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Comment on “Highly relativistic spin-gravity coupling for fermions”

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We exhibit difficulties of different sorts which appear when using the Mathisson-Papapetrou equations, in particular in the description of highly relativistic particles presented in R. Plyatsko and M. Fenyk [Phys. Rev. D **91**, 064033 (2015)]. We compare some results of this theory and of the aforementioned work with the ones obtained using a Lagrangian formulation for massive spinning particles and show that the issues mentioned in the preceding sentence do not appear in the Lagrangian treatment.

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This comment addresses some statements which appear in the article referred to in the title [1] which need clarification.

The article in question compares results obtained using two similar, but inequivalent, theories. In fact, [1] deals with a third-order set of the equations derived by Mathisson [2] and Papapetrou [3] (referred to as MPE in what follows) which describes massive test spinning particles moving on a background gravitational field. Matsyuk presents a Lagrangian treatment to obtain a set of third-order MPE [4]. Some of the results obtained in Ref. [1] are compared to the ones obtained in [5] where a different Lagrangian approach [referred to as the Lagrangian top theory (LTT)] is conceived to describe massive test spinning particles (tops) originally devised in the special relativistic framework by Hanson and Regge [6] and extended to consider the motion on gravitational fields using the general relativistic formalism in [5,7–12]. Similar studies in a Lagrangian formalism have been carried out in a quantum mechanical context [13–15].

Some of these statements we comment upon refer to the fact that there are solutions found in [5] which allow for superluminal motion of tops. The claim made in [1] is that this behavior is introduced spuriously by the inadequate choice of a constraint (details follow below). Our comment is based on problems arising from MPE concerning the following facts: (i) the MPE implicitly postulate timelike velocities, i.e., the statement that $u_\mu u^\mu - 1 = 0$ is essentially hidden in the MPE even without using any constraint at all, and (ii) the MPE do not predict that the Poincaré Casimir functions mass $P_\mu P^\mu$ and spin $S_{\mu\nu}S^{\mu\nu}$ are constants of motion. These two difficulties have as a consequence that some of the statements appearing in [1] have consistency issues, as is explicitly proved below.

A brief summary of MPE and concepts seems in order to present our comments. The MPE [2,3], which characterize the motion of massive spinning test particles in the presence of a given external gravitational field described in terms of a metric tensor $g_{\mu\nu}(x^\alpha)$, the Christoffel symbols connection $\Gamma^\mu{}_{\nu\rho}(x^\alpha)$ and the Riemann curvature tensor $R^\mu{}_{\nu\alpha\beta}(x^\gamma)$, are [1]

$$\frac{D}{Ds}\left(mu^\mu + u_\nu \frac{DS^{\mu\nu}}{Ds}\right) = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}u^\nu S^{\alpha\beta}, \qquad (1)$$

$$\frac{DS^{\mu\nu}}{Ds} + u^\mu u_\alpha \frac{DS^{\nu\alpha}}{Ds} - u^\nu u_\alpha \frac{DS^{\mu\alpha}}{Ds} = 0, \qquad (2)$$

where D/Ds is the s-parametrized covariant derivative and s is the proper time of the spinning particle. The above description is written in terms of position x^μ, velocity u^μ, and an antisymmetric spin tensor $S^{\mu\nu}$. Due to the appearance of the Riemann tensor in the right-hand side of (1) the spinning particles do not follow geodesics, in general. Even if no Lagrangian for the system has been defined, it is customary to define the (noncanonical) momentum vector P^μ by [1]

$$P^\mu = mu^\mu + u_\nu \frac{DS^{\mu\nu}}{Ds}. \qquad (3)$$

Note that, due to the fact that spin is related to angular velocity, the momentum so defined depends on second derivatives of dynamical variables and therefore (1) is a third-order differential equation. This is explicitly emphasized by Matsyuk in Ref. [4]. Also, this has been noticed in Refs. [8,16]. Besides, we remind the reader that the MPE were obtained as a limiting case of rotating fluids moving in gravitational fields. It is worth remarking that the MPE (1) and (2) as they stand are not reparametrization covariant. This represents a clear difference with other relativistic theories for classical particles. As is widely known, the

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relativistic spinless particle action and equations of motion are reparametrization invariant and reparametrization covariant, respectively. On the other hand, the LTT and its corresponding equations of motion for massive spinning particles [6,8] behave correctly under reparametrization. The dynamical equations derived from LTT are different from the MPE. They are [8]

$$\frac{DP^{\mu}}{D\lambda} = -\frac{1}{2} R^{\mu}{}_{\nu\alpha\beta} u^{\nu} S^{\alpha\beta},$$
$$\frac{DS^{\mu\nu}}{D\lambda} = S^{\mu\alpha}\sigma_{\alpha}{}^{\nu} - \sigma^{\mu\alpha} S_{\alpha}{}^{\nu} = P^{\mu}u^{\nu} - u^{\mu}P^{\nu}, \qquad (4)$$

where $D/D\lambda$ is the λ-parametrized covariant derivative, $u^{\mu} = dx^{\mu}/d\lambda$, and $P_{\mu} = \partial L/\partial u^{\mu}$ is the conjugated momentum vector obtained from the Lagrangian L. Note that λ is an *arbitrary* parameter. Also the antisymmetric spin tensor is $S_{\mu\nu} = \partial L/\partial\sigma^{\mu\nu}$, where $\sigma^{\mu\nu}$ is the antisymmetric angular velocity tensor. It is shown in Ref. [5] how to construct the proper Lagrangian to obtain the one-particle theory for spinning particles [note that Eqs. (4) are reparametrization covariant]. Theories which describe the dynamics of classical spinning massive particles (both MPE and the ones defined in [6,8]) are usually written in terms of variables whose number exceeds the one strictly needed to describe their orientation. In fact, they are written in terms of the six independent components of an orthonormal tetrad $e_{(\alpha)}{}^{\mu}$ as introduced in [5,7–11] or, equivalently, by the six parameters of a Lorentz transformation matrix as originally done by Hanson and Regge in their spinning top model [6]. Thereby, associated to the tetrad degrees of freedom, it appears $\sigma^{\mu\nu} = e^{(\alpha)\mu} D_{\lambda} e_{(\alpha)}{}^{\nu}$ and its canonically conjugated antisymmetric tensor $S_{\mu\nu}$.

Although the MPE (1) and (2) and the LTT equations (4) differ from one another (and produce different results), both theories require that the six (Lorentz transformation parameter) orientation degrees of freedom be constrained in order to be reduced to three rotational degrees of freedom only, to appropriately describe spin. Two types of constraints have been suggested in the literature. The first one, known as the Tulczyjew constraint [17], reads

$$P_{\mu} S^{\mu\nu} = 0. \qquad (5)$$

A second type, known as the Mathisson-Pirani constraint [18], is suggested to be

$$u_{\mu} S^{\mu\nu} = 0. \qquad (6)$$

The two previous constraints give rise to different dynamics for the particle. The Tulczyjew constraint (5) has been used frequently in the literature to properly define the right behavior of the spinning particle in the momentum rest frame (see for example Refs. [19–22]). Also it is important to point out that the Lagrangian formalism to obtain the third-order MPE developed in Ref. [4] is constructed to consider the Mathisson-Pirani constraint only.

The purpose of this comment is to show the severe problems, mentioned in the beginning, that arise in the MPE (1) and (2) [and that do not appear in the LTT (4)]. Recently, in Refs. [1,23], it was suggested that MPE correctly describe a classical spinning particle and that constraint (5) is not adequate to describe the spinning particle dynamics at velocities close to the speed of light [favoring constraint (6)], as the results derived from MPE (using the Tulczyjew constraint) seem contrary to physical intuition. The importance of the work developed in Refs. [1,23] is to remark that it is not possible to indistinctly use the two constraints (5) and (6), as they produce different results.

However, we show in the following that there are several limitations in the description of high velocity particles with MPE, as well as some problematic issues with MPE. First, let us focus on Eq. (2) for the evolution of the spin. Contracting that equation with u_{μ} we get the relation

$$u_{\mu}\frac{DS^{\mu\nu}}{Ds}(1 - u_{\alpha}u^{\alpha}) = 0. \qquad (7)$$

As $u_{\mu}DS^{\mu\nu}/Ds \neq 0$, Eq. (2) implies

$$u_{\mu}u^{\mu} = 1 \qquad (8)$$

(using the appropriate signature), and thus the MPE always describe timelike particles. This also has been highlighted by other authors working on MPE [4], where the timelike behavior of the velocity is used in the theories. Furthermore, the MPE imply the constraint $u_{\mu}D_{s}u^{\mu} = 0$ by consistency. This last condition seems to have remained unnoticed in Ref. [1]. This could appear as a desirable feature of the MPE, because it is a direct consequence of the MPE regardless of the constraint used. One may reasonably state that $u_{\mu}u^{\mu} = 1$ is a constituent part of MPE. Therefore, it is a result that is independent of the constraints (5) and (6). This is an important conclusion derived from the heart of MPE. In Ref. [1] an attempt to study the high-relativistic velocity motion of the spin particle is made using the Tulczyjew constraint (5). The authors of Ref. [1] claim to find an exact solution of the MPE that has physical inconsistencies as the velocity of the particle approaches or is larger than the speed of light. However, as it was simply shown in Eq. (7), the motion of the particle described by the MPE is always restricted to be subluminal in a consistent manner for any constraint. Therefore, the conclusion of Ref. [1], that superluminal behavior is obtained in the MPE when the Tulczyjew constraint (5) is used, is incorrect. Another way to see this point is by using the definition of the momentum (3) in the MPE. Contracting it with $S_{\alpha\mu}$, we obtain

$$S^{\alpha\mu}P_{\mu} = mS^{\alpha\mu}u_{\mu} + S^{\alpha\mu}u^{\beta}\frac{DS_{\mu\beta}}{Ds}. \qquad (9)$$

Now, using Eq. (2) in the right-hand side we obtain the relation

$$S^{\alpha\mu}P_\mu = mS^{\alpha\mu}u_\mu + (u_\nu u^\nu)S^{\alpha\mu}(P_\mu - mu_\mu), \tag{10}$$

which again implies a timelike behavior for the particle velocity, $u_\nu u^\nu = 1$.

Despite the previous analysis, we can also show that the momentum definition (3) induces severe physical problems. Contracting (3) with u_μ we obtain the relation

$$u_\mu P^\mu = mu_\mu u^\mu = m, \tag{11}$$

where the second equality is due to the previous results (in the framework of MPE). This is the mass definition according to Ref. [1], and it has already been used in that work without recognizing its origin in the timelike behavior of the velocity. In Ref. [1], it also has been recognized that m is a constant of motion when using the Mathisson-Pirani constraint. On the other hand, we can also contract (3) with P_μ to obtain

$$P_\mu P^\mu = m^2 + P_\mu u_\nu \frac{DS^{\mu\nu}}{Ds} = m^2 - \frac{D}{Ds}(P_\mu u_\nu)S^{\mu\nu}, \tag{12}$$

where we have used (11) and the fact that $P_\mu u_\nu S^{\mu\nu} = 0$ for any choice of the constraint (5) or (6). Now, notice that it is well known that a desirable feature of the physical momentum should be that $P_\mu P^\mu$ is a constant of motion, implying that the momentum vector remains timelike along the motion in all frames and that the energy density is positive [1,5,7–9,24]. Even more, in Ref. [1] it is also recognized that under constraint (5), the MPE should produce the constant of motion $P_\mu P^\mu$. Here two different problems emerge depending on what constraint is used. If we use constraint (6), the above equation is rewritten as

$$P_\mu P^\mu = m^2 - P_\mu \frac{Du_\nu}{Ds} S^{\mu\nu}. \tag{13}$$

As m is constant under this constraint, and $P_\mu D_s u_\nu S^{\mu\nu} \neq 0$ in general, $P_\mu P^\mu$ can hardly be a constant of motion when using the Mathisson-Pirani constraint. On the contrary, if we use constraint (5), m is no longer a constant of motion. Its evolution can be found from Eq. (1) [using the normalization of the velocity obtained from (2)] to be

$$\frac{Dm}{Ds} = \frac{Du_\nu}{Ds}\frac{D(u_\mu S^{\nu\mu})}{Ds}. \tag{14}$$

Besides, under the Tulczyjew constraint (5), Eq. (12) becomes

$$P_\mu P^\mu = m^2 + \frac{1}{2}R^\mu{}_{\alpha\beta\gamma}u^\alpha S^{\beta\gamma}u^\nu S_{\mu\nu}, \tag{15}$$

where we have made use of Eq. (1) for $D_s P_\mu$. It is clear that, using the MPE with the Tulczyjew constraint (5), $P_\mu P^\mu$ *is not* a constant of motion (which gives rise to a contradiction with the results discussed in Ref. [1]) unless m fulfils an extra constraint imposed by (14) to match exactly the right-hand side of Eq. (15). In other words, $P_\mu P^\mu =$ constant if

$$\frac{1}{4}\frac{D}{Ds}(R^\mu{}_{\alpha\beta\gamma}u^\alpha S^{\beta\gamma}u^\nu S_{\nu\mu}) = \frac{Du_\nu}{Ds}\frac{D(u_\mu S^{\nu\mu})}{Ds}\left(P_\alpha P^\alpha - \frac{1}{2}R^\lambda{}_{\alpha\beta\gamma}u^\alpha S^{\beta\gamma}u^\nu S_{\lambda\nu}\right)^{1/2} \tag{16}$$

is always satisfied. This means that, under the Tulczyjew constraint, the MPE (1) and (2) contain the extra constraint (16) that it is not usually mentioned, and that every exact solution of the MPE system should satisfy. It is very unlikely that constraint (16) will be satisfied in general for any metric.

As was discussed in Ref. [5], a correct formulation for the dynamics of a spinning particle is obtained using a Lagrangian treatment, and it gives rise to Eqs. (4). This formulation has been studied extensively [6,8,9]. Without a proper Lagrangian theory, a canonical momentum cannot be appropriately defined. With a Lagrangian theory, the lack of parallelism between velocity and momentum can be well studied and understood (which is a usual behavior in quantum theories). Of course, the momentum obtained from the Lagrangian formalism is nothing similar to (3). Interestingly enough, it has been shown in Ref. [5,8] that the Tulczyjew constraint emerges naturally from the LTT. On the other hand, the problems coming from constraint (6) have also been recognized in Ref. [24], in favor of constraint (5).

Equations (13)–(16), which stem from MPE, do not imply (for any constraint) that the two Casimir functions of the Poincaré group for massive spinning particles, namely the mass $m^2 \equiv P^\mu P_\mu$ and the spin $S^{\mu\nu}S_{\mu\nu}/2$, are constants of motion. Any model which attempts to describe tops reasonably should imply that these quantities are always conserved for a spinning particle moving on any curved background or otherwise the particle would change its identity (mass) as a consequence of its evolution in a (classical) dynamics context. The Lagrangian treatment always implies that the two Casimir functions are conserved, and in doing so, it ensures that the momentum is always timelike.

The relation between momentum and velocity in LTT is different from the one obtained in (3). The behavior of the velocity can be calculated from Eqs. (4). After some algebra, we obtain that

$$u_\mu u^\mu = \left(\frac{1}{m}u_\mu P^\mu\right)^2 - \frac{1}{2m^2}R^\nu{}_{s\gamma\delta}u^\mu S_{\mu\nu}u^s S^{\gamma\delta} + \frac{1}{m^2}\frac{Du_\mu}{D\lambda}P_\nu S^{\mu\nu}, \tag{17}$$

and thus $u^\mu u_\mu$ could be lightlike or spacelike [compare with (11)]. Notice that the last term vanishes for the Tulczyjew constraint. Thereby, the velocity is not restricted to be subluminal, and thus the Lagrangian theory could be the appropriated description to study highly relativistic fermions. However, the momentum maintains its desired timelike behavior, as in any well-behaved relativistic theory [Eq. (15) of the MPE implies that the momentum could have a meaningless lightlike or spacelike nature]. It is important to emphasize that superluminal behavior is not a new phenomenon in theories describing spinning particles [25].

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