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Pure Lovelock gravity and Chern-Simons theory

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We explore the possibility of finding pure Lovelock gravity as a particular limit of a Chern-Simons action for a specific expansion of the AdS algebra in odd dimensions. We derive in detail this relation at the level of the action in five and seven dimensions. We provide a general result for higher dimensions and discuss some issues arising from the obtained dynamics.

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I. INTRODUCTION

Lanczos-Lovelock theory [1,2] is the most natural generalization of the gravity theory in D dimensions that leads to the second-order field equations. Its action is built as a polynomial in the Riemann curvature with each term carrying an arbitrary coupling constant. As recently discussed in Ref. [3], the higher curvature terms can generate sectors in the space of solutions carrying different number of degrees of freedom. In particular, there might be degenerate sectors with no degrees of freedom and in that case the metric is not completely determined by the field equations, i.e., some components remain arbitrary. An explicit example was first provided in Refs. [4,5] where it was shown that the metric component g_{tt} of the static spherically symmetric solution stays arbitrary when the coupling constants cause the theory to admit a nonunique degenerate vacuum. In fact, metrics with the undetermined components were reported not only in the static case [6] but also when the time dependent metric functions [7–9] are considered or when torsional degrees of freedom are taken into account to find a charged solution [10].

The degeneracy problem mentioned above can be avoided in the static spherically symmetric sector for particular choices of the coupling constants in the Lovelock theory. The simplest case corresponds to the Einstein-Hilbert (EH) action with or without the cosmological constant term. The next one is given by the action containing the Einstein-Hilbert and Gauss-Bonnet terms in D dimensions [11]. Another possibility is to choose the coefficients as in the Chern-Simons (CS) or Born-Infeld (BI) theories. Recently in Refs. [12,13] the pure Lovelock (PL) theory was proposed as another way of fixing the constants such that the static metric is free of the degeneracy. Indeed, the PL theory has nondegenerate vacua in even dimensions and admits a unique nondegenerate (A)dS vacuum in the odd-dimensional case. In the first-order formalism with the vielbein e^a and the spin connection ω^{ab}, the considered action contains only two terms instead of the full Lovelock series,

$$I_{\mathrm{PL}}^{D} = \int (\alpha_0 \mathcal{L}_0 + \alpha_p \mathcal{L}_p), \tag{1}$$

where

$$\mathcal{L}_0 = \epsilon_{a_1 \dots a_D} e^{a_1} \cdots e^{a_D}, \tag{2}$$

$$\mathcal{L}_p = \epsilon_{a_1 \dots a_D} R^{a_1 a_2} \cdots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \cdots e^{a_D}. \tag{3}$$

As usual, the curvature and the torsion are defined as $R^{ab} = d\omega^{ab} + \omega^a{}_c \omega^{cb}$ and $T^a = de^a + \omega^a{}_b e^b$, where wedge product is assumed and the Lorentz indices a, b are running from 1 to D. This action naturally carries torsional degrees of freedom and usually, when the study of solutions is considered (see Ref. [14]), the vanishing of torsion is assumed.

The new coefficients, in terms of the cosmological constant related with the (A)dS radius ℓ

$$\Lambda = \frac{(\mp 1)^p (D-1)!}{2(D-2p-1)! \ell^{2p}}, \tag{4}$$

and the gravitational constant κ, will be given by

$$\alpha_0 = -\frac{2\Lambda\kappa}{D!} = -\frac{(\mp 1)^p \kappa}{D(D-2p-1)! \ell^{2p}},$$
$$\alpha_p = \frac{\kappa}{(D-2p)!}. \tag{5}$$

As was pointed out in Ref. [15], the (A)dS space in the PL theory is not directly related to the sign of a cosmological

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constant like in general relativity. Indeed, the definition of (A)dS space is due to the sign of curvature $R^{ab} = \mp\frac{1}{\ell^2}e^a e^b$, and not of the explicit Λ, which now has the dimension of the length^{-2p} (at the same time the dimension of the gravitational constant κ is length^{2p-D}).

It was shown in Ref. [13,16] that the corresponding PL black holes solutions have the remarkable property of being asymptotically indistinguishable from the AdS-Schwarzschild ones. In addition, for the maximal power $p = [\frac{D-1}{2}]$ they exhibit a peculiar thermodynamical behavior. From the dynamical point of view, the nonlinear character of this theory is responsible for the presence of sectors in the space of solutions carrying different number of degrees of freedom. The Hamiltonian analysis performed in Ref. [3] shows that the maximum possible number is the same as in the EH case.

It should be pointed out that a supersymmetric version of the general Lovelock theory is unknown, except for the EH Lagrangian and two very special cases in odd dimensions corresponding to the CS actions for the AdS and Poincaré symmetries. The existence of supergravity for the Einstein-Hilbert term coupled to a Lovelock quadratic curvature term in five dimensions was discussed in [17]; nevertheless, the explicit form remains unknown. Regarding the features of the PL theory mentioned earlier, it would be interesting to explore the possibility of finding its supersymmetric form. This is not a trivial task because of the lack of guidance principle with the kind of terms that must be considered to ensure the invariance under supersymmetric transformations, which also are not fully clear. However, if one could find the PL action as some kind of limit of a CS or BI theory for a special symmetry, then that result might be useful to obtain the supersymmetric pure Lovelock (sPL) theory. Something similar has been done using the Lie algebra expansion methods [18,19] to obtain general relativity from the Chern-Simons theory in odd dimension [20–22]. The same has also been achieved in even dimensions with the BI actions [23].

We will not deal with the full problem here, but only focus on the bosonic part in odd dimensions. Thus, as a first step, in Sec. II we give a brief review of the so-called Maxwell-like algebras and the S-expansion method [19] used for their construction, as it provides necessary building blocks for the actions. We choose one specific family of algebras that seems to have the best chance to establish the desired relation between PL and CS. In Sec. III, we explicitly construct the CS action in five dimensions and show how it relates with the PL in that context. For completeness, in Sec. IV, we present similar construction for seven dimensions, along with the extension of the result to $(2n+1)$ dimensions. Finally, after dealing with the number of obstacles (assuring the proper relative sign between the terms, overconstraining superposition of the field equations) we propose specific configuration that reproduces the correct PL dynamics.

II. $\mathfrak{C}_m$ ALGEBRAS AND S EXPANSION

Our task requires finding the symmetry for which the cosmological constant term and the desired Lovelock term effectively end up in one sector of the invariant tensor and unwanted terms in another. The particular symmetries offering such a feature originate from $\mathfrak{so}(D-1,2)\oplus\mathfrak{so}(D-1,1)$ algebra (so-called AdS-Lorentz introduced in Refs. [24–27]), where we equip the standard set $\{J_{ab}, P_a\}$ with a new generator Z_{ab}. The Abelian semigroup expansion procedure (also referred as the S-expansion) [19] has generalized it to the whole family, which we will denote as $\mathfrak{C}_m$, following notation proposed in [28] and where the subindex refers to $(m-1)$ types of generators. Although we extend the AdS algebra, it is worth to mention that for $m>4$ the new algebras do not contain it as a subalgebra. The $\mathfrak{C}_m$, as well as $\mathfrak{B}_m$ and $\mathfrak{D}_m$ families, introduced in Refs. [21,27,28], can be regarded as generalizations of the original Maxwell algebra [29,30] and thus they are all referred as Maxwell-like algebras. We focus on the $\mathfrak{C}_m$ family because, for its first representatives we are going to consider in five dimensions, one can arrive at the field equations resembling, at least at first sight, the wanted scheme (effectively giving the maximal PL-like equation $RR + eeee = 0$), whereas $\mathfrak{B}_m$ (producing $RR=0$ and $Ree=0$) and $\mathfrak{D}_m$ (producing $RR=0$ and $Ree+eeee=0$) explicitly exclude the maximal PL construction.

As we will see below, the S-expansion method is not only producing new algebras, but it will also deliver the corresponding invariant tensors needed to construct the actions. This makes it a very useful and important tool to relate different symmetries and (super)gravity theories [21,31–36]. For example, it was shown in Ref. [27] that the cosmological constant term can arise from the Born-Infeld gravity theory based on $\mathfrak{C}_m$ (see also [37]). Some further applications of the supersymmetric extension of $\mathfrak{C}_m$ can be found in Refs. [38–40].

Following Ref. [27], in this section we will review the explicit derivation of the $\mathfrak{C}_{m\geq 3}$ algebras from AdS by using a particular choice of a semigroup. The considered procedure starts from the decomposition of the original algebra $\mathfrak{g}$ into subspaces,

$$\mathfrak{g} = \mathfrak{so}(D-1,2) = \mathfrak{so}(D-1,1)\oplus\frac{\mathfrak{so}(D-1,2)}{\mathfrak{so}(D-1,1)} = V_0\oplus V_1,$$

where V_0 is spanned by the Lorentz generator $\tilde{J}_{ab}$ and V_1 by the AdS translation generator $\tilde{P}_a$. They satisfy the commutation relations

$$[\tilde{J}_{ab},\tilde{J}_{cd}] = \eta_{cb}\tilde{J}_{ad} - \eta_{ca}\tilde{J}_{bd} - \eta_{db}\tilde{J}_{ac} + \eta_{da}\tilde{J}_{bc},$$
$$[\tilde{J}_{ab},\tilde{P}_c] = \eta_{cb}\tilde{P}_a - \eta_{ca}\tilde{P}_b,$$
$$[\tilde{P}_a,\tilde{P}_b] = \tilde{J}_{ab}. \tag{6}$$

The subspace structure may be then written as

$$[V_0, V_0] \subset V_0, \qquad [V_0, V_1] \subset V_1, \qquad [V_1, V_1] \subset V_0. \quad (7)$$

We skip the two first representatives, namely $\mathfrak{C}_3$ and $\mathfrak{C}_4$, because they do not have necessary features, as will be explained in the next section. Thus, for our purposes we proceed with a detailed derivation of $\mathfrak{C}_5$, which we then generalize to an arbitrary m. We start by considering the Abelian semigroup $S_M^{(3)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ with the multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{if } \alpha+\beta \leq 3, \\ \lambda_{\alpha+\beta-4}, & \text{if } \alpha+\beta > 3, \end{cases} \quad (8)$$

and the subset decomposition $S_M^{(3)} = S_0 \cup S_1$, where $S_0 = \{\lambda_0, \lambda_2\}$ and $S_1 = \{\lambda_1, \lambda_3\}$. This decomposition satisfies

$$S_0 \cdot S_0 \subset S_0, \qquad S_0 \cdot S_1 \subset S_1, \qquad S_1 \cdot S_1 \subset S_0, \quad (9)$$

which is said to be resonant since it has the same structure as (7).

According to Theorem IV.2 from Ref. [19], we can say that $\mathfrak{g}_R = W_0 \oplus W_1$ is a resonant subalgebra of $S_M^{(3)} \times \mathfrak{g}$, where

$$W_0 = (S_0 \times V_0) = \{\lambda_0, \lambda_2\} \times \{\tilde{J}_{ab}\} = \{\lambda_0 \tilde{J}_{ab}, \lambda_2 \tilde{J}_{ab}\},$$
$$W_1 = (S_1 \times V_1) = \{\lambda_1, \lambda_3\} \times \{\tilde{P}_a\} = \{\lambda_1 \tilde{P}_a, \lambda_3 \tilde{P}_a\}. \quad (10)$$

The new $\mathfrak{C}_5$ algebra is given by the set of generators $\{J_{ab}, P_a, Z_{ab}, R_a\}$, defined as

$$J_{ab} = \lambda_0 \tilde{J}_{ab}, \qquad P_a = \lambda_1 \tilde{P}_a,$$
$$Z_{ab} = \lambda_2 \tilde{J}_{ab}, \qquad R_a = \lambda_3 \tilde{P}_a,$$

and satisfying

$$[P_a, P_b] = Z_{ab}, \qquad [J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b,$$
$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} + \eta_{ad} J_{bc} - \eta_{bd} J_{ac},$$
$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} + \eta_{ad} Z_{bc} - \eta_{bd} Z_{ac},$$
$$[Z_{ab}, Z_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} + \eta_{ad} J_{bc} - \eta_{bd} J_{ac},$$
$$[R_a, R_b] = Z_{ab}, \qquad [Z_{ab}, P_c] = \eta_{bc} R_a - \eta_{ac} R_b,$$
$$[R_a, P_b] = J_{ab}, \qquad [J_{ab}, R_c] = \eta_{bc} R_a - \eta_{ac} R_b,$$
$$[Z_{ab}, R_c] = \eta_{bc} P_a - \eta_{ac} P_b. \quad (11)$$

Analogously, we make a transition to the algebras containing arbitrary number of $(m-1)$ types of generators, with $m \geq 3$. Because there is no relation between spacetime dimension D and m, all the algebras $\mathfrak{C}_m$ can be used to construct the actions for a given dimension.[1] If the $\mathfrak{C}_m$-CS Lagrangian in odd dimensions $D \geq 5$ (with $m \geq D$) leads to the desired PL action limit, then $\mathfrak{C}_{m+1}$ offers exactly the same result. However, in the last case, the theory has more extra fields as the gauge symmetry is bigger. So, without loosing generality, we choose the minimal setting and only consider m being odd. The analysis might be extended for even dimensions $D \geq 6$ to obtain PL from BI gravities based on $\mathfrak{C}_m$ with m even, but we will not deal with that problem here (work in progress).

Thus, as we are restricting ourselves only to the odd values of the m index, we obtain $\mathfrak{C}_m$ algebras using the $S_M^{(m-2)} = \{\lambda_0, \lambda_1, \ldots, \lambda_{m-2}\}$ semigroup with the multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{if } \alpha+\beta \leq m-2, \\ \lambda_{\alpha+\beta-(m-1)}, & \text{if } \alpha+\beta > m-2, \end{cases} \quad (12)$$

and the resonant subset decomposition $S_M^{(m-2)} = S_0 \cup S_1$, with

$$S_0 = \{\lambda_{2i}\}, \quad \text{with} \quad i = 0, \ldots, \frac{m-3}{2},$$
$$S_1 = \{\lambda_{2i+1}\}, \quad \text{with} \quad i = 0, \ldots, \frac{m-3}{2}.$$

The new algebra will be spanned by $\{J_{ab,(i)}, P_{a,(i)}\}$, whose generators are related to the $\mathfrak{so}(D-1,2)$ ones through

$$J_{ab,(i)} = \lambda_{2i} \tilde{J}_{ab} \quad \text{and} \quad P_{a,(i)} = \lambda_{2i+1} \tilde{P}_a.$$

General commutation relations will be provided by

$$[J_{ab,(i)}, J_{cd,(j)}] = \eta_{bc} J_{ad,(i+j)\mathrm{mod}(\frac{m-1}{2})} - \eta_{ac} J_{bd,(i+j)\mathrm{mod}(\frac{m-1}{2})}$$
$$+ \eta_{ad} J_{bc,(i+j)\mathrm{mod}(\frac{m-1}{2})} - \eta_{bd} J_{ac,(i+j)\mathrm{mod}(\frac{m-1}{2})},$$
$$[J_{ab,(i)}, P_{a,(j)}] = \eta_{bc} P_{a,(i+j)\mathrm{mod}(\frac{m-1}{2})} - \eta_{ac} P_{b,(i+j)\mathrm{mod}(\frac{m-1}{2})},$$
$$[P_{a,(i)}, P_{b,(j)}] = J_{ab,(i+j+1)\mathrm{mod}(\frac{m-1}{2})}. \quad (13)$$

We can notice that the first nontrivial case is given by $\mathfrak{C}_4 = \mathfrak{so}(D-1,2) \oplus \mathfrak{so}(D-1,1)$, where $\mathfrak{C}_3$ trivially corresponds to the original $\mathfrak{so}(D-1,2)$ algebra.

As was mentioned before, quite interestingly, the S-expansion method allows us to express the invariant tensor of the final algebra in terms of the original one. Based on the definitions of Ref. [19], we can show that the

[1]The dimension of the Lie algebra depends on the spacetime dimension, which is encoded in the range of the Lorentz indices characterizing the generators.

only nonvanishing components of an invariant tensor in $D = 2n+1$ spacetime of order $n+1$ for the $\mathfrak{C}_m$ algebra (restricted only to the odd values of m) are given by

$$\langle J_{a_1 a_2,(i_1)} \cdots J_{a_{2n-1}a_{2n},(i_n)} P_{a_{2n+1},(i_{n+1})} \rangle = \frac{2^n}{n+1} \sigma_{2i+1} \delta^i_{j(i_1,i_2,\ldots,i_{n+1})} \epsilon_{a_1 a_2 \ldots a_{2n+1}}, \quad (14)$$

where the σ_j's are arbitrary dimensionless constants, and we have

$$j(i_1, i_2, \ldots, i_{n+1}) = (i_1 + i_2 + \cdots + i_{n+1}) \operatorname{mod}\left(\frac{m-1}{2}\right). \quad (15)$$

With these building blocks at hand, we can now turn to the construction of the corresponding CS actions.

III. CHERN-SIMONS GRAVITY AND PURE LOVELOCK ACTION IN $D = 5$

In the present section, we will show how the five-dimensional maximal pure Lovelock action (the first non-trivial example referred also as the pure Gauss-Bonnet [3,15]) could be obtained from a CS gravity theory for a particular choice of the $\mathfrak{C}_m$ algebra.

A CS action [41,42] is a quasiinvariant (i.e. invariant up to a boundary term) functional of a gauge connection one-form A, which is valued on a given Lie algebra $\mathfrak{g}$. In five dimensions, it is given by

$$I^{5\mathrm{D}}_{\mathrm{CS}} = k \int \left\langle A(dA)^2 + \frac{3}{2} A^3 dA + \frac{3}{5} A^5 \right\rangle, \quad (16)$$

where $\langle \cdots \rangle$ denotes the invariant tensor for a given algebra. Remarkably, the invariant tensor for the $\mathfrak{C}_m$ algebra [see Eq. (14)] splits various terms in the action into different sectors proportional to the arbitrary σ constants. As we are interested in obtaining the maximal PL action, the cosmological constant and the highest- (maximal-) order curvature terms both must be in the same sector. Use of $\mathfrak{C}_3 = \mathrm{AdS}$ and $\mathfrak{C}_4 = \mathrm{AdS} \oplus$ Lorentz algebras allow us to have these terms in the same sector, however, the Einstein-Hilbert term is also present there. In fact, these algebras correspond to the CS Lagrangians having all the gravitational terms (i.e. purely build from ω, e) proportional just to a single constant. As we will show now, this is not the case when $\mathfrak{C}_5$ is considered, making it an appropriate candidate to get the EH and the PL terms belonging to different sectors of the CS action.

To write down the action for the $\mathfrak{C}_5$ algebra from Eq. (11), we start from the connection one-form,

$$A = \frac{1}{2}\omega^{ab} J_{ab} + \frac{1}{l} e^a P_a + \frac{1}{2} k^{ab} Z_{ab} + \frac{1}{l} h^a R_a, \quad (17)$$

and the associated curvature two-form,

$$F = F^A T_A = \frac{1}{2}\mathcal{R}^{ab} J_{ab} + \frac{1}{l} \mathcal{T}^a P_a + \frac{1}{2} F^{ab} Z_{ab} + \frac{1}{l} H^a R_a, \quad (18)$$

where

$$\begin{aligned}
\mathcal{R}^{ab} &= d\omega^{ab} + \omega^a{}_c \omega^{cb} + k^a{}_c k^{cb} + \frac{2}{l^2} e^a h^b, \\
&= R^{ab} + k^a_c k^{cb} + \frac{2}{l^2} e^a h^b, \\
\mathcal{T}^a &= de^a + \omega^a{}_b e^b + k^a{}_b h^b = De^a + k^a{}_b h^b, \\
&= T^a + k^a{}_b h^b, \\
F^{ab} &= Dk^{ab} + \frac{1}{l^2} h^a h^b + \frac{1}{l^2} e^a e^b, \\
H^a &= Dh^a + k^a{}_b e^b.
\end{aligned}$$

Let us note that the covariant derivative $D = d + \omega$ is only with respect to the Lorentz part of the connection. See also that $\frac{1}{l^2} e^a e^b$ term appears in the F^{ab} curvature instead of being added to the Lorentz curvature as it usually happens in the AdS gravities. This is a particular feature of the Maxwell-like algebras, which all share the commutator $[P_a, P_b] = Z_{ab}$.

The presence of the l parameter in front of fields e^a and h^a in Eq. (17) is to make the dimensions right since our generators are dimensionless and so must be the gauge connection A. As usual in the general framework of (A)dS-CS theory [20], this parameter can be identified with the (A)dS radius ℓ introduced in Eq. (4), which characterizes the vacuum solutions in the theory (see also Appendix A). Similar identification can be made when we introduce a gauge connection A valued in other $\mathfrak{C}_m$ algebras. Thus, from now on, the scale parameter needed to make the connection dimensionless always will be give by the AdS radius ℓ.

Using Theorem VII.2 of Ref. [19], it is possible to show that the nonvanishing components of the invariant tensor for the $\mathfrak{C}_5$ algebra are given by

$$\begin{aligned}
\langle J_{ab} J_{cd} P_e \rangle &= \frac{4}{3}\sigma_1 \epsilon_{abcde}, & \langle J_{ab} J_{cd} Z_e \rangle &= \frac{4}{3}\sigma_3 \epsilon_{abcde}, \\
\langle J_{ab} Z_{cd} Z_e \rangle &= \frac{4}{3}\sigma_1 \epsilon_{abcde}, & \langle J_{ab} Z_{cd} P_e \rangle &= \frac{4}{3}\sigma_3 \epsilon_{abcde}, \\
\langle Z_{ab} Z_{cd} P_e \rangle &= \frac{4}{3}\sigma_1 \epsilon_{abcde}, & \langle Z_{ab} Z_{cd} Z_e \rangle &= \frac{4}{3}\sigma_3 \epsilon_{abcde},
\end{aligned} \quad (19)$$

where the factor $\frac{4}{3}$ comes from the original invariant tensor of the AdS algebra and, by the means of the expansion process, is transmitted to the other components. Note that σ_1 and σ_3 are arbitrary dimensionless constants allowing to separate the action into two different sectors. Then considering the gauge connection one-form (17) and the nonvanishing components of the invariant tensor in the general expression for the five-dimensional CS action we find

$$I^{5\mathrm{D}}_{\mathfrak{C}_5\text{-CS}} = k\int \sigma_1\left[\epsilon_{abcde}\left(\frac{1}{\ell}R^{ab}R^{cd}e^e + \frac{1}{5\ell^5}e^a e^b e^c e^d e^e\right) + \tilde{\mathcal{L}}_1(\omega, e, k, h,)\right] + \sigma_3\left[\epsilon_{abcde}\left(\frac{2}{3\ell^3}R^{ab}e^c e^d e^e\right) + \tilde{\mathcal{L}}_3(\omega, e, k, h)\right], \tag{20}$$

where $R^{ab} = d\omega^{ab} + \omega^a{}_c\omega^{cb}$ is usual Lorentz curvature and $\tilde{\mathcal{L}}_1$, $\tilde{\mathcal{L}}_3$ are explicitly given by

$$\begin{aligned}\tilde{\mathcal{L}}_1 = {}& \epsilon_{abcde}\left(\frac{2}{\ell}R^{ab}k^c{}_f k^{fd} + \frac{1}{\ell}k^a{}_f k^{fb}k^c{}_g k^{gd} + \frac{2}{\ell^5}h^a h^b e^c e^d + \frac{1}{\ell^5}h^a h^b h^c h^d + \frac{1}{\ell}Dk^{ab}Dk^{cd}\right.\\ &\left. + \frac{2}{3\ell^3}Dk^{ab}e^c e^d + \frac{2}{\ell^3}R^{ab}h^c e^d + \frac{2}{\ell^3}k^a{}_f k^{fb}h^c e^d + \frac{2}{\ell^3}Dk^{ab}h^c h^d\right)e^e\\ & + \epsilon_{abcde}\left(\frac{2}{\ell}Dk^{ab}R^{cd} + \frac{2}{\ell}Dk^{ab}k^c{}_f k^{fd} + \frac{2}{3\ell^3}R^{ab}h^c h^d + \frac{2}{3\ell^3}k^a{}_f k^{fb}h^c h^d\right)h^e\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathcal{L}}_3 = {}& \epsilon_{abcde}\left(\frac{2}{3\ell^3}k^a{}_f k^{fb}e^c e^d + \frac{1}{\ell^5}h^a e^b e^c e^d + \frac{2}{\ell^5}h^a h^b h^c e^d + \frac{2}{\ell^3}Dk^{ab}h^c e^d + \frac{2}{\ell^3}R^{ab}h^c h^d\right.\\ &\left. + \frac{2}{\ell^3}k^a{}_f k^{fb}h^c h^d + \frac{2}{\ell}Dk^{ab}R^{cd} + \frac{2}{\ell}Dk^{ab}k^c{}_f k^{fd}\right)e^e + \epsilon_{abcde}\left(\frac{1}{5\ell^5}h^a h^b h^c h^d\right.\\ &\left. + \frac{2}{3\ell^3}Dk^{ab}h^c h^d + \frac{1}{\ell}R^{ab}R^{cd} + \frac{2}{\ell}R^{ab}k^c{}_f k^{fd} + \frac{1}{\ell}k^a_f k^{fb}k^c_f k^{fd} + \frac{1}{\ell}Dk^{ab}Dk^{cd}\right)h^e.\end{aligned}$$

As we can see, the action (20) splits into two pieces. The part proportional to σ_1 contains the highest-order curvature term, the cosmological constant term and mixed terms with the new extra fields k^{ab} and h^a coupled to the spin connection and vielbein. On the other hand, the piece proportional to σ_3 contains the Einstein-Hilbert term and a part containing the new extra fields.

After choosing the constant σ_3 to vanish and in a matter-free configuration $(k^{ab} = h^a = 0)$ the action

$$I^{5\mathrm{D}} = k\int \sigma_1\epsilon_{abcde}\left(\frac{1}{\ell}R^{ab}R^{cd}e^e + \frac{1}{5\ell^5}e^a e^b e^c e^d e^e\right), \tag{21}$$

seems to resemble the structure of the maximal pure Lovelock action (1) for $p = 2$ in five dimensions with the identification $\frac{k\sigma_1}{\ell} = \kappa$. However, it does not match the sign of the constants in the desired action. Indeed, using Eq. (5) we see that these constants should be equal $\alpha_2 = \kappa$ and $\alpha_0 = -\frac{1}{5\ell^4}\kappa$. This relative sign difference makes the PL theory in odd dimensions and in the torsionless regime to have a unique nondegenerate dS and AdS vacuum (see Ref. [3]). This can be directly seen from the field equations

$$0 = \epsilon_{abcde}\left(R^{ab} - \frac{1}{\ell^2}e^a e^b\right)\left(R^{cd} + \frac{1}{\ell^2}e^c e^d\right). \tag{22}$$

One might ask if performing the expansion of the dS algebra might solve this sign problem but after looking at the full (A)dS Chern-Simons action in five dimensions,

$$I^{5\mathrm{D}}_{(\mathrm{A})\mathrm{dS\text{-}CS}} = \kappa\int \epsilon_{abcde}\left(\frac{1}{\ell}R^{ab}R^{cd}e^e \pm \frac{2}{3\ell^3}R^{ab}e^c e^d e^e + \frac{1}{5\ell^5}e^a e^b e^c e^d e^e\right), \tag{23}$$

it becomes clear that the cosmological term will have exactly the same sign for AdS and dS symmetries and we will end up with exactly the same action (21).

Naturally, the sign problem in our construction is inherited by the dynamics. When the previous conditions on the σ's and matter fields are imposed, the field equations of the action (20) read

$$0 = \epsilon_{abcde}\left(R^{ab}R^{cd} + \frac{1}{\ell^4}e^a e^b e^c e^d\right), \tag{24}$$

$$0 = \epsilon_{abcde}R^{cd}T^e, \tag{25}$$

$$0 = \epsilon_{abcde}R^{ab}e^c e^d, \tag{26}$$

$$0 = \epsilon_{abcde}e^c e^d T^e, \tag{27}$$

which do not correspond to the actual PL dynamics. The system does not even admit AdS space, $R^{ab} = -\frac{1}{\ell^2}e^a e^b$, as a vacuum solution. Moreover, the system does not have any maximally symmetric vacuum solutions. We can only recognize in the last line the torsionless condition.

The same results will be brought by considering the $\mathfrak{C}_6$ algebra, which contains more extra fields and it is generated by the set of generators $\{J_{ab}, P_a, Z_{ab,(1)}, Z_{ab,(2)}, R_{a,(1)}\}$ with the commutation relations given in Ref. [27].

All these facts about signs and dynamics, however, do not mean that the five-dimensional case is hopeless. To resolve it we should use a symmetry separating all the gravity terms into different sectors, instead of "gluing" some of them together. Thus, then we could put the whatever sign we want, simply by hand, since invariant tensors hold arbitrary values. In fact, we do not need to look very far: we can use for that purpose the $\mathfrak{C}_7$ algebra.

This can be treated as a generic method: for a particular dimension, we should use some $\mathfrak{C}_m$ algebra, with high enough value of the m index, to assure the separation of all purely (ω, e) terms. That will bring a much wider framework, not only producing the CS limit to the maximal PL action but also cases like the EH with Λ (see recent Ref. [43]) and a class of actions similar to the ones discussed in [17] having the form $R + LL$, where R is the curvature scalar and LL is understood as an arbitrary Lanczos-Lovelock term.

Following the definitions of Ref. [19], algebra $\mathfrak{C}_7$ is derived from $\mathfrak{so}(4,2)$ using $S_M^{(5)}$ as the relevant semigroup, whose elements satisfy Eq. (12). Its generators $\{J_{ab,(i)}, P_{a,(i)}\}$, with index $i = 0,1,2$ numerating different types, will be related to the $\mathfrak{so}(4,2)$ ones by

$$J_{ab,(i)} = \lambda_{2i}\tilde{J}_{ab},$$
$$P_{a,(i)} = \lambda_{2i+1}\tilde{P}_a,$$

and will satisfy Eq. (13). Let now A be the connection one-form for $\mathfrak{C}_7$,

$$A = \frac{1}{2}\omega^{ab,(i)}J_{ab,(i)} + \frac{1}{\ell}e^{a,(i)}P_{a,(i)}, \tag{28}$$

where the spin connection and vielbein being inside are defined as $\omega^{ab} = \omega^{ab,(0)}$ and $e^a = e^{a,(0)}$, respectively. The extra fields will be denoted as $k^{ab,(i)} = \omega^{ab,(i)}$ and $h^{a,(i)} = e^{a,(i)}$, with $i \neq 0$.

Using Theorem VII. 2 of Ref. [19], it is possible to show that the nonvanishing components of the invariant tensor are given by

$$\langle J_{ab,(q)}J_{cd,(r)}P_{e,(s)}\rangle = \frac{4}{3}\sigma_{2u+1}\delta^u_{j(q,r,s)}\epsilon_{abcde}, \tag{29}$$

where the σ's are arbitrary constants and $j(q,r,s)$ for $m = 7$ satisfies Eq. (15). We express the corresponding five-dimensional CS action as

$$I^{5D}_{\mathfrak{C}_7\text{-CS}} = k\int\sum_{i=0}^{2}\sigma_{2i+1}\mathcal{L}_{2i+1}, \tag{30}$$

where

$$\mathcal{L}_{2i+1} = \epsilon_{abcde}\left(\frac{\delta^i_{j(q,r,s)}}{\ell}R^{ab,(q)}R^{cd,(r)}e^{(s)} + \frac{2\delta^i_{j(q,r,s,u,1)}}{3\ell^3}R^{ab,(q)}e^{c,(r)}e^{d,(s)}e^{e,(u)} + \frac{\delta^i_{j(q,r,s,u,v,2)}}{5\ell^5}e^{a,(q)}e^{b,(r)}e^{c,(s)}e^{d,(u)}e^{e,(v)}\right), \tag{31}$$

and $j(\ldots)$ is satisfying Eq. (15). The explicit form of the action expression can be found in Appendix B. After writing the purely gravitational terms (ω, e) separately from those containing extra fields we obtain

$$I^{5D}_{\mathfrak{C}_7\text{-CS}} = k\int\sigma_1\left[\epsilon_{abcde}\left(\frac{1}{\ell}R^{ab}R^{cd}e^e\right) + \tilde{\mathcal{L}}_1(\omega^{(i)}, e^{(j)})\right] + \sigma_3\left[\epsilon_{abcde}\left(\frac{2}{3\ell^3}R^{ab}e^c e^d e^e\right) + \tilde{\mathcal{L}}_3(\omega^{(i)}, e^{(j)})\right] + \sigma_5\left[\epsilon_{abcde}\left(\frac{1}{5\ell^5}e^a e^b e^c e^d e^e\right) + \tilde{\mathcal{L}}_5(\omega^{(i)}, e^{(j)})\right]. \tag{32}$$

We can see that $\mathfrak{C}_7$ allows us to have each term in a different sector. When the σ_3 constant vanishes and $\sigma_5 = -\sigma_1$, the matter-free configuration limit $(k^{ab,(i)} = h^{a,(j)} = 0)$ leads to the maximal $p = 2$ pure Lovelock action

$$I^{5D}_{\text{CS}\to\text{PL}} = k\int\sigma_1\epsilon_{abcde}\left(\frac{1}{\ell}R^{ab}R^{cd}e^e - \frac{1}{5\ell^5}e^a e^b e^c e^d e^e\right). \tag{33}$$

An additional nontrivial choice of the constants to consider is $\sigma_1 = 0$ with $\sigma_3 = \sigma_5$. This particular choice leads to the $p = 1$ pure Lovelock corresponding to EH gravity with cosmological constant. Thus, we have shown that the PL actions of different orders can be related to the CS action for the $\mathfrak{C}_7$ algebra. We remind that these choices of the σ's are allowed since they are all arbitrary.

Unfortunately, although matching form of the actions, this still does not lead to the PL dynamics alone. When the constant σ_3 vanishes, a matter-free configuration is considered and $\sigma_5 = -\sigma_1$, the field equations read, respectively,

$$
\begin{aligned}
0 &= \epsilon_{abcde}\left(R^{ab}R^{cd} - \frac{1}{\ell^4}e^a e^b e^c e^d\right)\delta e^e, \\
0 &= \epsilon_{abcde}\left(\frac{2}{\ell^2}R^{ab}e^c e^d - \frac{1}{\ell^4}e^a e^b e^c e^d\right)\delta h^{e,(1)}, \\
0 &= \epsilon_{abcde}\left(R^{ab}R^{cd} - \frac{2}{\ell^2}R^{ab}e^c e^d\right)\delta h^{e,(2)},
\end{aligned} \tag{34}
$$

and

$$
\begin{aligned}
0 &= \epsilon_{abcde}(R^{cd}T^e)\delta\omega^{ab}, \\
0 &= \epsilon_{abcde}\left(\frac{2}{\ell^2}e^c e^d T^e\right)\delta k^{ab,(1)}, \\
0 &= \epsilon_{abcde}\left(R^{cd}T^e - \frac{2}{\ell^2}e^c e^d T^e\right)\delta k^{ab,(2)}.
\end{aligned} \tag{35}
$$

As it was pointed out in the Introduction, if we use the first-order formalism to describe PL gravity, then the vanishing of the torsion must be imposed by hand (or by introducing suitable Lagrange multiplier [3]). Here instead, this condition is coming from the variation of the extra $\delta k^{ab,(1)}$ field.

When the matter fields are switched off and torsionless condition is assumed the solution still has to simultaneously satisfy the PL and two other unusual equations. This problem has already appeared in Ref. [20], but there one finds superposition of vanishing of the RR and $Reee$ terms. Such restriction of the geometry rejected possibility of the spherical solutions but at least it was fulfilled by the pp-waves.

One can clearly see that proper handling of the full problem requires a more subtle treatment. Nevertheless, if we consider at the level of the action (32) some special extra fields identification among extra fields:

$$
h^{a,(1)} = h^{a,(2)} = h^a \quad \text{and} \quad k^{ab,(1)} = k^{ab,(2)} = k^{ab}, \tag{36}
$$

and later impose $\sigma_3 = 0$, $\sigma_5 = -\sigma_1$, then we will be able to find the following field equations in a matter-free configuration limit:

$$
\begin{aligned}
0 &= \epsilon_{abcde}\left(R^{ab}R^{cd} - \frac{1}{\ell^4}e^a e^b e^c e^d\right)\delta e^e, \\
0 &= \epsilon_{abcde}\left(R^{ab}R^{cd} - \frac{1}{\ell^4}e^a e^b e^c e^d\right)\delta h^e, \\
0 &= \epsilon_{abcde}(R^{cd}T^e)\delta\omega^{ab}, \\
0 &= \epsilon_{abcde}(R^{cd}T^e)\delta k^{ab}.
\end{aligned} \tag{37}
$$

Thus, we obtain the appropriate PL dynamics described in Refs. [12,13,16]. Note that although we are using the same σ's constants and eventually we enforce the extra matter free configuration limit, the PL dynamics is recovered only from the Lagrangian $\mathcal{L}(\omega, e, h, k)$ and not from the $\mathcal{L}(\omega, e, k^{(1)}, h^{(1)}, k^{(2)}, h^{(2)})$. Interestingly, proposed type of identification on the matter field from (36) can be applied not only in a five-dimensional CS action but also in higher-dimensional cases.

One could try to interpret h^a as the vielbein, but it is straightforward to see that the action (32) reproduces the PL action only when e^a is identified as the true vielbein. Moreover, the matter extra fields (h^a, k^{ab}) could allow to introduce a generalized cosmological term to the CS gravity action in an analogous way to the one introduced in the four-dimensional case [27,37].

For completeness, we will now supply the generalization of our construction to higher dimensions.

IV. THE HIGHER-DIMENSIONAL PURE LOVELOCK AND CHERN-SIMONS GRAVITY ACTIONS

In this section, we present the general setup to obtain the $(2n+1)$-dimensional PL action for any value of p from CS gravity theory using the Maxwell type $\mathfrak{C}_m$ family. In particular, we show that the $\mathfrak{C}_7$ algebra for seven dimensions allows us to recover the maximal $p = 3$ PL action, where the sign problem from the five-dimensional case does not occur.

Let us remind the generic form of the PL constants

$$
\alpha_p = \frac{1}{(D-2p)!}\kappa \quad \text{and} \quad \alpha_0 = -\frac{(\mp 1)^p}{D(D-2p-1)!\ell^{2p}}\kappa, \tag{38}
$$

with AdS ($-$sign) and the dS ($+$sign), and compare them with the generic (A)dS-Chern-Simons ones [44,45]

$$
\tilde{\beta}_p = \frac{(-\mathrm{sgn}(\Lambda))^p}{(D-2p)\ell^{D-2p}}\binom{\frac{D-1}{2}}{p}k \quad \text{and} \quad \tilde{\beta}_0 = \frac{1}{D\ell^D}k. \tag{39}
$$

Notice that for the PL constants p is fixed, while for the CS constants index p is running from 0 to the maximum power of the curvature $N = [\frac{D-1}{2}]$. The construction of the

$D = 2n + 1$-dimensional $\mathfrak{C}_m$ Chern-Simons has to take into account also the arbitrary constants σ,

$$\beta_p = \frac{(-\mathrm{sgn}(\Lambda))^p}{(D-2p)\ell^{D-2p}} \binom{\frac{D-1}{2}}{p} k\sigma_{2i+1}\delta^i_{j(i_1,\ldots,i_{D-p},\frac{D-1}{2}-p)} \quad \text{and}$$

$$\beta_0 = \frac{1}{D\ell^D} k\sigma_{2i+1}\delta^i_{j(i_1,\ldots,i_D,\frac{D-1}{2})},$$

where algebraic m dependence appears explicitly in the definition of

$$j(i_1, i_2, \ldots, i_{n+1}) = (i_1 + i_2 + \cdots + i_{n+1}) \mathrm{mod}\left(\frac{m-1}{2}\right)$$

and is responsible for shifting terms into different sectors of the invariant tensor. Then restricting only to the purely gravitational terms we obtain

$$\mathrm{PL}^{(A)\mathrm{dS}}: \ \frac{\alpha_0}{\alpha_p} = -(\mp 1)^p \frac{(D-2p)}{D} \frac{1}{\ell^{2p}},$$

$$\mathrm{CS}^{\mathfrak{C}_m}: \ \frac{\beta_0}{\beta_p} = (-\mathrm{sgn}(\Lambda))^p \frac{(D-2p)}{D} \frac{1}{\ell^{2p}} \frac{1}{\binom{\frac{D-1}{2}}{p}} \times \frac{\delta^i_{n \bmod \frac{m-1}{2}}}{\delta^j_{(n-p) \bmod \frac{m-1}{2}}} \frac{\sigma_{2i+1}}{\sigma_{2j+1}},$$

from where we see that

$$(-\mathrm{sgn}(\Lambda))^p_{\mathrm{CS}} \frac{1}{\binom{n}{p}} \frac{\sigma_{2(n \bmod \frac{m-1}{2})+1}}{\sigma_{2((n-p) \bmod \frac{m-1}{2})+1}} = -(\mp 1)^p_{\mathrm{PL}}. \qquad (40)$$

For the same σ's, whose cancellation forces $\binom{n}{p} = 1$, we are restricted only to the maximal $p = N$ PL. Then we see that $p = even$ leads to a sign contradiction, as we have found explicitly in the previous section. This problem does not appear when the maximal order happens to be $p = odd$, which is possible only for $D = 3, 7, 11, \ldots, 4k-1$. For other situations, it is necessary to absorb the problematic sign and the numerical factors into a definition of one of the σ's.

Indeed, in Sec. II we have shown that the five-dimensional CS action constructed using $\mathfrak{C}_5$ could not lead to the maximal $p = 2$ PL action with the proper sign but could only produce the right $p = 1$ PL action, which is just the Einstein-Hilbert action with Λ. Finally, we managed to show that $\mathfrak{C}_7$ algebra in that context allowed us to accommodate minus sign by the means of setting $\sigma_5 = -\sigma_1$, which effectively provides good maximal $p = 2$ PL action.

On the other hand, the right sign for the maximal $p = 3$ PL in seven dimensions will be obtained in a straightforward way, simply by using $\mathfrak{C}_7$. Additionally, by using $\mathfrak{C}_9$ algebra, we can derive other seven-dimensional PL actions, $p = 2$ and $p = 1$. Naturally $p = 3$ can be also establish within that bigger algebra. However, taking into account growing complexity with the transition to the higher value of m (see for example Appendix B), we will be always targeting in the minimal setup leading to the right result.

Then the full picture in odd dimensions becomes clear and it can be summarized in the following way:

(i) for $D = 4k-1$, the smallest representative in the $\mathfrak{C}_m$ family that allows to obtain maximal $p = N$ PL action is given by $\mathfrak{C}_D$, whereas for any other order of PL $p = 1, \ldots, (N-1)$ one needs to use $\mathfrak{C}_{D+2}$,

(ii) for $D \neq 4k-1$, the smallest representative in the $\mathfrak{C}_m$ family that allows us to obtain arbitrary order of PL action for $p = 1, \ldots, N$ is given by $\mathfrak{C}_{D+2}$.

Obviously, all these PL actions could be recovered as a limit using even bigger $\mathfrak{C}_m$ algebras, but then one needs to introduce more extra fields and conditions on the σ's.

Although the PL actions can be obtained from the CS theory, their right dynamical limit requires appropriate identifications of the extra fields at the level of the action in order to reproduce the PL dynamics:

$$h^{a,(1)} = h^{a,(2i+1)} = (\pm 1)^p h^{a,(2i)},$$
$$k^{ab,(1)} = k^{ab,(2i+1)} = (\pm 1)^p k^{ab,(2i)}, \qquad (41)$$

for all $i = 1, 2, \ldots, \frac{m-3}{2}$.

One should notice that even if we consider the dS case as a starting point, it would lead to the same sign problems, as was mentioned in the five-dimensional case.

A. Chern-Simons gravity and pure Lovelock action in $D = 7$

The generic form of a CS action in seven dimensions [41,42] is given by

$$I^{7\mathrm{D}}_{\mathrm{CS}} = k \int \left\langle A(dA)^3 + \frac{8}{5} A^3 (dA)^2 + \frac{4}{5} A (dA) A^2 (dA) + 2A^5 dA + \frac{4}{7} A^7 \right\rangle. \qquad (42)$$

Let us first consider the $\mathfrak{C}_7 = \{J_{ab,(i)}, P_{a,(i)}\}$ algebra, whose generators satisfy Eq. (13), with $i = 0, 1, 2$. The associated connection one-form is defined as

$$A = \frac{1}{2} \omega^{ab,(i)} J_{ab,(i)} + \frac{1}{\ell} e^{a,(i)} P_{a,(i)}. \qquad (43)$$

From Theorem VII. 2 of Ref. [19] and according to Eq. (14), it is possible to show that the nonvanishing components of the invariant tensor of order 4 for $\mathfrak{C}_7$ are given by

$$\langle J_{ab,(q)} J_{cd,(r)} J_{ef,(s)} P_{g,(u)} \rangle = 2\sigma_{2v+1} \delta^v_{j(q,r,s,u)} \epsilon_{abcdefg}, \qquad (44)$$

where the σ's are arbitrary constants and $j(q,r,s,u)$ satisfies Eq. (15). Then using the invariant tensor for $\mathfrak{C}_7$ (29) and introducing the gauge connection one-form (43) in the general expression for the seven-dimensional CS action gives

$$I^{7\mathrm{D}}_{\mathfrak{C}_7\text{-CS}} = k\int \sum_{i=0}^{2} \sigma_{2i+1}\mathcal{L}_{2i+1}, \tag{45}$$

where

$$\begin{aligned}\mathcal{L}_{2i+1} = \epsilon_{abcdefg}\Bigg(&\frac{\delta^{i}_{j(q,r,s,u)}}{\ell} R^{ab,(q)}R^{cd,(r)}R^{ef,(s)}e^{g,(u)} + \frac{\delta^{i}_{j(q,r,s,u,v,1)}}{3\ell^3} R^{ab,(q)}R^{cd,(r)}e^{e,(s)}e^{f,(u)}e^{g,(v)}\\ &+\frac{3\delta^{i}_{j(q,r,s,u,v,w,2)}}{5\ell^5} R^{ab,(q)}e^{a,(r)}e^{b,(s)}e^{c,(u)}e^{d,(v)}e^{e,(w)}\\ &+\frac{\delta^{i}_{j(q,r,s,u,v,w,o,3)}}{7\ell^7} e^{a,(q)}e^{b,(r)}e^{c,(s)}e^{d,(u)}e^{e,(v)}e^{f,(w)}e^{g,(o)}\Bigg).\end{aligned}$$

Separating the purely gravitational terms from those containing extra fields, the action can be written as

$$\begin{aligned}I^{7\mathrm{D}}_{\mathfrak{C}_7\text{-CS}} = k\int \sigma_1&\left[\epsilon_{abcdefg}\left(\frac{1}{\ell}R^{ab}R^{cd}R^{ef}e^g + \frac{1}{7\ell^7}e^a e^b e^c e^d e^e e^f e^g\right) + \tilde{\mathcal{L}}_1(\omega^{(i)}, e^{(j)})\right]\\ &+\sigma_3\left[\epsilon_{abcdefg}\left(\frac{1}{\ell^3}R^{ab}R^{cd}e^e e^f e^g\right) + \tilde{\mathcal{L}}_3(\omega^{(i)}, e^{(j)})\right]\\ &+\sigma_5\left[\epsilon_{abcdefg}\left(\frac{3}{5\ell^5}R^{ab}e^c e^d e^e e^f e^g\right) + \tilde{\mathcal{L}}_5(\omega^{(i)}, e^{(j)})\right].\end{aligned} \tag{46}$$

Let us notice that the action (46) splits into three pieces. The part proportional to σ_1 contains the maximal PL Lagrangian and mixed terms containing extra fields $k^{ab,(i)} = \omega^{ab,(i)}$ and $h^{a,(i)} = e^{a,(i)}$, with $i \neq 0$. Unlike the five-dimensional case, when the constants σ_3, σ_5 vanish a matter-free configuration ($k^{ab,(i)} = h^{a,(i)} = 0$) leads to the action corresponding to the maximal $p = 3$ PL:

$$I^{7\mathrm{D}}_{\mathrm{CS}\to\mathrm{PL}} = k\int \sigma_1\epsilon_{abcdefg}\left(\frac{1}{\ell}R^{ab}R^{cd}R^{ef}e^g + \frac{1}{7\ell^7}e^a e^b e^c e^d e^e e^f e^g\right). \tag{47}$$

Following the same procedure for the bigger $\mathfrak{C}_9$ algebra, we extend the form of the connection (43) to the range of $i = 0, 1, 2, 3$, which also makes

$$I^{7\mathrm{D}}_{\mathfrak{C}_9\text{-CS}} = k\int \sum_{i=0}^{3} \sigma_{2i+1}\mathcal{L}_{2i+1}.$$

That gives the $\mathfrak{C}_9$-CS action, which now splits into four terms

$$\begin{aligned}I^{7\mathrm{D}}_{\mathfrak{C}_9\text{-CS}} = k\int \sigma_1&\left[\epsilon_{abcdefg}\left(\frac{1}{\ell}R^{ab}R^{cd}R^{ef}e^g\right) + \tilde{\mathcal{L}}_1(\omega^{(i)}, e^{(j)})\right] + \sigma_3\left[\epsilon_{abcdefg}\left(\frac{1}{\ell^3}R^{ab}R^{cd}e^e e^f e^g\right) + \tilde{\mathcal{L}}_3(\omega^{(i)}, e^{(j)})\right]\\ &+\sigma_5\left[\epsilon_{abcdefg}\left(\frac{3}{5\ell^5}R^{ab}e^c e^d e^e e^f e^g\right) + \tilde{\mathcal{L}}_5(\omega^{(i)}, e^{(j)})\right] + \sigma_7\left[\epsilon_{abcdefg}\left(\frac{1}{7\ell^7}e^a e^b e^c e^d e^e e^f e^g\right) + \tilde{\mathcal{L}}_7(\omega^{(i)}, e^{(j)})\right].\end{aligned} \tag{48}$$

Interestingly, each term of the original seven-dimensional AdS-CS action appears in a different sector of the $\mathfrak{C}_9$-CS action. This feature allows us to reproduce any p order of the PL theory, as it happened for $\mathfrak{C}_7$ algebra in the five-dimensional case. Each order can be derived in a matter-free configuration, after imposing the following conditions:

$$p = 1\colon\ \sigma_1 = \sigma_3 = 0, \qquad \sigma_5 = \sigma_7, \tag{49}$$

$$p = 2\colon\ \sigma_1 = \sigma_5 = 0, \qquad -\sigma_3 = \sigma_7, \tag{50}$$

$$p = 3\colon\ \sigma_3 = \sigma_5 = 0, \qquad \sigma_1 = \sigma_7. \tag{51}$$

The first case reproduces the EH with cosmological constant, the second represents first nontrivial seven-dimensional PL, while third one gives the maximal PL. Note that last result was already achieved by the smaller $\mathfrak{C}_7$ algebra, as the spacetime dimension is a particular case of $D = 4k - 1$.

When we turn to the dynamical limit of (49)–(51) we face very complicated superposition of the field equations. Only minimal and maximal cases, $p = 1$ and $p = 3$ automatically assure the torsionless condition. For the other intermediate case, $p = 2$, vanishing of torsion is not coming from field equations. On the other hand, the following identification in the action (48):

$$h^{a,(1)} = h^{a,(3)} = (\pm 1)^p h^{a,(2)},$$
$$k^{ab,(1)} = k^{ab,(3)} = (\pm 1)^p k^{ab,(2)}, \tag{52}$$

reproduces the desired PL dynamics for any p order after considering the appropriate conditions on the σ's and in a matter-free configuration limit.

One might try to look for a modification of the $\mathfrak{C}_m$ algebra that, besides giving the right limit for the action, it leads also to the right dynamical limit without any identification on the extra fields. However, we will leave that task for a future work.

It is important to point out that other nontrivial conditions on the σ's can lead to other interesting gravity theories. In fact, after choosing $\sigma_5 = \sigma_1$ or $\sigma_5 = \sigma_3$ and killing all other σ constants, a matter-free configuration will reproduce the class of actions discussed in Ref. [17] having the form $EH + LL$, where LL corresponds to an arbitrary Lanczos-Lovelock term.

B. Chern-Simons gravity and Pure Lovelock action in $D = 2n + 1$

To assure the arbitrary p-order PL action resulting from $D = 2n + 1$ CS gravity action, we require the use of $\mathfrak{C}_{D+2}$ algebra. As previously, we start with the connection one-form,

$$A = \frac{1}{2}\omega^{ab,(i)} J_{ab,(i)} + \frac{1}{\ell} e^{a,(i)} P_{a,(i)}, \tag{53}$$

with $i = 0, \ldots, n$ and the nonvanishing components of the invariant tensor given by Eq. (14). It is possible to show that the $(2n+1)$-dimensional CS action for the $\mathfrak{C}_{D+2}$ algebra is then given by

$$I^{2n+1}_{\mathfrak{C}_{D+2}\text{-CS}} = k \int \epsilon_{a_1 a_2 \ldots a_{2n+1}} \sum_{p=0}^{n} \frac{\ell^{2(p-n)-1}}{2(n-p)+1} \binom{n}{p} \sigma_{2i+1} \times \delta^{i}_{j(i_1,\ldots,i_{2n+1-p},n-p)} R^{a_1 a_2,(i_1)} \cdots R^{a_{2p-1} a_{2p},(i_p)} \times e^{a_{2p+1},(i_{p+1})} \cdots e^{a_{2n+1},(i_{2n+1-p})}, \tag{54}$$

where we identify $\omega^{ab} = \omega^{ab,(0)}$, $e^a = e^{a,(0)}$ and function $j(\ldots)$ in the Kronecker delta satisfies Eq. (15). The action (54) is separated into $n+1$ pieces proportional to the different σ's and under specific conditions $(\sigma_{2n+1-2p} = (\pm 1)^{p+1}\sigma_{2n+1}$, vanishing all other σ constants and of course applying the free-matter configuration limit) gives the p-order PL action:

$$I^{2n+1}_{\text{PL}} = k \int \sigma_{2n+1-2p} \epsilon_{a_1 a_2 \ldots a_{2n+1}} \times \left(\frac{\ell^{2(p-n)-1}}{2(n-p)+1} \binom{n}{p} R^{a_1 a_2} \cdots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \cdots e^{a_{2n+1}} + \frac{\ell^{-2n-1}}{(2n+1)} e^{a_1} e^{a_2} \cdots e^{a_{2n+1}} \right). \tag{55}$$

Naturally, for $D = 4k - 1$ to derive the maximal PL action from CS gravity theory, it is enough to use the $\mathfrak{C}_D$ algebra. The only difference will appear in the range of the indices $i = 0, \ldots, n-1$ enumerating various types of generators and, consequently, in the modulo function inside the definition of $j(\ldots)$ present in Eq. (54).

Interestingly, we observe that the extra field content $k^{ab,(i)}$ for minimal $p = 1$ (EH) and maximal $p = N$ PL always leads to the explicit torsionless condition, whereas for other situations the torsion remains still involved with the rest of the field equations.

Arbitrariness of the σ's allows us to construct a wide class of actions, where we can couple arbitrary LL terms together. One of the notable examples could be the Einstein-Hilbert term equipped by the arbitrary p-order LL term [17] (coming from setting $\sigma_{2n+1-2p} = \sigma_{2n-1}$ for $p \neq 0$ and vanishing all others σ's):

$$I^{2n+1}_{\text{EH+LL}} = k \int \sigma_{2n+1-2p} \epsilon_{a_1 a_2 \ldots a_{2n+1}} \left(\frac{n\ell^{1-2n}}{2n-1} R^{a_1 a_2} e^{a_3} \cdots e^{a_{2n+1}} + \frac{\ell^{2(p-n)-1}}{2(n-p)+1} \binom{n}{p} \times R^{a_1 a_2} \cdots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \cdots e^{a_{2n+1}} \right). \tag{56}$$

V. CONCLUSION

We have managed to explore applications of the $\mathfrak{C}_m$ algebras in a CS gravity theory to incorporate the PL actions as a special limit. It was achieved by the fact that the $\mathfrak{C}_m$ algebra is shifting various powers of the Riemann tensor into different sectors of the invariant tensor. Although it allowed us to obtain the proper p-order PL action in arbitrary spacetime dimension (by particular choice of the algebra and by manipulating the constants), we faced a problem in the dynamical limit caused by additional field equations, similarly as happens in Ref. [20] in the context of CS and GR. For the maximal and minimal case we see that the extra fields give the explicit torsionless condition, which can be an interesting feature. Finally to overcome the nontrivial superposition of the field equations overconstraining standard solutions we have proposed a particular identification of the matter fields at the level of the action allowing to reproduce the correct PL dynamics for any value of p.

Altogether, the general treatment involves for arbitrary odd D-dimensional spacetime the use of $\mathfrak{C}_{D+2}$ as it allows us to fully manipulate the gravitational terms through imposing suitable conditions on the σ constants. In this way, we obtained not only the maximal but also the arbitrary order of the PL action along with the gravity theories similar to those discussed in Ref. [17]. Some cases can be achieved with less effort. For example, in $D = 4k-1$ dimensions, it is sufficient to use the $\mathfrak{C}_D$ algebra to get the maximal PL.

The possibility of finding a suitable symmetry leading also to the right dynamical limit with less conditions on the extra fields still remains as an open problem. This will be approached in a future work with techniques developed in Refs. [46,47]. Finally, the mechanism presented here could be useful to derive the supersymmetric version of the PL theory.

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APPENDIX A: RELATION OF THE CONSTANTS

Here we present the explicit relation between the constant k appearing in the five-dimensional CS action and the invariant tensor constants.

The pure Lovelock theory is described by the action

$$I_{\rm PL}^{\rm 5D} = \kappa \int \epsilon_{abcde}\left(\frac{1}{\ell}R^{ab}R^{cd}e^e - \frac{1}{5\ell^5}e^a e^b e^c e^d e^e\right), \qquad \text{(A1)}$$

which is equivalent to the tensorial action

$$I_{\rm PL}^{\rm 5D} = -\kappa \int \sqrt{-g}\left(\frac{1}{4}R^{\rho\sigma}_{\mu\nu}R^{\gamma\lambda}_{\alpha\beta}\delta^{\mu\nu\alpha\beta}_{\rho\sigma\gamma\lambda} - 2\Lambda\right)d^5x. \qquad \text{(A2)}$$

When dealing with solutions on this theory [12,13,16], in both formalisms, the vanishing of torsion is usually imposed by hand, either by $T^a = 0$ or $T^\alpha_{\mu\nu} = 0$.

We, however, are interested in finding this action as a special limit of a CS action constructed for a specific symmetry,

$$I_{\rm CS}^{\rm 5D} = k \int \left\langle A(dA)^2 + \frac{3}{2}A^3 dA + \frac{3}{5}A^5\right\rangle,$$

where A is the gauge connection valued on the corresponding algebra and k (different, in principle, from κ) is a constant that makes this functional of A have dimension of the action. We notice that all the factors are dimensionless, as the connection is constructed in a way that is dimensionless too. When the algebra is chosen to be $\mathfrak{C}_7$, then the connection is given by

$$A = \frac{1}{2}\omega^{ab}J_{ab} + \frac{1}{l}e^a P_a + \frac{1}{2}k^{ab,(1)}Z_{ab,(1)} + \frac{1}{l}h^{a,(1)}R_{a,(1)} + \frac{1}{2}k^{ab,(2)}Z_{ab,(2)} + \frac{1}{l}h^{a,(2)}R_{a,(2)},$$

where the length parameter l assures that A is indeed dimensionless (remember that the vielbein has dimension of length, and then extra fields like $h^{a,(i)}$ acquire the same dimension).

To analyze the relation between $\mathfrak{C}_7$-CS and PL actions, we need to identify l with the AdS radius ℓ characterizing a vacuum solution of the theory, as it is usually done in (A) dS-CS theory [20]. In Sec. III, we showed that, effectively, this leads to

$$I_{\rm CS\to PL}^{\rm 5D} = k\sigma_1 \int \epsilon_{abcde}\left(\frac{1}{\ell}R^{ab}R^{cd}e^e - \frac{1}{5\ell^5}e^a e^b e^c e^d e^e\right), \qquad \text{(A3)}$$

where σ_1 is a dimensionless arbitrary constant and the factor k carries suitable dimensions making the functional have units of an action. Comparing this with Eq. (A1) written as

$$I_{\rm PL}^{\rm 5D} = \ell\kappa \int \epsilon_{abcde}\left(\frac{1}{\ell}R^{ab}R^{cd}e^e - \frac{1}{5\ell^5}e^a e^b e^c e^d e^e\right), \qquad \text{(A4)}$$

we immediately recognize that

$$k = \sigma_1^{-1}\ell\kappa, \qquad \text{(A5)}$$

which is clearly dimensionless.

The same line of reasoning was performed in Sec. IV to analyze other examples of $\mathfrak{C}_m$ in higher odd dimensions.

APPENDIX B: 5D CS ACTION

In this appendix, we present the explicit expression of the five-dimensional $\mathfrak{C}_7$-CS action. To this end, let us consider the explicit connection one-form

$$A=\frac{1}{2}\omega^{ab}J_{ab}+\frac{1}{l}e^{a}P_{a}+\frac{1}{2}k^{ab,(1)}Z_{ab,(1)}+\frac{1}{l}h^{a,(1)}R_{a,(1)}+\frac{1}{2}k^{ab,(2)}Z_{ab,(2)}+\frac{1}{l}h^{a,(2)}Z_{a,(2)}, \tag{B1}$$

and the 18 nonvanishing components of the invariant tensor for the $\mathfrak{C}_7$ algebra:

$$\langle J_{ab,(q)}J_{cd,(r)}P_{e,(s)}\rangle=\frac{4}{3}\sigma_{2u+1}\delta^{u}_{j(q,r,s)}\epsilon_{abcde}. \tag{B2}$$

After using the connection one-form (B1) and the nonvanishing components of the invariant tensor in the general expression for the five-dimensional CS action (16), we obtain

$$\begin{aligned}I^{5D}_{\mathfrak{C}_7\text{-CS}}=k\int\sigma_1&\left[\epsilon_{abcde}\frac{1}{\ell}R^{ab}R^{cd}e^{e}+\tilde{\mathcal{L}}_1(\omega,e,k^{(1)},h^{(1)},k^{(2)},h^{(2)})\right]+\sigma_3\left[\epsilon_{abcde}\frac{2}{3\ell^3}R^{ab}e^{c}e^{d}e^{e}+\tilde{\mathcal{L}}_3(\omega,e,k^{(1)},h^{(1)},k^{(2)},h^{(2)})\right]\\&+\sigma_5\left[\epsilon_{abcde}\frac{1}{5\ell^5}e^{a}e^{b}e^{c}e^{d}e^{e}+\tilde{\mathcal{L}}_5(\omega,e,k^{(1)},h^{(1)},k^{(2)},h^{(2)})\right],\end{aligned} \tag{B3}$$

where $\tilde{\mathcal{L}}_1$, $\tilde{\mathcal{L}}_3$, $\tilde{\mathcal{L}}_5$ are explicitly given by

$$\begin{aligned}\tilde{\mathcal{L}}_1=\epsilon_{abcde}&\left(\frac{4}{\ell}R^{ab}k^{c,(1)}_{f}k^{fd,(2)}+\frac{4}{\ell}k^{a,(1)}_{f}k^{fb,(2)}k^{c,(1)}_{g}k^{gd,(2)}+\frac{2}{\ell}R^{ab,(1)}R^{cd,(2)}+\frac{2}{\ell^3}R^{ab}h^{c,(1)}h^{d,(1)}\right.\\&+\frac{2}{\ell^3}R^{ab,(1)}h^{c,(1)}e^{d}+\frac{2}{\ell^3}R^{ab,(1)}h^{c,(2)}h^{d,(2)}+\frac{2}{3\ell^3}R^{ab,(2)}e^{c}e^{d}+\frac{1}{\ell^5}h^{a,(1)}e^{b}e^{c}e^{d}\\&+\frac{1}{\ell^5}h^{a,(1)}h^{b,(1)}h^{c,(1)}h^{d,(1)}+\frac{2}{\ell^5}h^{a,(2)}h^{b,(2)}e^{c}e^{d}+\frac{6}{\ell^5}h^{a,(1)}h^{b,(1)}h^{c,(2)}e^{d}\\&\left.+\frac{4}{\ell^5}h^{a,(1)}h^{b,(2)}h^{c,(2)}h^{d,(2)}\right)e^{e}+\epsilon_{abcde}\left(\frac{2}{\ell}R^{ab}R^{cd,(2)}+\frac{1}{\ell}R^{ab,(1)}R^{cd,(1)}+\frac{4}{\ell}R^{ab,(2)}k^{c,(1)}_{g}k^{gd,(2)}\right.\\&+\frac{2}{3\ell^3}R^{ab,(2)}h^{c,(1)}h^{d,(1)}+\frac{6}{\ell^3}R^{ab,(2)}h^{c,(2)}e^{d}+\frac{2}{\ell^3}R^{ab,(1)}h^{c,(1)}h^{d,(2)}+\frac{2}{\ell^3}R^{ab}h^{c,(2)}h^{d,(2)}\\&\left.+\frac{2}{\ell^5}h^{a,(2)}h^{b,(2)}h^{c,(1)}h^{d,(1)}\right)h^{e,(1)}+\epsilon_{abcde}\left(\frac{2}{\ell}R^{ab}R^{cd,(1)}+\frac{1}{\ell}R^{ab,(2)}R^{cd,(2)}+\frac{4}{\ell}R^{ab,(1)}k^{c,(1)}_{g}k^{gd,(2)}\right.\\&+\frac{2}{\ell^3}R^{ab}e^{c}e^{d}+\frac{2}{\ell^3}R^{ab}h^{c,(1)}h^{d,(2)}+\frac{2}{\ell^3}R^{ab,(1)}h^{c,(1)}h^{d,(1)}+\frac{6}{\ell^3}R^{ab,(2)}h^{c,(1)}e^{d}\\&\left.+\frac{2}{3\ell^3}R^{ab,(2)}h^{c,(2)}h^{d,(2)}+\frac{1}{5\ell^5}h^{a,(2)}h^{b,(2)}h^{c,(2)}h^{d,(2)}\right)h^{e,(2)},\end{aligned}$$

$$\begin{aligned}\tilde{\mathcal{L}}_3=\epsilon_{abcde}&\left(\frac{2}{\ell}R^{ab}R^{cd,(1)}+\frac{1}{\ell}R^{ab,(2)}R^{cd,(2)}+\frac{4}{\ell}R^{ab,(1)}k^{c,(1)}_{g}k^{gd,(2)}+\frac{6}{\ell^3}R^{ab}h^{c,(1)}h^{d,(2)}+\frac{2}{\ell^3}R^{ab,(1)}h^{c,(1)}h^{d,(1)}\right.\\&+\frac{2}{\ell^3}R^{ab,(1)}h^{c,(2)}e^{d}+\frac{2}{\ell^3}R^{ab,(2)}h^{c,(1)}e^{d}+\frac{2}{\ell^3}R^{ab,(2)}h^{c,(2)}h^{d,(2)}+\frac{2}{\ell^5}h^{a,(1)}h^{b,(1)}e^{c}e^{d}+\frac{1}{\ell^5}h^{a,(2)}e^{b}e^{c}e^{d}\\&\left.+\frac{1}{\ell^5}h^{a,(2)}h^{b,(2)}h^{c,(2)}h^{d,(2)}+\frac{4}{\ell^5}h^{a,(1)}h^{b,(2)}h^{c,(1)}h^{d,(1)}+\frac{6}{\ell^5}h^{a,(1)}h^{b,(2)}h^{c,(2)}e^{d}\right)e^{e}\\&+\epsilon_{abcde}\left(\frac{1}{\ell}R^{ab}R^{cd}+\frac{2}{\ell}R^{ab,(1)}R^{cd,(2)}+\frac{4}{\ell}R^{ab}k^{c,(1)}_{g}k^{gd,(2)}+\frac{2}{3\ell^3}R^{ab}h^{c,(1)}h^{d,(1)}+\frac{2}{\ell^3}R^{ab,(1)}h^{c,(2)}h^{d,(2)}\right.\\&\left.+\frac{2}{\ell^3}R^{ab,(2)}h^{c,(1)}h^{d,(2)}+\frac{1}{5\ell^5}h^{a,(1)}h^{b,(1)}h^{c,(1)}h^{d,(1)}+\frac{2}{\ell^5}h^{a,(2)}h^{b,(2)}h^{c,(2)}h^{d,(1)}\right)h^{e,(1)}\\&+\epsilon_{abcde}\left(\frac{2}{\ell}R^{ab}R^{cd,(2)}+\frac{1}{\ell}R^{ab,(1)}R^{cd,(1)}+\frac{4}{\ell}R^{ab,(2)}k^{c,(1)}_{g}k^{gd,(2)}+\frac{2}{\ell^3}R^{ab,(2)}h^{c,(1)}h^{d,(1)}+\frac{2}{\ell^3}R^{ab,(1)}h^{c,(1)}h^{d,(2)}\right.\\&\left.+\frac{2}{3\ell^3}R^{ab}h^{c,(2)}h^{d,(2)}\right)h^{e,(2)},\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathcal{L}}_5 = \epsilon_{abcde}\bigg(&\frac{2}{\ell}R^{ab}R^{cd,(2)} + \frac{1}{\ell}R^{ab,(1)}R^{cd,(1)} + \frac{4}{\ell}R^{ab,(2)}k^{c,(1)}_{\ g}k^{gd,(2)} + \frac{2}{\ell^3}R^{ab}h^{c,(2)}h^{d,(2)} \\
&+ \frac{2}{3\ell^3}R^{ab,(1)}e^c e^d + \frac{2}{\ell^3}R^{ab,(2)}h^{c,(1)}h^{d,(1)} + \frac{2}{\ell^3}R^{ab,(2)}h^{c,(2)}e^d + \frac{2}{\ell^5}h^{a,(1)}h^{b,(1)}h^{c,(1)}e^d \\
&+ \frac{2}{\ell^5}h^{a,(2)}h^{b,(2)}h^{c,(2)}e^d + \frac{4}{\ell^5}h^{a,(1)}h^{b,(2)}e^c e^d + \frac{6}{\ell^5}h^{a,(1)}h^{b,(2)}h^{c,(2)}h^{d,(1)}\bigg)e^e \\
&+ \epsilon_{abcde}\bigg(\frac{2}{\ell}R^{ab}R^{cd,(1)} + \frac{1}{\ell}R^{ab,(2)}R^{cd,(2)} + \frac{4}{\ell}R^{ab,(1)}k^{c,(1)}_{\ g}k^{gd,(2)} + \frac{2}{\ell^3}R^{ab}e^c e^d \\
&+ \frac{2}{\ell^3}R^{ab}h^{c,(1)}h^{d,(2)} + \frac{6}{\ell^3}R^{ab,(1)}h^{c,(2)}e^d + \frac{2}{3\ell^3}R^{ab,(1)}h^{c,(1)}h^{d,(1)} + \frac{2}{\ell^3}R^{ab,(2)}h^{c,(2)}h^{d,(2)} \\
&+ \frac{1}{\ell^5}h^{a,(2)}h^{b,(1)}h^{c,(1)}h^{d,(1)} + \frac{1}{\ell^5}h^{a,(2)}h^{b,(2)}h^{c,(2)}h^{d,(2)}\bigg)h^{e,(1)} \\
&+ \epsilon_{abcde}\bigg(\frac{1}{\ell}R^{ab}R^{cd} + \frac{2}{\ell}R^{ab,(1)}R^{cd,(2)} + \frac{4}{\ell}R^{ab}k^{c,(1)}_{\ g}k^{gd,(2)} + \frac{2}{\ell^3}R^{ab}h^{c,(1)}h^{d,(1)} \\
&+ \frac{6}{\ell^3}R^{ab,(1)}h^{c,(1)}e^d + \frac{2}{3\ell^3}R^{ab,(1)}h^{c,(2)}h^{d,(2)} + \frac{2}{\ell^3}R^{ab,(2)}h^{c,(1)}h^{d,(2)}\bigg)h^{e,(2)},
\end{aligned}$$

where we have defined

$$R^{ab,(1)} = Dk^{ab,(1)} + k^{a,(2)}_{\ c}k^{cb,(2)},$$
$$R^{ab,(2)} = Dk^{ab,(2)} + k^{a,(1)}_{\ c}k^{cb,(1)}.$$

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