at this, juncture to discuss the significance what we have just calculated:

(1) The induced transparency, measured by the decrease in the effective plasma frequency, is caused by a product of two effects: (1) the particle mass enhanced via the EM field renormalization (through Γ_f) and (2) the further particle inertia enhancement through thermal effects manifesting through Γ_{th}. In the laser fusion literature, the transparency due to Γ_f is, often, called relativistic transparency and is attributed to the relativistic mass increase: this interpretation aught to be modified in the context of the current calculation, which clearly shows that the origin of Γ_f is entirely due to the inertia contributed by the electromagnetic field to the particle-electromagnetic system, and has does not directly reflect the kinematic $\gamma = \sqrt{1+K^2/m^2}$ (which measures the standard kinematic mass increase).

(2) The thermal enhancement factor, though a somewhat complicated function of the argument $\zeta = \sqrt{(m^2+q^2A^2)}/T$, registers the field enhanced inertia $\sqrt{(m^2+q^2A^2)}$ rather than the "bare" mass m. This is a profound change emerging from this formalism; the thermal effects are cognizant of the strength of the field in which the particle is immersed.

(3) The fact that Γ_{th} is a function of $\sqrt{(m^2+q^2A^2)}/T$ rather than m/T is profound because it resets the scale of temperature. In the standard framework, the temperature would be considered super-relativistic if $T \gg m$ and non-relativistic if $T \ll m$. In the former (latter) case, one could use the small (large) argument expansion to obtain the limiting expressions. But in the current framework, $\mathcal{M}$ replaces m, and a 10 MeV electron ($m = 0.5$ MeV) temperature (for instance) that would be super relativistic for the standard case becomes rather "sub-relativistic" when it is embedded, say, in a field with $|qA|/m = 100$. For the field renormalized particle, its effective temperature goes down to $Tm/\mathcal{M}$ which for strong fields can be $\ll T$.

(4) The induced current and hence the induced transparency have come out to be independent of the anisotropy factor a^2. This, however, is a consequence of our approximations; the anisotropy will appear when we use the exact solution of the eigenvalue problem (the Mathieu equation (17)) to evaluate the induced current.

B. Statistical averages of Dirac solution

Similarly, we can perform macroscopic averages for the Dirac particles using Fermi-Dirac statistics[26] and the approximate energy relation (C25). Invoking the Fermi-Dirac distribution function ($Z = \exp(\mu/T)$, μ is the chemical potential)

$$f = \frac{1}{Z^{-1}\exp(E/T)+1}, \tag{53}$$

we first calculate the macroscopic density $n = n_+ + n_-$, comprising the spin up $(+)$ and the spin down$(-)$ contributions[26]

$$n_\pm = \int d^3K \frac{1}{Z^{-1}\exp(E_\pm/T)+1} = \frac{\mathcal{M}_\pm^3 D_1(Z,\zeta_\pm)}{a^2\sqrt{1\pm\beta}}, \tag{54}$$

where we have set the normalization constant as unity, $\zeta_\pm = \mathcal{M}_\pm/T$, $\mathcal{M}_\pm = \mathcal{M}\sqrt{1\pm 2\beta}$, $\beta = k/(2\mathcal{M}) \ll 1$, and the function D_1 is defined in Eq. (D1) (Appendix D). Note that the spin up-down degeneracy has been removed by the electromagnetic field.

Repeating the procedure followed in Section IV A 2, we can calculate the energy density, $\mathcal{E} = \mathcal{E}_+ + \mathcal{E}_-$, for the Dirac collection:[26]

$$\mathcal{E}_\pm = \int d^3K \frac{E_\pm}{Z^{-1}\exp(E_\pm/T)+1} = \mathcal{M}_\pm n_\pm \frac{D_2(Z,\zeta_\pm)}{D_1(Z,\zeta_\pm)}, \quad (55)$$

where D_2 is defined in (D2). Finally, the parallel and perpendicular pressures for each state are

$$P_{\parallel\pm} = \frac{1}{3}\int d^3K \frac{K_z^2}{E_\pm}\frac{1}{Z^{-1}\exp(E_\pm/T)+1} = \frac{\mathcal{M}_\pm n_\pm}{3(1\pm\beta)}\frac{D_3(Z,\zeta_\pm)}{D_1(Z,\zeta_\pm)}, \quad (56)$$

and

$$P_{\perp\pm} = \frac{1}{3}\int d^3K \frac{K_\perp^2}{E_\pm}\frac{1}{Z^{-1}\exp(E_\pm/T)+1} = \frac{\mathcal{M}_\pm n_\pm}{3a^2}\frac{D_3(Z,\zeta_\pm)}{D_1(Z,\zeta_\pm)}, \quad (57)$$

where D_3 is defined in (D3) (see Appendix D).

1. Weakly degenerate states

When $Z < e^{\zeta_\pm}$, the Dirac system will be weakly degenerate. In this limit, the integrals simplify (see Eqs. (D6)–(D8)), and the relevant physical quantities are expressed as

$$\mathcal{E}_\pm \approx \frac{\mathcal{M}_\pm n_\pm \zeta_\pm}{K_2(\zeta_\pm)}\frac{d^2}{d\zeta_\pm^2}\left(\frac{K_1(\zeta_\pm)}{\zeta_\pm}\right) + \frac{Z\mathcal{M}_\pm n_\pm}{2K_2(\zeta_\pm)} \times\left(\frac{K_3(\zeta_\pm)K_2(2\zeta_\pm)}{K_2(\zeta_\pm)} - K_3(2\zeta_\pm) - \frac{K_2(2\zeta_\pm)}{2\zeta_\pm}\right), \quad (58)$$

$$P_{\parallel\pm} \approx \frac{n_\pm T}{1\pm\beta}\left(1+\frac{ZK_2(2\zeta_\pm)}{4K_2(\zeta_\pm)}\right), \quad (59)$$

$$P_{\perp\pm} \approx \frac{n_\pm T}{a^2}\left(1+\frac{ZK_2(2\zeta_\pm)}{4K_2(\zeta_\pm)}\right), \quad (60)$$

where K_n are the modified Bessel function of second kind of order n. Notice that each one of the formulas has the appropriate K-G like term and a correction, proportional to Z, reflecting the (weak) degeneracy effect.

When $\beta \ll 1$, the physical quantities corresponding to the two spin states differ by a factor proportional to β. We list here the differences in the density, energy, and pressure contributions

$$\frac{n_+ - n_-}{n_+ + n_-} \approx \frac{5\beta}{2}, \quad (61)$$

$$\frac{\mathcal{E}_+ - \mathcal{E}_-}{\mathcal{E}_+ + \mathcal{E}_-} \approx \frac{7\beta}{2}, \quad (62)$$

$$\frac{P_{\parallel+} - P_{\parallel-}}{P_{\parallel+} + P_{\parallel-}} \approx 5\beta, \quad \frac{P_{\perp+} - P_{\perp-}}{P_{\perp+} + P_{\perp-}} \approx 7\beta. \quad (63)$$

2. Degenerate states

Approximate expressions for the integrals can also be worked out for the strongly degenerate Dirac gas defined by $Z \gg e^{\zeta_\pm}$. In this limit (see (D9)–(D11)), the density is calculated to be

$$n_\pm \approx \frac{\mathcal{M}_\pm^3}{3\zeta_\pm^3 a^2\sqrt{1\pm\beta}}\left[(\ln Z)^2 - \zeta_\pm^2\right]^{3/2} \approx \frac{T^3}{3a^2\sqrt{1\pm\beta}}\left[(\ln Z)^2 - \zeta_\pm^2\right]^{3/2}, \quad (64)$$

implying that the Fermi energy (chemical potential at zero temperature) for the two-states $\varepsilon_{F\pm}$ is related to the density as ($\ln Z = \varepsilon_F/T$)

$$\varepsilon_{F\pm}^2 - \mathcal{M}_\pm^2 \propto (n_\pm)^{2/3}. \quad (65)$$

This is, indeed, the correct limit for a relativistic degenerate electron gas, but now it comes with the field-modified effective mass $\mathcal{M}_\pm$.

Similarly, the macroscopic energy densities (55) simplify to

$$\mathcal{E}_\pm \approx \frac{3}{4}n_\pm T\left(\ln Z + \frac{\zeta_\pm^2}{2\ln Z}\right) = \frac{3}{4}n_\pm\varepsilon_{F\pm}\left(1+\frac{\mathcal{M}_\pm^2}{2\varepsilon_{F\pm}^2}\right). \quad (66)$$

Since the Fermi energy can be calculated exactly from (64), above expression evaluates the energies completely in terms of the defining parameters of the assembly of Dirac particles dressed by the electromagnetic field; the latter imparts an effective mass $\mathcal{M}_+(\mathcal{M}_-)$ to the spin up (down) states. Similarly, the parallel and perpendicular pressures are now

$$P_{\parallel\pm} \approx \frac{n_\pm T}{4(1\pm\beta)}\left(\ln Z - \frac{3\zeta_\pm^2}{2\ln Z}\right) = \frac{n_\pm\varepsilon_{F\pm}}{4(1\pm\beta)}\left(1-\frac{3\mathcal{M}_\pm^2}{2\varepsilon_{F\pm}^2}\right), \quad (67)$$

$$P_{\perp\pm} \approx \frac{n_\pm T}{4a^2}\left(\ln Z - \frac{3\zeta_\pm^2}{2\ln Z}\right) = \frac{n_\pm\varepsilon_{F\pm}}{4a^2}\left(1-\frac{3\mathcal{M}_\pm^2}{2\varepsilon_{F\pm}^2}\right). \quad (68)$$

From Eqs. (66)–(68), we note that in, the standard limit of an isotropic, extreme relativistic gas, the pressure $P \to \mathcal{E}/3$, as it should.

In this limit, the spin up and down contributions differ as

$$\frac{n_+ - n_-}{n_+ + n_-} \approx \beta, \quad \frac{\mathcal{E}_+ - \mathcal{E}_-}{\mathcal{E}_+ + \mathcal{E}_-} \approx \beta, \quad (69)$$

$$\frac{P_{\parallel+} - P_{\parallel-}}{P_{\parallel+} + P_{\parallel-}} \approx -\beta, \quad \frac{P_{\perp+} - P_{\perp-}}{P_{\perp+} + P_{\perp-}} \approx \beta. \quad (70)$$

3. General energy momentum tensor—Quantum Dirac fluid in CPEM

Using the values (55)–(57) for the energy density and pressures (or their respective limits), the energy-momentum tensor for a Dirac fluid can be constructed in the same mathematical form as (47).

4. Dirac current

To obtain the statistical averaged current densities for the Dirac fluid (integrating (36)), we will need the simplified expressions derived previously (and in Appendix C 2)

$$\hat{n}_0 \approx \left(P_1^2+Q_1^2\right)\left(\frac{2E}{E+m}\right) \approx 2\left(P_1^2+Q_1^2\right). \tag{71}$$

The perpendicular current density, thus, becomes

$$\mathbf{J}_D^\perp = \frac{q\hat{n}_0\lambda}{m}\mathbf{A}^\perp + \frac{\hat{\Lambda}\hat{n}_0}{mA}\mathbf{A}^\perp, \tag{72}$$

where $\lambda = (P_1^2 - P_2^2 + Q_1^2 - Q_2^2)/\hat{n}_0$, and $\hat{\Lambda} = \Lambda/\hat{n}_0$, with Λ given in (37). The current $\mathbf{J}_{D\pm}^\perp$ reflects all four states of the Dirac electron. One clearly sees that the current associated with the negative energy solution (positron current), measured by $P_2^2 + Q_2^2$, opposes the positive energy or electron current proportional to $P_1^2 + Q_1^2$.

To view the current more explicitly, we make further approximations (see Appendix C) yielding

$$\lambda \approx \frac{m}{E_\pm}, \quad \hat{\Lambda} \approx \mp\frac{mqA(\omega^2-k^2)}{2kE_\pm^2}, \tag{73}$$

in the limit $qA \gg K_z$. Statistical averaging now leads to

$$\langle J_{D\pm}^0\rangle = n_\pm \hat{n}_0,$$
$$\langle \mathbf{J}_{D\pm}^\perp\rangle = \frac{\hat{n}_0 n_\pm q\mathbf{A}^\perp}{D_1(Z,\varrho_\pm)\mathcal{M}_\pm}\left(D_4(Z,\varrho_\pm) \mp \frac{\omega^2-k^2}{2k\mathcal{M}_\pm}D_5(Z,\varrho_\pm)\right), \tag{74}$$

where the functions $D_4(Z,\varrho)$ and $D_5(Z,\varrho)$ are defined in (D4) and (D5), respectively. Notice that the effective plasma density is given by $\langle J_{D\pm}^0\rangle$.

When substituted in the perpendicular components of the Maxwell equations, this current will lead to the modified dispersion relation

$$\omega^2 - k^2 = \left(\frac{\omega_{p+}^2}{\Theta_+} + \frac{\omega_{p-}^2}{\Theta_-}\right)\left(1 + \frac{\omega_{p+}^2}{2k\theta_+} - \frac{\omega_{p-}^2}{2k\theta_-}\right)^{-1}, \tag{75}$$

in which every term, on the right hand side, depends on the spin state. The plasma frequencies

$$\omega_{p\pm}^2 = \frac{4\pi q^2\hat{n}_0 n_\pm}{\mathcal{M}} \tag{76}$$

represent, respectively, the contributions of the spin up (down) component, and the factors

$$\Theta_\pm = \frac{\sqrt{1\pm 2\beta}D_1(Z,\varrho_\pm)}{D_4(Z,\varrho_\pm)}, \quad \theta_\pm = \frac{(1\pm 2\beta)\mathcal{M}D_1(Z,\varrho_\pm)}{D_5(Z,\varrho_\pm)} \tag{77}$$

reflect additional spin induced modification of the dispersion relation. When the difference in the two spin states is neglected, i.e., the differentiating parameter $\beta \to 0$, the spin contribution to the dispersion relation (75) vanishes as $\omega_{p+} = \omega_{p-}$, $\theta_+ = \theta_-$, $\Theta_+ = \Theta_-$. We will investigate the importance of the spin effects in a forthcoming paper.

It should also be noted that the plasma frequencies are defined in terms of the field renormalized effective mass $\mathcal{M}$.

V. CONCLUSIONS

We have developed a formalism to explore the dynamics of an assembly of weakly interacting, arbitrary energy quantum particles (Klein Gordon and Dirac) subjected to an arbitrary amplitude electromagnetic wave. We name the particle dressed by the electromagnetic field as combined QPF system. The quantum many body QPF dynamics is created in two major steps: (1) By directly solving the relevant quantum equations as eigenvalue problems: finding the microscopic energy spectrum and the corresponding eigenfunctions and (2) based on the calculated energy spectrum, developing a statistical description of weakly interacting QPF. Statistical averages, using appropriate distribution functions, lead us to the construction of the relevant macroscopic physical quantity (fluxes, currents, and energy momentum tensor etc). Naturally, the "Boltzmann" factor, deciding the probability of a given micro state, will be controlled by the energy spectrum calculated for the QPF. Preliminary and partial investigations of this formalism reveal highly interesting features:

(a) The electromagnetic field re-normalizes the mass of the quantum particle; the QPF acquires an effective mass $M^2_{field-renormalized} \equiv \mathcal{M}^2 = m^2 + q^2A^2$, a combination of the particle inertial mass (m) and the inertia contributed by the electromagnetic field $|qA|$, could be much larger than the rest mass. Transformation from m to $\mathcal{M}$ has profound consequences for setting the temperature scale in statistical averaging.

(b) Since the propagating electromagnetic wave breaks the rotational symmetry by introducing a preferred direction, the energy spectrum is anisotropic, i.e., the momenta parallel and perpendicular to the direction of the wave propagation appear with different weights. The extent of the anisotropy, just like the effective mass $\mathcal{M}$, depends on the wave intensity, measured by q^2A^2; the anisotropy is also a function of other wave characteristics like ω and k. The anisotropy in the energy spectrum (implying an anisotropic pressure/temperature) for the QPF system spells fundamental consequences especially for realistic temperatures; it can, for instance, translate into differential propagation characteristics for different polarizations of an incident electromagnetic pulse. This phenomenon was illustrated, spectacularly, in PIC (particle-in-cell) simulations of a temperature-anisotropy induced relativistic plasma polarizer.[28]

(c) For the more complicated Dirac case, the interaction with the electromagnetic wave, in addition, splits the energy spectrum; the degeneracy between the positive and negative energy states and between the spin up and spin down states in either class is broken. Thus, the statistical mechanics of the QPF will, again, be profoundly different from that of the standard Dirac electron (without the field).

The consequences of these newly revealed features will be investigated in a series of future papers. One very interesting consequence, for instance, is that the dispersion relation of an electromagnetic wave is fundamentally changed (for both K-G and Dirac QPF) by the replacement of m by $\mathcal{M}$, and the appearance of the spin induced contribution to the cutoff frequency (see (52) and (75)).

It is expected that this formulation will be particularly useful in laboratory as well as astrophysical systems where one encounters arbitrarily large electromagnetic fields.

ACKNOWLEDGMENTS

S.M.M.'s work was supported by the US-DOE Grant No. DE-FG02-04ER54742. F.A.A. thanks to CONICyT for Funding No. 79130002.

APPENDIX A: KLEIN-GORDON EQUATION WITH LINEARLY POLARIZED EM WAVE

The Klein-Gordon particle in an arbitrary amplitude linearly polarized EM wave obeys a generalized Mathieu or Hill equation. Assuming, without any loss of generality,

$$\mathbf{A} = \hat{e}_x A\cos(\omega t - kz), \tag{A1}$$

the KG equation (after Fourier transform in x, y) is

$$\partial_t^2\psi - \partial_z^2\psi + [m^2 + K_\perp^2 + q^2A^2\cos^2(\omega t - kz)]\psi + 2qAK_x\cos(\omega t - kz)\psi = 0, \tag{A2}$$

where $K_\perp^2 = K_x^2 + K_y^2$. Going to through the same procedure as the CPEM, we can rewrite the previous equation as

$$\frac{d^2\psi}{d\zeta^2} + [\mu + \lambda_1\cos(2\zeta) + \lambda_2\cos\zeta]\psi = 0, \tag{A3}$$

where

$$\mu = K_\rho^2 + \frac{m^2 + K_\perp^2 + q^2A^2/2}{\omega^2 - k^2}, \tag{A4}$$

$$\lambda_1 = \frac{q^2A^2}{2(\omega^2 - k^2)}, \tag{A5}$$

$$\lambda_2 = \frac{2qAK_x}{\omega^2 - k^2}. \tag{A6}$$

Equation (A3) can be analyzed as a Hill equation. This problem, along with a detailed analysis of the CPEM Mathieu equation, will be presented in a forthcoming paper.

APPENDIX B: WKB SOLUTION FOR MATHIEU EQUATION

For the equation

$$\psi'' + (\mu + \lambda\cos\zeta)\psi = 0, \tag{B1}$$

the WKB solution is

$$\psi = \frac{C}{(\mu + \lambda\cos\zeta)^{1/4}}\exp\left[i\int^{\zeta}(\mu + \lambda\cos\zeta')d\zeta'\right], \tag{B2}$$

where C is a normalization constant to be determined by normalization. Choosing to normalize on a finite interval in ζ, the total wavefunction is

$$\psi_{tot} = \frac{C}{(\mu + \lambda\cos\zeta)^{1/4}}\exp\left[i\mathbf{K}_\perp\cdot\mathbf{x}_\perp + iK_\rho(kt - \omega z) + i\int^{\zeta}(\mu + \lambda\cos\zeta')d\zeta'\right]. \tag{B3}$$

Assuming a trivial normalization in $\mathbf{x}_\perp$ and in $(kt - \omega z)$, we have

$$1 = \int_{-\pi}^{\pi}\psi^*\psi d\zeta = C^2\int_{-\pi}^{\pi}\frac{d\zeta}{(\mu + \lambda\cos\zeta)^{1/2}}, \tag{B4}$$

which implies that

$$C^2 = \frac{(\lambda + \mu)^{1/2}}{4K(\gamma)}, \tag{B5}$$

where $\gamma = \sqrt{2\lambda/(\lambda + \mu)}$ and K is the complete elliptic function of the first kind.

Notice that we could write the eikonal

$$\int(\mu + \lambda\cos\zeta)^{1/2}d\zeta = 2\sqrt{\lambda + \mu}\,\mathbf{E}\left(\frac{\zeta}{2}, \gamma\right), \tag{B6}$$

but we will keep (B2) as it is.

Now

$$E = i\int\left(\psi_t^*\frac{\partial\psi_t}{\partial t}\right)d\zeta, \tag{B7}$$

$$= C^2\left\langle -\frac{K_\rho k}{(\mu + \lambda\cos\zeta)^{1/2}} - \omega\frac{(\mu + \lambda\cos\zeta)^{1/2}}{(\mu + \lambda\cos\zeta)^{1/2}}\right\rangle, \tag{B8}$$

$$= -K_\rho k - 2\pi\omega C^2 = -K_\rho k - \omega\pi\frac{(\lambda + \mu)^{1/2}}{2K(\gamma)}. \tag{B9}$$

Similarly,

$$K_z = -i\int\left(\psi_t^*\frac{\partial\psi_t}{\partial z}\right)d\zeta = -K_\rho\omega - k\pi\frac{(\lambda + \mu)^{1/2}}{2K(\gamma)}. \tag{B10}$$

From the perturbative Mathieu analysis, in the limit $\lambda/\mu \to 0$, terms of order λ/μ vanish and the corrections are of order λ^2.

Now, in the limit of $\gamma \to 0$,

$$\frac{2}{\pi}K(\gamma) \approx 1 + \frac{\lambda}{2\mu} + \frac{\lambda^2}{16\mu^2}. \tag{B11}$$

Therefore, in the same limit

$$\frac{\pi(\lambda + \mu)^{1/2}}{2\mu^{1/2}K(\gamma)} \approx 1 - \frac{3\lambda^2}{16\mu^2}. \tag{B12}$$

Then, the dispersion relation becomes

$$E^2 - K_z^2 = (\omega^2 - k^2)\left[-K_\rho^2 + \mu - \frac{3\lambda^2}{8\mu}\right]. \tag{B13}$$

It is important to realize that

$$\frac{\lambda}{\mu} = \frac{2qAK_\perp}{q^2A^2 + K_\perp^2 + K_\perp^2(\omega^2 - k^2)} \le 1 \tag{B14}$$

always and we can use perturbative analysis.

APPENDIX C: DIRAC EQUATION IN CPEM

The Dirac equation (29), embedded in a CPEM wave,

$$\gamma^\mu(i\partial_\mu + qA_\mu)\Psi = m\Psi, \quad \Psi = \begin{pmatrix} \varphi_1 \\ \chi_1 \\ \varphi_2 \\ \chi_2 \end{pmatrix} \tag{C1}$$

can be solved in one spatial dimension (the wave function is a function of z and t, $\partial_1 = 0 = \partial_2$) by the wave function

$$\begin{aligned} \varphi_1 &= P_1 e^{i\zeta/2 - i(Et - K_z z)}, \\ \varphi_2 &= P_2 e^{i\zeta/2 - i(Et - K_z z)}, \\ \chi_1 &= Q_1 e^{-i\zeta/2 - i(Et - K_z z)}, \\ \chi_2 &= Q_2 e^{-i\zeta/2 - i(Et - K_z z)}, \end{aligned} \tag{C2}$$

yielding the homogeneous set of equations for the amplitudes (P_1, P_2, Q_1, Q_2)

$$\begin{aligned} (W_- - m)P_1 - K_- P_2 &= qAQ_2, \\ (W_+ - m)Q_1 + K_+ Q_2 &= qAP_2, \\ (W_+ - m)P_2 + K_- P_1 &= qAQ_1, \\ (W_+ + m)Q_2 + K_+ Q_1 &= qAP_1. \end{aligned} \tag{C3}$$

The solvability condition for (C3) yields the eigenvalue condition

$$(W_-^2 - K_-^2 - \mathcal{M}^2)(W_+^2 - K_+^2 - \mathcal{M}^2) = q^2A^2(k^2 - \omega^2), \tag{C4}$$

where $W_\pm = E \pm \omega/2$ and $K_\pm = K_z \pm k/2$.

1. General solution

The primary purpose of this Appendix is to manipulate the system so that we can derive expressions for the particle current induced by the electromagnetic wave, i.e., to evaluate (36). For this purpose, an equivalent form of Eq. (33) may be more useful

$$G^2 - H^2 = m^2(\omega^2 - k^2), \tag{C5}$$

where

$$G = W_+W_- - K_+K_- - \mathcal{M}^2, \quad H = kE - K_z\omega. \tag{C6}$$

In terms of G and H, (32) becomes

$$(G - m\omega)P_1 = -(H + mk)P_2, \tag{C7}$$

$$(H - mk)P_1 = -(G + m\omega)P_2, \tag{C8}$$

$$(G + m\omega)Q_1 = -(H + mk)Q_2, \tag{C9}$$

$$(H - mk)Q_1 = -(G - m\omega)Q_2, \tag{C10}$$

from where we derive the relations

$$P_2^2 + Q_2^2 = \frac{1}{(H + mk)^2}[(G^2 + m^2\omega^2)(P_1^2 + Q_1^2) - 2Gm\omega(P_1^2 - Q_1^2)], \tag{C11}$$

$$P_2Q_1 - P_1Q_2 = \frac{2m\omega}{H + mk}P_1Q_1, \tag{C12}$$

$$P_1Q_1 - P_2Q_2 = \frac{2mk}{H + mk}P_1Q_1, \tag{C13}$$

such that the one-particle dispersion relation becomes

$$\omega^2 - k^2 = \frac{\lambda(\omega, k)\omega_p^2}{1 - \mu(\omega, k)\omega_p^2}, \tag{C14}$$

with

$$\lambda(\omega, k) = \left[1 - \frac{G^2 + m^2\omega^2}{(H + mk)^2} + \frac{2Gm\omega(1 - \zeta^2)}{(H + mk)^2(1 + \zeta^2)}\right] \times \left[1 + \frac{G^2 + m^2\omega^2}{(H + mk)^2} - \frac{2Gm\omega(1 - \zeta^2)}{(H + mk)^2(1 + \zeta^2)}\right]^{-1}, \tag{C15}$$

with

$$\zeta = \frac{Q_1}{P_1} = \frac{1}{qA}\left[\frac{(m - W_-)(G - m\omega)}{H + mk} - K_-\right], \tag{C16}$$

which can be calculated from (32) and (C7). Also, we find

$$\mu(\omega, k) = \frac{2m\zeta}{qA(H + mk)}\left(\left[1 + \frac{G^2 + m^2\omega^2}{(H + mk)^2}\right](1 + \zeta^2) - \frac{2Gm\omega}{(H + mk)^2}(1 - \zeta^2)\right)^{-1}. \tag{C17}$$

2. Approximations

From (32), we get

$$P_2^2 + Q_2^2 = \left(\frac{K_+P_1 - qAQ_1}{W_+ + m}\right)^2 + \left(\frac{K_-Q_1 + qAP_1}{W_- + m}\right)^2. \tag{C18}$$

Let us analyze the natural case of relativistic electrons with $E \gg m$ but $k, \omega \ll m, E$. Then, $W_+ + m \approx E + m \approx W_- + m$. Also, we approximate $K_+ \approx K_- \approx K_z$. Then, the above equation becomes

$$P_2^2 + Q_2^2 = \frac{q^2A^2 + K_z^2}{(E + m)^2}(P_1^2 + Q_1^2). \tag{C19}$$

Also, (33) can be approximated as

$$E^2 = K_z^2 + m^2 + q^2A^2. \tag{C20}$$

Therefore

$$P_2^2 + Q_2^2 = \frac{E - m}{E + m}(P_1^2 + Q_1^2), \tag{C21}$$

and the λ factor becomes

$$\lambda = \frac{m}{E} = \left(1 + \frac{K_z^2 + q^2 A^2}{m^2}\right)^{-1/2}. \qquad \text{(C22)}$$

A further approximation at first order of k can be obtained. In this case, we can solve Eq. (33) by assuming a light-wave approximation $\omega \sim k$. We can find two different energies due to the contributions of the two spin orientations

$$E^2 - K_z^2 \pm k(E - k) - \mathcal{M}^2 = 0, \qquad \text{(C23)}$$

which at first order on k has the solution

$$E_\pm^2 = K_z^2(1 \pm \beta) + \mathcal{M}_\pm^2, \qquad \text{(C24)}$$

using the definitions after Eq. (54) and where we have assumed that at first order approximation $\mathcal{M} \gg K$. Notice that the two states have different effective masses $\mathcal{M}_\pm$.

3. 3D case

The study of a complete 3D case of the Dirac equation in CPEM is difficult. However, we can estimate phenomenologically how the energies should change. Analogously to the K-G case, the energy of the spin states should behave as

$$E_\pm^2 \approx K_z^2(1 \pm \beta) + \mathcal{M}_\pm^2 + a^2 K_\perp^2. \qquad \text{(C25)}$$

APPENDIX D: DEFINITIONS FOR INTEGRALS

The integral functions used in Section IV B are defined as

$$D_1(Z,\zeta) = \int dq \frac{q^2}{Z^{-1}\exp\left(\zeta\sqrt{1+q^2}\right) + 1}, \qquad \text{(D1)}$$

$$D_2(Z,\zeta) = \int dq \frac{q^2\sqrt{1+q^2}}{Z^{-1}\exp\left(\zeta\sqrt{1+q^2}\right) + 1}, \qquad \text{(D2)}$$

$$D_3(Z,\zeta) = \int dq \frac{q^4}{\sqrt{1+q^2}} \frac{1}{Z^{-1}\exp\left(\zeta\sqrt{1+q^2}\right) + 1}, \qquad \text{(D3)}$$

$$D_4(Z,\zeta) = \int dq \frac{q^2}{\sqrt{1+q^2}} \frac{1}{Z^{-1}\exp\left(\zeta\sqrt{1+q^2}\right) + 1}, \qquad \text{(D4)}$$

$$D_5(Z,\zeta) = \int dq \frac{q^2}{1+q^2} \frac{1}{Z^{-1}\exp\left(\zeta\sqrt{1+q^2}\right) + 1}. \qquad \text{(D5)}$$

In the case of $Z < e^\zeta$ (non-degenerate states), the previous integrals have the expansion[26]

$$D_1(Z,\zeta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\zeta} Z^n K_2(n\zeta), \qquad \text{(D6)}$$

$$D_2(Z,\zeta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\zeta} Z^n \left[K_3(n\zeta) - \frac{K_2(n\zeta)}{n\zeta}\right], \qquad \text{(D7)}$$

$$D_3(Z,\zeta) = 3\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n\zeta)^2} Z^n K_2(n\zeta), \qquad \text{(D8)}$$

where K_n are the modified Bessel function of second kind of order n.

In the case of $Z \gg e^\zeta$ (degenerate states), the integrals (D1)–(D3) can be approximated as[26]

$$D_1(Z,\zeta) \approx \frac{1}{3\zeta^3}\left[(\ln Z)^2 - \zeta^2\right]^{3/2}, \qquad \text{(D9)}$$

$$D_2(Z,\zeta) \approx \frac{\ln Z}{8\zeta^4}\left[2(\ln Z)^2 - \zeta^2\right]\left[(\ln Z)^2 - \zeta^2\right]^{1/2}, \qquad \text{(D10)}$$

$$D_3(Z,\zeta) \approx \frac{\ln Z}{8\zeta^4}\left[2(\ln Z)^2 - 5\zeta^2\right]\left[(\ln Z)^2 - \zeta^2\right]^{1/2}. \qquad \text{(D11)}$$

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