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Explicitly covariant dispersion relations and self-induced transparency

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Explicitly covariant dispersion relations for a variety of plasma waves in unmagnetized and magnetized plasmas are derived in a systematic manner from a fully covariant plasma formulation. One needs to invoke relatively little known invariant combinations constructed from the ambient electromagnetic fields and the wave vector to accomplish the program. The implication of this work applied to the self-induced transparency effect is discussed. Some problems arising from the inconsistent use of relativity are pointed out.

Key words: magnetized plasmas, plasma waves

1. Introduction

Investigations of relativistic plasmas, both classical (for instance, (Gedalin & Oiberman 1995; Elsässer & Pope 1997; Tajima & Shibata 1997; Gedalin & Melrose 2001; Marklund *et al.* 2003) and references therein) and quantum (Hakim & Heyvaerts 1978, 1980; Hayes & Melrose 1984; Sivak 1985; Bialynicki-Birula, Górnicki & Rafelski 1991; Melrose 2000, 2005; Melrose, Weise & McOrist 2006; Asenjo *et al.* 2011; Eliasson & Shukla 2011; Mendonça 2011), have been steadily gaining speed in the recent past. This effort is motivated by the desire to deeply understand the nature of physical systems as diverse as: laser produced plasmas, the plasma (MeV) epoch in the evolution of the early universe, plasmas in the vicinity of highly compact objects (black holes, galactic nuclei), plasmas in neutron stars and magnetars. It is natural that fully covariant formulations for plasma dynamics are being put together (Melrose 1973, 1982, 2008). This work will concentrate on the covariant treatment of waves in a plasma.

One of the bastions of plasma physics is the investigation of various low-amplitude (linear) modes in which a plasma can oscillate. One does expect that special relativity will certainly affect the dispersion relations that these modes obey even if it is only through the relativistic increase in the particle mass, and the density increase as these quantities are measured in frames not at rest with respect to the plasma fluid. However to be consistent and systematic, it may be of essence that all calculations be done in an explicitly covariant manner so that the spirit of special relativity is fully respected.

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Although such calculations are aesthetically pleasing, the more important reason is that such procedures will help us avoid errors that may creep in due to a purely intuitively driven program of 'relativization'.

Lorentz invariance demands that all physically acceptable equations must be equalities between tensors of the same kind (zero being a general tensor) in order to guarantee frame independence. We remind the reader that wave dispersion relations are consistency conditions governed by wave character, for instance, the wave four-vector $K^\mu = [\omega, \boldsymbol{k}]$, the plasma parameters (the plasmas ambient four-velocity $U_0^\mu = [\gamma, \gamma \boldsymbol{v}]$ where γ is the relativistic Lorentz factor, the density, the charge, the mass etc. leading to various quantities like the plasma frequency) and the attributes of the equilibrium embedding field ($F_0^{\mu\nu}$, the Faraday tensor) if any. For a dispersion relation to be physically meaningful, it must be composed entirely of Lorentz scalars – that is – besides the explicit Lorentz scalars like the charge q or the rest mass m, the dispersion relation must be built entirely from the invariant scalars constructed from the wave, particle and the field attributes.

In this paper we derive explicitly covariant dispersion relations, using an explicitly covariant formalism for a variety of standard plasma waves both in unmagnetized and magnetized plasmas. The basic equations employed in this study are the fully fluid covariant set of Mahajan (2003) derived for a plasma that is relativistic both in directed velocity as well as in temperature. This paper is, thus, limited to fluid theories and does not capture kinetic results such as Landau dampimg. For a kinetic treatment, we refer the reader to the already mentioned Melrose textbook. There do exist earlier works in which fluid theories have been studied (Roth 1969; Achterberg & Wiersma 2007); this study, however, is more encompassing and complete in a more modern and transparent setting. It also raises important questions of how one can be led astray when enough attention is not paid to Lorentz invariance. Thus, the aim of the paper is partly presenting new results and partly pedagogic. It is important to note the difference between (i) a relativistically correct theory, (ii) a covariant dispersion relation and (iii) the use of a covariant formalism to derive covariant dispersion relations, in particular between the last two. Any given dispersion relation (derived in, say, a non-covariant formalism) can be written in a covariant form. Take, for example, the frequency ω, not being a Lorentz invariant, it has to be the limiting form of some invariant. It could, in fact, be $\sqrt{-K_\mu K^\mu} = \sqrt{\omega^2 - \boldsymbol{k}\cdot\boldsymbol{k}}$ or it could be $K_\mu U^\mu = -\gamma\omega + \gamma \boldsymbol{k}\cdot\boldsymbol{v}$. Thus simply 'writing' a dispersion relation in a covariant form is not unique and cannot substitute for a direct derivation of the correct and unique dispersion relation from a proper covariant theory.

Finally, in light of the above, it is shown that the standard theoretical picture of self-induced transparency for an electromagnetic wave, based on the relativistic decrease in plasma frequency, violates Lorentz invariance. A consistent covariant understanding of the dispersion relation for electromagnetic waves propagating in a relativistic hot plasma is presented to show that the directed (kinematic) and thermal motions affect the dynamics in profoundly different ways; the thermal effects, as distinct from the kinematic ones, bring about self-induced transparency by actually lowering the magnitude of the plasma frequency.

2. Relativistic fluid plasma

In the local rest frame, the energy momentum tensor of a perfect isotropic fluid is fully determined by the two Lorentz scalars: the energy density ${T_R}^{00} = \epsilon$ and the pressure ${T_R}^{ii} = p$. The corresponding covariant energy–momentum tensor of the fluid, then, may then be written as (Misner, Thorne & Wheeler 1973)

$$T^{\mu\nu} = hU^{\mu}U^{\nu} + p\eta^{\mu\nu}, \tag{2.1}$$

where $h = \epsilon + p$ is the enthalpy density, $\eta^{\mu\nu}$ is the metric tensor and U_{μ} is the normalized four-velocity of the fluid ($U_{\mu}U^{\mu} = -1$).

Importantly, all the thermodynamical properties of the fluid are defined in its rest frame. We can model the plasma as a relativistic gas constituting N identical, non-interacting and non-degenerate relativistic particles with mass m and momentum p_i. Its thermodynamical properties can be easily obtained through the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N} \sqrt{m^2c^4 + p_i^2c^2} = p^0, \tag{2.2}$$

which could provide the appropriate Boltzmann factor $\exp(-H/k_BT)$ in defining the local rest-frame distribution function, also known as the relativistic Maxwell–Jutnner distribution. The covariant distribution function, then, is explicitly proportional to $\exp(p^{\mu}U_{\mu}/k_BT)$, where the temperature T must be a Lorentz scalar parameter (k_B is the Boltzmann constant). This parameter coincides with the plasma temperature in the rest frame. Though several relativistic temperature transformations have been proposed (Ter Haar & Wergeland 1971), it is perfectly consistent that all thermodynamical quantities will be defined and measured in the rest frame defined at the beginning of this section.

The partition function of such an N particle gas ($Z_N(T, V) = [Z_1(T, V)]^N/N!$, where V is the rest-frame volume) is readily calculated by first noticing that (Greiner *et al.* 1995)

$$Z_1(T, V) = 4\pi V \left(\frac{m}{2\pi\hbar}\right)^3 \frac{K_2(u)}{u}, \tag{2.3}$$

where $u \equiv mc^2/k_BT$ is the ratio between the rest-mass particle energy and the rest-frame thermodynamical energy and K_j is the modified Bessel function of order j. From the free energy $F = -T\ln Z_N$, we deduce the pressure

$$p = -\left.\frac{\partial F}{\partial V}\right|_{T,N} = n_R T, \tag{2.4}$$

where $n_R = N/V$ is the rest-frame density. Similarly, the entropy density, which is a Lorentz scalar, can be obtained as

$$\sigma = -\frac{1}{V}\left.\frac{\partial F}{\partial T}\right|_{N,V} = n_R \ln\left[\frac{4\pi}{n_R}\left(\frac{m}{2\pi\hbar}\right)^3 \frac{K_2(u)}{u}\right] + 4n_R + n_R u\frac{K_1(u)}{K_2(u)}. \tag{2.5}$$

Finally, the enthalpy density (another Lorentz scalar) is $h = F/V + T\sigma + p = n_R m f(u)$, where f takes the succinct form

$$f(u) = \frac{K_3(mc^2/k_BT)}{K_2(mc^2/k_BT)}. \tag{2.6}$$

Notice that for the Maxwell–Juttner gas, f is a function of temperature T alone.

It was shown in Mahajan (2003) that the relativistic dynamics of a covariant plasma (2.1) could be cast in the form (in natural units $k_B = 1 = c$)

$$T\partial^\nu \sigma = qU_\mu \left(F^{\mu\nu} + \frac{m}{q} S^{\mu\nu} \right), \tag{2.7}$$

where q is the particle charge, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic field tensor and

$$S^{\mu\nu} = \partial^\mu (fU^\nu) - \partial^\nu (fU^\mu), \tag{2.8}$$

is an equivalent fluid tensor combining the kinematic and statistical attributes of the plasma through f.

The entropy density σ satisfies the standard thermodynamic relation

$$n_R T\partial^\nu \sigma = \partial^\nu p - mn_R \partial^\nu f. \tag{2.9}$$

Due to the antisymmetry of the electromagnetic and fluid tensors, equation (2.7) leads to the isentropic condition

$$U_\nu \partial^\nu \sigma = 0, \tag{2.10}$$

that is, the plasma entropy density is constant along the streamlines. The complete plasma dynamics is contained in (2.7), and the Maxwell equation

$$\partial_\mu F^{\mu\nu} = -4\pi J^\nu, \tag{2.11}$$

where the four-current, $J^\mu = qn_R U^\mu$, obeying the continuity equation

$$\partial_\mu J^\mu = 0, \tag{2.12}$$

has to be computed from (2.7). All thermodynamical densities are defined per unit proper (rest-frame) volume.

For the simplest case of a homentropic plasma,

$$\partial^\nu \sigma = 0, \tag{2.13}$$

entropy density does not appear directly in the equation of motion (2.7). However, the temperature can be arbitrary, even relativistically large. For such a system, the equation of motion reduces to

$$U_\mu F^{\mu\nu} = -\frac{m}{q} U_\mu \partial^\mu (fU^\nu) - \frac{m}{q} \partial^\nu f, \tag{2.14}$$

and can be split into the time component

$$\boldsymbol{v} \cdot \boldsymbol{E} + \frac{m}{q\gamma} \frac{\partial f}{\partial t} = \frac{m}{q} \left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \right) (f\gamma), \tag{2.15}$$

and the spatial component

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} - \frac{m}{q\gamma} \boldsymbol{\nabla} f = \frac{m}{q} \left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \right) (f\gamma \boldsymbol{v}), \tag{2.16}$$

where $\gamma = (1 - v^2)^{-1/2}$ is the relativistic factor and $v = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$.

One of the primary function of this paper is the covariant 'derivation' and discussion of the dispersion relations governing many of the standard plasma waves. We illustrate this purpose with the study of some of the well-known dispersion modes.

3. Electrostatic waves

We begin with the simplest field free (no equilibrium electric and magnetic fields), zero pressure system where the ions are fixed and provide a neutralizing background, and the electrons are the only dynamic species. We further assume a constant equilibrium velocity $\boldsymbol{v}_0 = \boldsymbol{v}$, such that the equilibrium relativistic factor $\gamma_0 = \gamma$ is a constant. Besides, we assume a constant temperature and therefore a constant function f.

The linearized perturbed electrostatic system, then, is described by

$$\boldsymbol{v}\cdot\delta\boldsymbol{E} = -\mathrm{i}\frac{\bar{m}}{q}(\omega - \boldsymbol{k}\cdot\boldsymbol{v})(\gamma^3\boldsymbol{v}\cdot\delta\boldsymbol{v}), \tag{3.1}$$

$$\mathrm{i}\boldsymbol{k}\cdot\delta\boldsymbol{E} = 4\pi q\delta n, \tag{3.2}$$

$$\frac{\delta n}{n} = \frac{\boldsymbol{k}\cdot\delta\boldsymbol{v}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}, \tag{3.3}$$

where $\boldsymbol{k}$ is the wave vector, ω is the wave frequency, $\bar{m} = mf$ and $n = \gamma n_R^0$ is the constant equilibrium density in the laboratory frame; δn, thus, is the perturbed density in the laboratory frame.

Using (3.2) and (3.3), we relate the electric field with the perturbed velocity

$$\delta\boldsymbol{E} = -\mathrm{i}\frac{4\pi qn}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\delta\boldsymbol{v}, \tag{3.4}$$

which, when substituted in (3.1), yields the dispersion relation

$$(\omega - \boldsymbol{k}\cdot\boldsymbol{v})^2 = \frac{4\pi q^2 n}{\bar{m}\gamma^3}. \tag{3.5}$$

This dispersion relation looks somewhat awkward, but it is fully covariant. Remembering that the laboratory-frame density, $n = \gamma n_R^0$, and defining the wave four-vector

$$K^\mu \equiv [\omega, \boldsymbol{k}], \quad K_\mu K^\mu = -\omega^2 + \boldsymbol{k}\cdot\boldsymbol{k}, \tag{3.6a,b}$$

we cast the dispersion relation (3.5) in the desired form

$$(K^\mu U_\mu)^2 = \varOmega_p^2, \tag{3.7}$$

with

$$\varOmega_p = \frac{\omega_p}{\sqrt{f}}, \tag{3.8}$$

where $\omega_p = \sqrt{4\pi q^2 n_R^0/m}$ is the usual scalar plasma frequency with constant n_R^0. There are three important points to notice:

(i) the effective local rest-frame plasmas frequency is decreased by the temperature factor f. We shall discuss this effect in detail later,

(ii) the dispersion relation, as required by special relativity, comes out to be an equality between two Lorentz scalars, and therefore is frame independent,

(iii) the familiar cold plasma wave dispersion relation $\omega^2 = \omega_p^2$ follows in the limit $\boldsymbol{k}\cdot\boldsymbol{v} \ll \omega$. The non-relativistic limit breaks the covariance by terms of order $\boldsymbol{k}\cdot\boldsymbol{v}/\omega \sim v/c$. Though it is a fully consistent limit, one must note that the standard dispersion relation is correct only to order zero in v/c.

Before proceeding further, it is advised that we make a comment about the fundamental plasma parameter $\omega_p = \sqrt{4\pi q^2 n_R^0/m}$. This parameter is a Lorentz scalar, its value is the same in all frames. Since in a moving frame n_R goes to $n = \gamma n_R$ along with the rest mass m going to γm, the frequency ω_p remains constant in any arbitrary moving frame. Notice that, though n_R^0 is a constant for this simple dispersion, it is always a Lorentz scalar even when it is variable.

4. Bohm–Gross dispersion relation

When the plasma is hot and pressure perturbations are permitted, the simple plasma wave become considerably more interesting. The relativistic dispersion relation is obtained from the linearized equation of motion (equivalent of (3.1)) which has an additional term proportional to δf (in the homentropic case, temperature appears only through f)

$$q\boldsymbol{v}\cdot\delta\boldsymbol{E} = \mathrm{i}m\gamma v\,(k-\omega v)\,\delta f - \mathrm{i}mf(\omega - \boldsymbol{k}\cdot\boldsymbol{v})(\gamma^3\boldsymbol{v}\cdot\delta\boldsymbol{v}), \tag{4.1}$$

where $k = \sqrt{\boldsymbol{k}\cdot\boldsymbol{k}}$. The homentropic condition, $\delta\sigma = 0$, translating as

$$\delta p = m n_R^0 \delta f, \tag{4.2}$$

and the perturbed relation (from the definition $n = \gamma n_R$)

$$\frac{\delta n_R}{n_R^0} = \frac{\delta n}{n} - \frac{\delta\gamma}{\gamma}, \tag{4.3}$$

may be combined with the perturbed equation of state (Γ is the adiabatic index)*

$$\delta p = \Gamma T \delta n_R, \tag{4.4}$$

and the continuity equation (3.2) to derive a relation between δf and $\delta\boldsymbol{v}$

$$\delta f = \frac{\Gamma T}{m}\left(\frac{\boldsymbol{k}\cdot\delta\boldsymbol{v}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}} - \gamma^2\boldsymbol{v}\cdot\delta\boldsymbol{v}\right). \tag{4.5}$$

For the longitudinal modes, $\boldsymbol{k}$, $\boldsymbol{v}$ and $\delta\boldsymbol{E}$ are parallel to one another. Equation (4.5), then, simplifies to

$$\delta f = \frac{\Gamma T}{m}\gamma^2\,(k-\omega v)\,\frac{\boldsymbol{v}\cdot\delta\boldsymbol{v}}{v(\omega - kv)}. \tag{4.6}$$

Combining the Poisson equation (3.4), (4.1) and (4.6), we derive the relation

$$\Omega_p^2 = \gamma^2\left[(\omega - kv)^2 - V_s^2(\omega v - k)^2\right], \tag{4.7}$$

where $V_s = v_s/\sqrt{f}$, the relativistic sound speed, is the f modulated thermal speed $v_s = \sqrt{\Gamma T/m}$.

We can write the above dispersion relation in an explicitly covariant form invoking the identity between the four-vectors, $\gamma^2(\omega v - k)^2 = (K_\mu U^\mu)^2 + K_\mu K^\mu$. Thus, the dispersion relation will be

$$\Omega_p^2 = (1 - V_s^2)(K_\mu U^\mu)^2 - V_s^2 K_\mu K^\mu, \tag{4.8}$$

where only the Lorentz scalars $K_\mu U^\mu$ and $K_\mu K^\mu$ participate. Notice that the scalar invariant $K_\mu K^\mu$ was needed to represent the finite-temperature modifications to the cold plasmas dispersion. To the best of our knowledge, this is the first derivation of a relativistic covariant dispersion relation for a longitudinal plasma wave propagating in a relativistically hot plasma. The standard Bohm–Gross form can be obtained in the appropriate limit.

*$n_R T = p \propto n_r^\Gamma$. In the non-relativistic limit $\Gamma = 5/3$. In the ultra-relativistic limit $\Gamma = 4/3$.

5. Two-stream instability

We now proceed to write down the covariant dispersion relation relevant to the two-stream instability in a plasma consisting of beams of relativistic electrons and ions. For simplicity, we assume equal and constant temperatures. The electric field is parallel to the fluid velocity and to the propagation vector. Thus, we could calculate the electron (specie e) and ion (specie i) density perturbations from (3.1) and (3.3),

$$\frac{\delta n_e}{n_e} = \frac{-iek}{m_e f \gamma_e^3 (\omega - k v_e)^2} \delta E, \tag{5.1}$$

and

$$\frac{\delta n_i}{n_i} = \frac{iek}{m_i f \gamma_i^3 (\omega - k v_i)^2} \delta E. \tag{5.2}$$

Both the electrons and ions have relativistic drift velocities v_e and v_i with their associated Lorentz factors $\gamma_e = (1 - v_e^2)^{-1/2}$ and $\gamma_i = (1 - v_i^2)^{-1/2}$. The densities $n_e = \gamma_e n_{Re}^0$ and $n_i = \gamma_i n_{Ri}^0$ are measured in the laboratory frame (n_{Re}^0 and n_{Ri}^0 are the local rest-frame densities).

Substitution in the Poisson equation $ik\delta E = 4\pi e(\delta n_i - \delta n_e)$, leads to the dispersion relation

$$1 = \frac{\Omega_{pi}^2}{\gamma_i^2 (\omega - \boldsymbol{k} \cdot \boldsymbol{v}_i)^2} + \frac{\Omega_{pe}^2}{\gamma_e^2 (\omega - \boldsymbol{k} \cdot \boldsymbol{v}_e)^2}, \tag{5.3}$$

that has the covariant form

$$1 = \frac{\Omega_{pi}^2}{(U_i^\mu K_\mu)^2} + \frac{\Omega_{pe}^2}{(U_i^\mu K_\mu)^2}, \tag{5.4}$$

where $U_e^\mu = [\gamma_e, \gamma_e \boldsymbol{v}_e]$ and $U_i^\mu = [\gamma_i, \gamma_i \boldsymbol{v}_i]$ are the four-velocity of species, and the appropriately modified plasma frequencies for electrons and ions are $\Omega_{pi}^2 = 4\pi e^2 n_{Ri}^0 / m_i f$ and $\Omega_{pe}^2 = 4\pi e^2 n_{Re}^0 / m_e f$ respectively. The two-stream instability appears on analysing this dispersion.

6. Compressional Alfvén waves

Next we investigate electromagnetic waves in a relativistic electron–positron plasma with constant temperature embedded in an external electromagnetic field. The self consistent system consists of the equations of motions (2.16) for each specie (electrons e and positrons p)

$$\boldsymbol{E} + \boldsymbol{v}_e \times \boldsymbol{B} = -\frac{\bar{m}}{e} \left(\frac{\partial}{\partial t} + \boldsymbol{v}_e \cdot \nabla \right) \gamma_e \boldsymbol{v}_e, \tag{6.1}$$

$$\boldsymbol{E} + \boldsymbol{v}_p \times \boldsymbol{B} = \frac{\bar{m}}{e} \left(\frac{\partial}{\partial t} + \boldsymbol{v}_p \cdot \nabla \right) \gamma_p \boldsymbol{v}_p, \tag{6.2}$$

and the Maxwell equation

$$\nabla \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = 4\pi n_R e (\gamma_p \boldsymbol{v}_p - \gamma_e \boldsymbol{v}_e). \tag{6.3}$$

The choice of external electromagnetic fields suitable for a covariant theory requires some care. One may, in principle, choose arbitrary fields $\boldsymbol{E}_0$ and $\boldsymbol{B}_0$. However we will restrict our calculation to the field choice that, in the standard non-relativistic limit, reduces to the so called a magnetic or magnetizable system. For this class of fields, the two relativistic invariants must obey

$$F_{\mu\nu}F^{\mu\nu} \equiv |\boldsymbol{B}|^2 - |\boldsymbol{E}|^2 > 0, \tag{6.4}$$

$$\mathcal{F}_{\mu\nu}F^{\mu\nu} \equiv \boldsymbol{E}\boldsymbol{\cdot}\boldsymbol{B} = 0, \tag{6.5}$$

where $\mathcal{F}_{\mu\nu} = (1/2)\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ is the dual of the electromagnetic tensor. Equations (6.4) and (6.5) insure that the electric and magnetic fields are perpendicular, and there exists a frame which contains only the magnetic field ($\boldsymbol{E} = 0$). In fact we will do our calculations exactly in such a frame and then apply a boost to go to an arbitrary frame.

A simple consistent equilibrium solution is made up of constant fields ($\boldsymbol{E}_0$, $\boldsymbol{B}_0$), and constant and equal flows for the two species: $\boldsymbol{v}_e = \boldsymbol{v}_p = \boldsymbol{v}$, $\gamma_e = \gamma_p = \gamma = (1 - \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{v})^{-1/2}$. Notice that this kind of an equilibrium pertains to a specific frame. We will further assume that the calculation frame is defined by $\boldsymbol{E}_{00} = 0$, and $\boldsymbol{B}_{00} = B_{00}\hat{z}$. We had used the subscript 00 for the magnetic field in the special equilibrium frame ($\boldsymbol{B}_{00} = B_{00}\hat{z}$, $\boldsymbol{E}_{00} = 0$) to distinguish it from an arbitrary equilibrium frame with subscript 0 ($\boldsymbol{B}_0$, $\boldsymbol{E}_0$). The linearized set of equations for studying wave propagation, then, are:

$$\delta\boldsymbol{E} + \boldsymbol{v}\times\delta\boldsymbol{b} + \delta\boldsymbol{v}_e\times\hat{z} = \frac{\mathrm{i}}{\varOmega_c}\,(\omega - \boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v})\,\delta(\gamma_e\boldsymbol{v}_e), \tag{6.6}$$

$$\delta\boldsymbol{E} + \boldsymbol{v}\times\delta\boldsymbol{b} + \delta\boldsymbol{v}_p\times\hat{z} = -\frac{\mathrm{i}}{\varOmega_c}\,(\omega - \boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v})\,\delta(\gamma_p\boldsymbol{v}_p), \tag{6.7}$$

$$\boldsymbol{k}\times\delta\boldsymbol{b} + \omega\delta\boldsymbol{E} = -\mathrm{i}\frac{\varOmega_p}{V_A}\delta(\gamma_p\boldsymbol{v}_p - \gamma_e\boldsymbol{v}_e), \tag{6.8}$$

where $\delta\boldsymbol{E}$ and $\delta\boldsymbol{b}$ are the perturbed electromagnetic fields normalized to B_{00}. We must emphasize that both the cyclotron frequency and the Alfvén velocity are defined as Lorentz scalars with their rest mass enhanced only by the scalar temperature (thru the statistical factor, $\bar{m} = mf$)

$$\varOmega_c = \frac{eB_{00}}{\bar{m}}, \quad V_A = \frac{B_{00}}{\sqrt{4\pi n_R^0\bar{m}}}, \tag{6.9a,b}$$

and n_R^0 is the background constant density in the local rest frame.

In general $\delta(\gamma_e\boldsymbol{v}_e) = \gamma\delta\boldsymbol{v}_e + \boldsymbol{v}_e\gamma^3\boldsymbol{v}_e\boldsymbol{\cdot}\delta\boldsymbol{v}_e$ (same for positrons), but for perturbations perpendicular to the equilibrium flow, it follows that $\delta(\gamma_e\boldsymbol{v}_e) = \gamma\delta\boldsymbol{v}_e$. For this calculation, we restrict ourselves to such perturbations. Adding and subtracting (6.6) and (6.7), we rewrite the linearized system

$$\delta\boldsymbol{E} + \boldsymbol{v}\times\delta\boldsymbol{b} + \tfrac{1}{2}\boldsymbol{V}\times\hat{z} = -\mathrm{i}P\boldsymbol{j}, \tag{6.10}$$

$$\boldsymbol{j}\times\hat{z} = -2\mathrm{i}P\boldsymbol{V}, \tag{6.11}$$

$$\boldsymbol{k}\times\delta\boldsymbol{b} + \omega\delta\boldsymbol{E} = -\mathrm{i}\frac{\varOmega_p}{V_A}\gamma\boldsymbol{j}, \tag{6.12}$$

in terms of the perturbed magnetohydrodynamic (MHD) variables: the mean fluid velocity $\boldsymbol{V} = \delta\boldsymbol{v}_p + \delta\boldsymbol{v}_e$, and the current $\boldsymbol{j} = \delta\boldsymbol{v}_p - \delta\boldsymbol{v}_e$. The symbol $P = \gamma\,(\omega - \boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{v})\,/2\varOmega_c$ is introduced for notational convenience.

The linearized system can be manipulated similarly to Alfvén waves in the MHD theory. Let us first derive the dispersion relation whose non-relativistic limit is the compressional Alfvén wave. These waves are transverse, $\boldsymbol{k}\cdot\delta\boldsymbol{b}=0$ and devoid of fluid perturbations along the field lines $\hat{z}\cdot\delta\boldsymbol{v}=0=\hat{z}\cdot\boldsymbol{V}$ (no sound wave, we have neglected pressure perturbations).

By taking the curl of (6.10) and (6.12), we obtain the following equations

$$2(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\delta\boldsymbol{b}+k_z\boldsymbol{V}-\hat{z}(\boldsymbol{V}\cdot\boldsymbol{k})=-2\mathrm{i}P\boldsymbol{k}\times\boldsymbol{j}, \tag{6.13}$$

$$\boldsymbol{k}\times\boldsymbol{j}=\mathrm{i}Q\delta\boldsymbol{b}, \tag{6.14}$$

where $k_z=\boldsymbol{k}\cdot\hat{z}$ and $Q=V_A(\omega^2-k^2)/\gamma\varOmega_p$. From them, we can derive the relation

$$2(\omega-\boldsymbol{k}\cdot\boldsymbol{v}-PQ)\delta\boldsymbol{b}=\hat{z}(\boldsymbol{V}\cdot\boldsymbol{k})-\boldsymbol{V}k_z. \tag{6.15}$$

Similarly, combining (6.14) and (6.11) yields

$$Q\hat{z}\cdot\delta\boldsymbol{b}=-2P\boldsymbol{V}\cdot\boldsymbol{k}. \tag{6.16}$$

With the aid of (6.16), the $\hat{z}$ component of (6.15) ($\hat{z}\cdot\boldsymbol{V}=0$) leads to the dispersion relation

$$\gamma^2(\omega-\boldsymbol{k}\cdot\boldsymbol{v})^2\left(1+\frac{k^2-\omega^2}{2\varOmega_p^2}\right)=\frac{V_A^2}{2}(k^2-\omega^2). \tag{6.17}$$

It is comforting that, in the non-relativistic limit and for $\omega^2\ll k^2$, we recover the standard compressional wave dispersion relation $\omega^2=k^2V_A^2/(2+k^2/\varOmega_p^2)$. The factor 2 appears because the both the electron and positron fluids contribute equally; it could be absorbed in a redefinition of the Alfvén speed.

But for the occurrence of B_{00} in the definition of V_A, equation (6.17) will be fully covariant. This is readily fixed if we recall that the calculations were done in a frame in which the electric field, $\boldsymbol{E}_{00}=0$. All that we need to do is to replace $\boldsymbol{B}_{00}^2$ by the invariant $F_{0\mu\nu}F_0{}^{\mu\nu}=\boldsymbol{B}_0^2-\boldsymbol{E}_0^2$, since $F_{0\mu\nu}F_0{}^{\mu\nu}\equiv\boldsymbol{B}_{00}^2$. When the Alfvén speed is expressed as

$$V_A^2=\frac{B_{00}^2}{4\pi n_R^0\bar{m}}\longrightarrow\frac{F_{0\mu\nu}F_0{}^{\mu\nu}}{4\pi n_R^0\bar{m}}, \tag{6.18}$$

the dispersion relation (6.17) assumes the fully covariant form

$$\left(U^\mu K_\mu\right)^2\left(1+\frac{K_\nu K^\nu}{2\varOmega_p^2}\right)=\left(K_\mu K^\mu\right)\frac{F_{0\alpha\nu}F_0{}^{\alpha\nu}}{8\pi n_R^0 mf}. \tag{6.19}$$

It is important to remind the reader that the explicitly covariant expression for the cyclotron frequency $\varOmega_c=eB_{00}/\bar{m}$ will be $\varOmega_c=e(F_{0\mu\nu}F_0{}^{\mu\nu})^{1/2}/\bar{m}$.

When the plasma is unmagnetized ($F_{0\mu\nu}=0$) , the dispersion relation (6.19) reduces to $K_\mu K^\mu=-2\varOmega_p^2$, which may be recognized as the electromagnetic wave dispersion for an electron–positron plasma. We will discuss this mode in considerable detail in § 8.

7. Shear Alfvén waves

We now turn to the relativistic version of shear Alfvén wave for which the perturbations are perpendicular to the background magnetic field, and to the wave vector. With $\boldsymbol{k}\cdot\boldsymbol{V}=0$, (6.15) reduces to

$$2(\omega-\boldsymbol{k}\cdot\boldsymbol{v}-PQ)\delta\boldsymbol{b}=-k_z\boldsymbol{V}. \tag{7.1}$$

For perturbations perpendicular to the ambient field, $\delta\boldsymbol{b}\cdot\hat{z}=0=\delta\boldsymbol{E}\cdot\hat{z}$, equation (6.12) yields

$$\hat{z}\cdot(\boldsymbol{k}\times\delta\boldsymbol{b})=-\mathrm{i}\frac{\Omega_p}{V_A}\gamma j_z, \tag{7.2}$$

where $j_z=\boldsymbol{j}\cdot\hat{z}$. Invoking $\boldsymbol{k}\cdot\boldsymbol{j}=0$, we obtain from (6.11) and (6.14), two equations relating j_z to $\hat{z}\cdot(\boldsymbol{k}\times\delta\boldsymbol{b})$ and $\hat{z}\cdot(\boldsymbol{k}\times\boldsymbol{V})$

$$j_z k_z^2=-2\mathrm{i}Pk_z\hat{z}\cdot(\boldsymbol{k}\times\boldsymbol{V}), \tag{7.3}$$

$$-j_z k^2=\mathrm{i}Q\hat{z}\cdot(\boldsymbol{k}\times\delta\boldsymbol{b}). \tag{7.4}$$

From the preceding three equations (7.2)–(7.4), one derives

$$\frac{V_A}{\Omega_p\gamma}(\omega^2-k_z^2)\hat{z}\cdot(\boldsymbol{k}\times\delta\boldsymbol{b})=2k_zP\hat{z}\cdot(\boldsymbol{k}\times\boldsymbol{V}). \tag{7.5}$$

Another independent relation between $\hat{z}\cdot(\boldsymbol{k}\times\delta\boldsymbol{b})$ and $\hat{z}\cdot(\boldsymbol{k}\times\boldsymbol{V})$ is derived by applying the operator $(\hat{z}\cdot\boldsymbol{k}\times)$ to (7.1)

$$\frac{K^\mu U_\mu}{\gamma}\left(1+\frac{k^2-\omega^2}{2\Omega_p^2}\right)\hat{z}\cdot(\boldsymbol{k}\times\delta\boldsymbol{b})=-k_z\hat{z}\cdot(\boldsymbol{k}\times\boldsymbol{V}). \tag{7.6}$$

By combining equations (7.5) and (7.6), we, finally arrive at the dispersion relation

$$\frac{V_A^2}{2}(k_z^2-\omega^2)=(U^\mu K_\mu)^2\left(1+\frac{K^\nu K_\nu}{2\Omega_p^2}\right). \tag{7.7}$$

The dispersion relation (7.7), clearly is not explicitly covariant because of the presence of $(k_z^2-\omega^2)$, the very hallmark of the shear Alfvén wave. The problem is beautifully solved by figuring out that this quantity is, indeed, the correct translation, in the frame of calculation ($\boldsymbol{E}_0=0$), for an invariant $\mathcal{P}^\mu\mathcal{P}_\mu$. In an arbitrary frame, this four-vector, constructed from the duel $\mathcal{F}_0^{\mu\nu}$ of the equilibrium Faraday tensor, and the wave four-vector, spells out as

$$\mathcal{P}^\mu=\mathcal{F}_0{}^{\mu\nu}K_\nu\equiv\{\boldsymbol{k}\cdot\boldsymbol{B}_0,\omega\boldsymbol{B}_0-\boldsymbol{k}\times\boldsymbol{E}_0\}. \tag{7.8}$$

And in the $\boldsymbol{E}_0=0$ frame, the invariant $\mathcal{P}^\mu\mathcal{P}_\mu$ takes exactly the form

$$\mathcal{P}^\mu\mathcal{P}_\mu=(\boldsymbol{k}\cdot\boldsymbol{B}_0)^2-\omega^2B_0^2=(k_z^2-\omega^2)B_0^2, \tag{7.9}$$

that appears in the dispersion relation. Thus a fully explicitly covariant dispersion relation which contains the shear Alfvén wave in its appropriate non-relativistic limit appears as

$$(U^\mu K_\mu)^2\left(1+\frac{K^\nu K_\nu}{2\Omega_p^2}\right)=\frac{\mathcal{P}^\mu\mathcal{P}_\mu}{8\pi n_R^0 mf}, \tag{7.10}$$

in which all combinations are Lorentz scalars.

8. Electromagnetic waves and self-induced transparency

Because of the widespread applications of the dispersion relation for electromagnetic waves, particularly in laser produced relativistic plasmas, we deal with it in a little more detail. We will derive this dispersion relation in a slightly different way because, in the absence of density perturbations, the electromagnetic wave (under well-defined conditions) is an exact solution of the system. We begin by rewriting the Maxwell equation (2.11) in the covariant Lorentz gauge ($\partial_\mu A^\mu = 0$)

$$\partial_\nu \partial^\nu A^\mu = -4\pi J^\mu = -4\pi q n_R U^\mu, \tag{8.1}$$

from which follows the spatial component

$$\partial_\nu \partial^\nu \boldsymbol{A} = \left(-\frac{\partial^2}{\partial t^2} + \nabla^2 \right) \boldsymbol{A} = -4\pi q \gamma n_R \, \boldsymbol{v}. \tag{8.2}$$

We, then, take the curl of the spatial part of the equation of motion (2.16) and rearrange to find

$$\frac{\partial}{\partial t} (\boldsymbol{\nabla} \times \boldsymbol{P}) = \boldsymbol{\nabla} \times [\boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{P})], \tag{8.3}$$

where $\boldsymbol{P} = (\bar{m}/q)\gamma \boldsymbol{v} + \boldsymbol{A}$ is proportional to the canonical momentum appropriately modified to reflect the relativistic kinematic as well as thermal motion. Equation (8.3) represents the evolution of the generalized vorticity $\boldsymbol{\nabla} \times \boldsymbol{P}$. A very interesting feature of the vorticity evolution equation is that if $\boldsymbol{\nabla} \times \boldsymbol{P}$ is zero anytime, it always remains so. We shall exploit this special property to assume the simplest solution $\boldsymbol{P} = 0$

$$\boldsymbol{v} = -\frac{q}{\bar{m}\gamma} \boldsymbol{A}, \tag{8.4}$$

relating the velocity and the magnetic field. Substitution in (8.2) leads to

$$\partial_\nu \partial^\nu \boldsymbol{A} = \Omega_p^2 \boldsymbol{A}, \tag{8.5}$$

where Ω_p^2 is the temperature-corrected plasma frequency defined in (3.8). Since $\partial_\nu \partial^\nu$ is a Lorentz-invariant operator, both sides of (8.5) transform like $\boldsymbol{A}$ implying that the equation is frame independent. Equation (8.5) is manifestly a linear equation, and is readily Fourier analysed to yield the covariant dispersion relation

$$-K_\mu K^\mu = \omega^2 - \boldsymbol{k} \cdot \boldsymbol{k} = \Omega_p^2. \tag{8.6}$$

As dictated by the demands of covariance, no explicit γ factor can or does appears on the right-hand side of (8.6). This is because of the fact that the plasma frequency is constructed from the invariant combination n/M, where the density n and the mass M (the rest mass is denoted by m) must refer to the same frame. The numerical value of $n/M = n_R/m$ is a characteristic scalar of the given fluid, where the number density is a varying function governed by the continuity equation in any given frame. Hence, n_R/m is a characteristic local scalar that varies self-consistently with field. In the literature there exists a serious conceptual problem: the relativistic dispersion relation is often written as $\omega^2 - k^2 = \Omega_p^2/\gamma$ with Ω_p defined with n rather than n_R (it is as if we decided to allow for a relativistic increase in mass but suppressed the relativistic increase in density). At the very least this is in violation of the spirit of relativity, but this confusion has much more serious consequences. For instance, the mixing of the frames has led many authors to treat (8.5) as a nonlinear equation when it is entirely linear (Akhiezer & Polovin 1956; Kaw & Dawson 1970; Guérin *et al.* 1995; Lefebvre & Bonnaud 1995; Guérin *et al.* 1996; Cattani *et al.* 2000; Goloviznin & Schep 2000; Emerin, Korzhimanov & Kim 2010). One of the more exciting predictions of the alleged relativistic decrease in Ω^2 is the self-induced transparency (SIT) in relativistic plasmas. Conventionally, using equation $k^2 = \omega^2 - \Omega_p^2/\gamma$, one may deduce that, for a given frequency radiation, a plasma which was initially opaque can become transparent since the critical frequency decreases by γ as electrons gain speed and become heavy. Such a conclusion, however, could not be drawn if one looked at the covariant dispersion relation (8.6), which tells us that for a given rest-frame density n_R, the 'plasma frequency' is independent of γ, the relativistic mass increase is, and must be, fully cancelled by the density increase.

In order to demonstrate the confusion that may arise when Lorentz invariance is not used as a guide, let us rewrite the dispersion relation (8.6) as

$$\omega^2 - \boldsymbol{k}\cdot\boldsymbol{k} = \frac{4\pi q^2 n}{\bar{m}\gamma}, \tag{8.7}$$

where n is the laboratory-frame density, and we have used that $n_R = n/\gamma$. Now, by using (8.4), it is straightforward to obtain ($\boldsymbol{E} = -(\partial/\partial t)\boldsymbol{A}$ is the electric field)

$$\gamma^2 = 1 + \frac{q^2E^2}{\omega^2\bar{m}^2}, \tag{8.8}$$

where E is the amplitude of the electromagnetic wave. This allow us to write the dispersion relation (8.7) as

$$\omega^2 - \boldsymbol{k}\cdot\boldsymbol{k} = \frac{4\pi q^2 n}{mf}\left(1 + \frac{q^2E^2}{\omega^2 m^2 f^2}\right)^{-1/2}, \tag{8.9}$$

which could be viewed as the finite-temperature generalization of the dispersion relation for pure transverse electromagnetic waves appearing, for example, in Kaw & Dawson (1970). However, this explicit appearance of $\gamma(E^2)$ is just an artefact; this $\gamma(E^2)$ should exactly cancel the $\gamma(E^2)$ implicit in $n = n_R\gamma(E^2)$, and leave only n_R that has no direct cognizance of E^2. Notice that local rest-frame density evolves dynamically and self-consistently with the electromagnetic fields obeying a continuity equation through the conserved current density $J^\mu = qn_RU^\mu$. The correct Lorentz-invariant dynamical equation (8.5) does not require that the rest-frame density be constant. One could go back to the dynamical equation (8.5) (rewritten using the relation (8.4), and replacing n_R by n/γ)

$$\partial_\nu\partial^\nu \boldsymbol{A} = \frac{4\pi q^2 n\boldsymbol{A}}{\bar{m}\sqrt{1 + (q^2/\bar{m}^2)\boldsymbol{A}^2}} \tag{8.10}$$

and get the impression that it is a nonlinear equation in $\boldsymbol{A}$. However the moment one puts back γn_R for n, the nonlinearity through γ disappears and we go back to (8.5), the correct Lorentz-invariant equation. Mixing frames violates special relativity; equations (8.9) and (8.10) should be construed as a bad and misleading representations of (8.6) (or (8.5)), and should be avoided.

One can, thus, conclude that the phenomena of SIT in relativistic plasmas has a subtler origin; in fact, it must be attributed to what has been called the field renormalized mass (Mahajan & Asenjo 2016). What has been 'propagated' in literature for over fifty years does not pass the covariance muster. Fortunately, the prospects of discovering exciting relativistic phenomena like the SIT, relativistic self-focusing and profile steepening, have stimulated a great amount of research. The literature is replete with studies and investigations of laser–plasma interactions; the laser intensities have, recently, become so large that the electron motions under their influence can be strongly relativistic (Mourou, Tajima & Bulanov 2006; Umstadter 2013). However, it seems that there does not exist definitive experimental evidence for conventional understanding of SIT (Goloviznin & Schep 1999). In fact there is some experimental 'evidence' that propagation and penetration in overdense relativistic plasmas may be due to effects different from what could be termed conventional SIT (Giulietti *et al.* 1997, 1998). In this context the reader is referred to Mahajan & Asenjo (2016).

For a hot plasma, however, enhanced penetration (compared to predictions from the simple cold plasma dispersion relation (Cairns, Rau & Airila 2000)) is somewhat easier to understand. Simulations (Pukhov, Sheng & Meyer-ter Vehn 1999) find that the temperature T of the electrons goes up as $T \sim \sqrt{I}$ with the intensity I of the laser beam. When temperature effects are properly included, the effective momentum of the charged fluid becomes $\boldsymbol{p} = mf\gamma\boldsymbol{v}$ (see (2.16)), where $f \equiv f(T)$ is the scalar function given in (2.6). Dispersion relation (8.6) for electromagnetic waves in a hot relativistic electron plasma implies that strong thermal motion, as distinct from the directed motion, does actually lead to a lower effective plasma frequency, by $1/f$. Thus a strong laser beam will, indeed, be able to make a plasma more transparent as it transfers more and more energy to raise the plasma temperature.

The thermal self-induced transparency effect is clearly amenable to the conventional interpretation since it directly reflects an 'actual' decrease in the plasma frequency due to thermally enhanced electron inertia mf. Notice that at low temperatures, f approaches unity.

9. Conclusions

Using the fundamental Lorentz tensors defining the plasma, the ambient magnetic field and the wave characteristics, we have systematically derived explicit covariant forms of some of the well-known dispersion relations in magnetized as well as unmagnetized plasmas. It is difficult to overemphasize the importance of finding covariant dispersion when one deals with plasmas at high velocities and temperatures. It is only through such a route that one can insure physically acceptable solutions, as an intuitive relativization of the dispersion relations may be misleading. This paper's content should be as significant as the interesting and elegant covariant forms for the dispersion relations.

This formalism allow us to identify a fully consistent interpretation for SIT, that is that the plasma temperature increase the effective average inertia of the fluid by the factor f. Hence, this factor is responsible for bringing down the effective plasma frequency, while the rest-frame density evolves self-consistently in the plasma system. We hope that this interpretational change will make it easier to understand the experimental and simulation results.

Finally we would like to point out that, for mildly relativistic plasmas, thermally induced transparency effects are somewhat stronger than the kinematically induced ones. For example, for a $T \approx 20$ KeV ($T/m \approx 0.04$) plasma, the thermally induced change in critical density, $f - 1 \simeq (5/2)T/m = (5/4){v_{th}}^2 = 0.1$, while, for $v = v_{th}$, the kinematically induced difference, $\gamma - 1 \simeq v^2/2 = 2/50 \simeq 0.04$.

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