


# Class of Exact Solutions for a Cosmological Model of Unified Gravitational and Quintessence Fields

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**Abstract** A new approach to tackle Einstein equations for an isotropic and homogeneous Friedmann–Robertson–Walker Universe in the presence of a quintessence scalar field is devised. It provides a way to get a simple exact solution to these equations. This solution determines the quintessence potential uniquely and it differs from solutions which have been used to study inflation previously. It relays on a unification of geometry and dark matter implemented through the definition of a functional relation between the scale factor of the Universe and the quintessence field. For a positive curvature Universe, this solution produces perpetual accelerated expansion rate of the Universe, while the Hubble parameter increases abruptly, attains a maximum value and decreases thereafter. The behavior of this cosmological solution is discussed and its main features are displayed. The formalism is extended to include matter and radiation.

**Keywords** Quintessence · Exact solution · Unification of geometry and dark matter

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## 1 Introduction

One way to study the effect of dark energy in a cosmological model is by describing it by means of a scalar field with a potential energy. This field, sometimes called quintessence field or phantom scalar field, has been widely explored in inflationary theory (the literature is vast, see for example Refs. [1–7]). Several models for quintessence fields have been explored, for example, scalar fields with general potential [8] exponential potentials [9, 10], non minimally coupled scalar-tensor theories [11], Brans–Dicke theory [12], dark matter [13–15], or complex scalar fields [16], among others.

The equations which describe the model can be solved approximately by considering that the potential energy of the field is always much larger than its kinetic energy, producing inflation. During the past years, exact analytical solutions were studied for models with perfect fluids [17–21], matter or dust fluids [22–28], for non-minimal coupling [29, 30], and for non-spatially flat Friedmann–Robertson–Walker spacetimes [31, 32].

Our approach also studies an analytic exact solution for quintessence field models, although it is different in its approach. It is based on a simple and interesting feature of the Friedmann–Robertson–Walker–Quintessence (FRWQ) equations. A direct identification between the scale factor of the Universe and the quintessence field allows for a unique determination of the quintessence potential and produces equations that may be solved exactly. The solution shows an accelerated expansion of the Universe different from other inflationary theories. It can be also proved that this solutions admits the inclusion of matter or radiation in a consistent fashion.

First we start studying the solution for quintessence field highlighting the main features of this approach. Then, in Sec. III, we include a component to the Universe showing that the same anstaz allow us to solve the cosmological equations.

To write the equations for the FRWQ model, we start with the Lagrangian density  $\mathcal{L}$  for the evolution of the spacetime metric  $g_{\mu\nu}(x^\alpha)$  in interaction with a scalar field  $\phi(x^\beta)$ . This is  $\mathcal{L} = \sqrt{-g} [(R - 2\Lambda)/2\mathcal{G} + \mathcal{L}_\phi]$ , where  $g$  stands for the determinant of the metric  $g_{\mu\nu}$ ,  $\mathcal{G} = 8\pi G/c^4$  (with  $G$  as the gravitational constant and  $c$  the speed of light) and  $\Lambda$  is the cosmological constant. Here,  $\mathcal{L}_\phi$  is the Lagrangian density for the scalar field

$$\mathcal{L}_\phi = \epsilon \left( \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - \mathcal{V}(\phi) \right), \quad (1)$$

where  $\mathcal{V}(\phi)$  is (up to now) an unspecified potential for the scalar field  $\phi$ , and  $\epsilon$  is a parameter that classifies the nature of the scalar field, i.e.,  $\epsilon = 1$  yields the Lagrangian density for usual scalar fields while  $\epsilon = -1$  defines the Lagrangian for a quintessence field [33]. The line element for an isotropic and homogeneous Friedmann–Robertson–Walker (FRW) spacetime is [1]

$$ds^2 = dt^2 - a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \quad (2)$$

that defines the metric, where  $a(t)$  is the scale factor of the Universe, and  $k$  is the curvature constant that takes the values  $k = -1, 0, 1$  denoting negative, zero or positive

curvature of the Universe. Furthermore, it is worth noting that the quintessence field is described by the same Lagrangian density as the tlaplon field [34] which describes propagating torsion.

The action  $S = \int \mathcal{L} d^4x$ , gives rise (upon variation with respect to the metric tensor  $g_{\mu\nu}$ ) to gauge invariant (generally covariant) and constrained Einstein field equations coupled to matter  $G_{\mu\nu} + \Lambda g_{\mu\nu} = \mathcal{G} T_{\mu\nu}$ , where  $G^{\mu\nu}$  is the Einstein tensor and  $T^{\mu\nu}$  is the energy–momentum tensor of matter  $T_{\mu\nu} = g_{\mu\nu} \mathcal{L}_\phi - 2 \partial \mathcal{L}_\phi / \partial g^{\mu\nu}$ . Assuming that the scalar field  $\phi$  depends on time only, the Einstein equations, in terms of the line element (2), get two second order dynamical equations and one first order constraint. The dynamical equations read (setting  $\mathcal{G} = 1$ )

$$2 \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \epsilon \left(\frac{1}{2} \dot{\phi}^2 - V(\phi)\right) = 0, \tag{3}$$

and

$$\ddot{\phi} + 3 \frac{\dot{a}}{a} \dot{\phi} + \frac{dV(\phi)}{d\phi} = 0, \tag{4}$$

where we have introduced  $V(\phi) = \mathcal{V}(\phi) + \epsilon \Lambda$ . The constraint equation is

$$3 \left(\frac{\dot{a}}{a}\right)^2 + 3 \frac{k}{a^2} + \epsilon \left(\frac{1}{2} \dot{\phi}^2 + V(\phi)\right) = 0. \tag{5}$$

Equation (4) is obtained by varying the action  $S$  with respect to  $\phi$ , producing the Klein–Gordon equation for the scalar field  $\square\phi + d\mathcal{V}(\phi)/d\phi = 0$ . On the other hand, another useful equation can be obtained manipulating Eqs. (3) and (5) to give  $3\ddot{a}/a = \epsilon(\dot{\phi}^2 - V)$ .

For  $\epsilon = -1$ , the set (3)–(5) becomes the FRWQ system. This set of equations has been already studied for different scenarios and matter configurations [35–37]. The FRWQ system is usually employed to study inflation under standard approximations  $\dot{\phi}^2 \ll V(\phi)$  and  $\ddot{\phi} \ll dV/d\phi$  [1]. That solution represents exponential (inflationary) expansion of the Universe.

## 2 Exact Solution

The purpose of this manuscript is to show that there is an exact solution that may be obtained by defining a specific relation between the space–time curvature (represented by the scale factor  $a$ ) and dark energy (represented by the quintessence field  $\phi$ ).

As the scalar field depends on time, then  $\phi \equiv \phi(a)$ ,  $V = V(\phi) \equiv V(a)$ , and  $\dot{\phi} = (d\phi/da)\dot{a}$ . Therefore, Eq. (5) becomes

$$\dot{a}^2 \left[ \frac{3}{a^2} + \frac{\epsilon}{2} \left(\frac{d\phi}{da}\right)^2 \right] = -\epsilon V(a) - 3 \frac{k}{a^2}. \tag{6}$$

To solve this equation, one needs a specific form of the scalar field potential. However, notice that for  $\epsilon = -1$  a non-trivial solution can be obtained if the left-hand side of the previous equation is forced to vanish. This choice will be justified *a posteriori*.

Using this ansatz, the quintessence scalar field  $\phi$  is related to the scale factor  $a$  by

$$\phi = \sqrt{6} \ln a, \quad (7)$$

that also determines the quintessence potential as

$$V = \frac{3k}{a^2}. \quad (8)$$

The reader must realize that this kind of ansatz cannot be performed for usual scalar fields  $\epsilon = 1$ , as in this case, is not possible to solve the left-hand side for real scalar fields. We need to mention that other attempts of relating the quintessence field to the metric have been proposed [38], where the derivatives of scalar fields define a general metric. This approach does not produce the kind of unified dynamics studied here.

It is a straightforward matter to show that under the solutions given by Eqs.(7)–(8), both Eqs. (3) and (4) give rise to the same following equation

$$\frac{\ddot{a}}{a} + 2 \left( \frac{\dot{a}}{a} \right)^2 - \frac{k}{a^2} = 0, \quad (9)$$

in terms of  $a$  (or an equivalent equation written in terms of  $\phi$ ). This huge compactification in the dynamics is the justification on the ansatz that allows to solve Eq. (6). In other words, the whole set of Eqs. (3)–(5) describing FRWQ is now unified in the sole Eq. (9), where every dynamical field is expressed in terms of the scale factor.

Remarkably, Eq. (9) can be solved exactly, describing a cosmological model where gravity and dark energy are intimately intertwined into a single structure (7). This result is due to the fact that (5) is, loosely speaking, the “energy conservation equation” derived from the dynamical equations (3) and (4).

We can infer directly from Eq. (9) that this cosmological model presents accelerated expansion  $\ddot{a} > 0$  for  $k = 1$  if the (positive) initial expansion velocity  $\dot{a}(0)$  is restricted to be

$$0 < \dot{a}(t=0) < \frac{1}{\sqrt{2}}, \quad (10)$$

being a particular initial condition of the system. Notice that in this case, the quintessence scalar field becomes a dilaton field with the potential

$$V(\phi) = 3 \exp\left(-\frac{2\phi}{\sqrt{6}}\right). \quad (11)$$

For the other cases, either  $k = 0$  or  $k = -1$ , this cosmological model does not produce accelerated expansion. Although with different cosmological models with respect to

the one presented in this work, several potentials with the form of (11) have been considered extensively in the literature [22,23,39–41], showing that these models are integrable even in the case when the Hubble parameter is written in terms of the quintessence scalar field [24,39,42,43].

On the other hand, Eq. (9) can be integrated to find a conservation law. This result to be

$$E = a^4 \left( \frac{k}{2} - \dot{a}^2 \right), \tag{12}$$

for general  $k$ , and where  $E$  is a constant. In the case  $E = 0$ , then the dynamics is restricted to  $\ddot{a} = 0$ . We can see that for  $k = 1$  and under the initial condition (10), the constant  $E$  is positive. The previous equation can be integrated to find the general solution of Eq. (9)

$$t = \int_0^a \frac{a^2 da}{(ka^4/2 - E)^{1/2}}, \tag{13}$$

that can be exactly written in terms of elliptical integrals for  $k = 1$ . As an aside, notice that defining  $u = a^3/3$ , Eq. (9) becomes

$$\ddot{u} = k (3u)^{1/3}, \tag{14}$$

showing that this kind of cosmology behaves as a particle [44] driven by to the force  $k (3u)^{1/3}$ .

Let us study the behavior of the solutions of  $\dot{a}$ ,  $\ddot{a}$  and the Hubble parameter  $H = \dot{a}/a$  for certain initial conditions  $a(t = 0) \ll 1$  and  $\dot{a}(t = 0) \ll 1$ . Let us start the analysis for  $k = 1$ . First, we notice that from Eq. (12) the initial conditions of  $a$  and  $\dot{a}$  (or  $H$ ) determine  $E$ . As initially  $\dot{a}$  is small, then from Eq. (12) we get that  $0 < E \ll 1$ . Also, from Eq. (13) we see that our solution has the lower limit  $a > (2E)^{1/4}$ . From the evolution equation (9), we see that if the expansion is accelerated  $\ddot{a} \geq 0$ , then  $\dot{a}$  grows with time reaching the value  $1/\sqrt{2}$  asymptotically, where the accelerated expansion ends. Close to  $1/\sqrt{2}$ , the scale factor grows linearly with time. Thereby, in previous times, the time derivative of the scale factor is always positive. The exact behavior of the expansion is obtained from (12)

$$\dot{a}(t) = \sqrt{\frac{1}{2} - \frac{E}{a(t)^4}}, \tag{15}$$

where the time-dependence of  $a$  is obtained from Eq. (13). The behavior of  $\ddot{a}$  can be obtained from Eqs. (9) and (12). As long as  $k = 1$  and  $\dot{a}(0) < 1/\sqrt{2}$ , the expansion is accelerated  $\ddot{a} > 0$ . Using Eqs. (9) and (12), we can obtain that

$$\ddot{a} = \frac{2E}{a^5}, \tag{16}$$

and the initial condition for  $\ddot{a}(0)$  is determined by  $a(0)$ . Solution (16) shows that although in this model the Universe is subjected to an accelerated expansion, the acceleration decreases as  $a^{-5}$ . Finally, we can study the Hubble parameter, obtaining from Eq. (15) that

$$H = \sqrt{\frac{1}{2a^2} - \frac{E}{a^6}}. \quad (17)$$

As  $\dot{a} > 0$ , then the Hubble parameter grows reaching the maximum value  $(54 E)^{-1/4}$  for  $a = (6E)^{1/4}$ . In this point,  $\dot{a} = 1/\sqrt{3}$  and  $\ddot{a} = (486 E)^{-1/4}$ . Then, it decreases with time as  $\dot{a}$  approaches  $1/\sqrt{2}$ . An approximated solution of Eqs. (9) and (12) can be obtained using the above information. As  $E \ll 1$ , it is straightforward to obtain that

$$a = \frac{t}{\sqrt{2}} + \frac{2\sqrt{2}E}{3t^3} + \mathcal{O}(E^2). \quad (18)$$

This shows that the acceleration of the expansion for  $k = 1$  goes as

$$\ddot{a} \approx \frac{8\sqrt{2}E}{t^5} > 0. \quad (19)$$

On the other hand, similar analysis can be performed to the  $k = 0$  case. From Eqs. (9) and (12) we infer that  $E < 0$ . In this case we can direct integrate to obtain

$$a = \left(3\sqrt{-E} t\right)^{1/3}, \quad (20)$$

producing a desaccelerating expansion rate given by

$$\ddot{a} = -\frac{2(3\sqrt{-E})^{1/3}}{9t^{5/3}}. \quad (21)$$

Lastly, for the  $k = -1$  model, we get from Eqs. (9) and (12) that  $E < 0$ . Eq. (12) also gives the upper limit to the dynamics  $(-2E)^{1/4} > a$ . As  $E$  is small by initial conditions, the dynamics of this case is not very interesting.

### 3 Model of Universe with Different Components

One may wonder if this kind of solution holds for a Universe with single components. For these cases, the Eqs. (3)–(5) must be extended to include the effect of matter or radiation. The pertinent equations for such purposes are

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \epsilon \left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right) = -p, \quad (22)$$

$$3 \left(\frac{\dot{a}}{a}\right)^2 + 3\frac{k}{a^2} + \epsilon \left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) = \epsilon, \tag{23}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0, \tag{24}$$

and

$$\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + p) = 0, \tag{25}$$

where  $\epsilon$  is the energy density of the corresponding component, whereas  $p$  is its pressure. The whole previous system is consistent, and the closure is given by the equation of state  $p = w\epsilon$ , with  $w$  a dimensionless number. For matter-dominated era of the Universe  $w = 0$ , while for the radiation-dominated era,  $w = 1/3$ .

Newly, the system can be written in unified form for the gravitational and quintessence fields if we take the solution (7) [ $\epsilon = -1$  and  $\phi = \sqrt{6} \ln a$ ]. In this way, the quintessence potential acquires the form

$$V = \frac{3k}{a^2} - \epsilon. \tag{26}$$

This ansatz solves identically Eq. (23). Besides, Eqs. (22) and (24) become the following equation

$$\frac{\ddot{a}}{a} + 2 \left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2} + \frac{\epsilon}{2}(1 + w) = 0, \tag{27}$$

where we have made use of Eq. (25). Besides, the solution of Eq. (25) is

$$\epsilon = \epsilon_0 a^{-3(1+w)}, \tag{28}$$

with the initial value of  $\epsilon_0$ . Then, this FRWQ model with single component is determined by the equation

$$\frac{\ddot{a}}{a} + 2 \left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2} + \frac{\epsilon_0(1 + w)}{2a^{3(1+w)}} = 0. \tag{29}$$

The previous dynamics can be integrated for  $w \neq 1$  to give

$$E = a^4 \left[ \frac{k}{2} - \dot{a}^2 - \frac{\epsilon_0}{3a^{1+3w}} \left( \frac{1 + w}{1 - w} \right) \right], \tag{30}$$

where  $E$  is again a constant. The solution can be obtained through the integral (which can be solved in terms of elliptical integrals)

$$t = \int da \left[ \frac{k}{2} - \frac{E}{a^4} - \frac{\epsilon_0}{3a^{1+3w}} \left( \frac{1 + w}{1 - w} \right) \right]^{-1/2}, \tag{31}$$

for matter-dominated Universe ( $w = 0$ ) and radiation-dominated Universe ( $w = 1/3$ ).

Again we can study the behavior of the solutions for initial conditions  $a(t = 0) \ll 1$  and  $\dot{a}(t = 0) \ll 1$ . From Eq. (29) we infer that  $E \ll 1$  for  $w = 0$  and  $w = 1/3$ . We can readily see that the only case that produce accelerated expansion  $\ddot{a} > 0$  is for  $k = 1$ . For this case is enough to choose  $E > 0$ , and the dynamics is restricted to a lowest limit given by

$$a \left[ 1 - \frac{2\varepsilon_0}{3a^{1+3w}} \left( \frac{1+w}{1-w} \right) \right] > (2E)^{1/4}. \quad (32)$$

Similar to the previous case, as  $E \ll 1$ , an approximated solution can be found  $a(t) = a_1(t) + E a_2(t) + \mathcal{O}(E^2)$ , where  $a_1$  can be found as the solution of integral (31) with  $k = 1$  and  $E = 0$

$$\dot{a}_1 \approx \sqrt{\frac{1}{2} - \frac{\varepsilon_0}{3a^{1+3w}} \left( \frac{1+w}{1-w} \right)}, \quad (33)$$

whereas  $a_2$  satisfies the equation

$$\left( a_1^4 \dot{a}_1 \right) \dot{a}_2 = \frac{\varepsilon_0 a_2}{2a_1^{-2+3w}} \left( \frac{1+w}{1-w} \right) \left( \frac{1}{3} + w \right) - \frac{1}{2}. \quad (34)$$

In an opposite way, for the  $k = 0$  and  $k = -1$  cases the dynamics do not present accelerated expansion.

## 4 Conclusions

We have shown that the previous solutions produce a functional relation between the Universe scale factor  $a$  and the dark matter quintessence field, uniquely determining the scalar field  $\phi = \phi(a)$  (Eq. (7)), and the quintessence potential (Eqs. (8), (26)). This approach constitutes a new solution of the FRWQ system, as implies a new ansatz that can produce a new dynamic different to previous ones where the scalar field and the potential are reconstructed from the equations of motion [45]. This solution is different to previous models as the relation between spacetime and the fields is a priori invoked to solve the dynamics.

For the case of only quintessence fields, it can be easily seen that the evolution of the scale factor  $a$  for  $k = 1$  is such that its velocity reaches asymptotically the value  $1/\sqrt{2}$ . Therefore, the evolution is always positively accelerated, while the Hubble parameter grows until it reaches a maximum value and it decreases from then on. This behavior is appropriated for an initial value of  $\dot{a}$  constrained to be less than  $1/\sqrt{2}$ . In this case, the Universe attains the terminal velocity  $1/\sqrt{2}$ , at  $a \rightarrow \infty$  behaving as an attractor point for the Universe. If  $\dot{a}$  is larger than  $1/\sqrt{2}$  initially, then the Universe experiences a decelerated expansion with  $\dot{a}$  also approaching to  $1/\sqrt{2}$  at the attractor point ( $a \rightarrow \infty$ ,  $\dot{a} \rightarrow 1/\sqrt{2}$ ). On the contrary, we can see from (9) that for  $k \neq 1$  the Universe always experiences a decelerated expansion  $\ddot{a} < 0$ .



Also, we have shown that the ansatz (7) and (8) still holds when matter or radiation are included in the scheme. Again the most interesting dynamics is achieved for  $k = 1$  showing that the system presents positive acceleration and that it can be solved for  $E \ll 1$ .

The current approach takes advantage of the simplest ansatz we can assume to solve the cosmological equations exactly for a quintessence field without using the slow-roll approximation. This ansatz allows us to obtain some properties of the model in a straightforward way. For example, for the case discussed in Sec. II, the e-foldings during inflation can be obtained as

$$N = \int_{t_i}^{t_f} H dt = \int_{\phi_i}^{\phi_f} \frac{H}{\dot{\phi}} d\phi \approx -\sqrt{6}\phi_i, \quad (35)$$

as  $\phi_f \ll \phi_i$  and  $\dot{\phi} = \sqrt{6}H$  by Eq. (7). As we can find the solution  $a \approx t/\sqrt{2}$  given by (18) for  $E \ll 1$ , then  $\phi_i \approx \sqrt{6} \ln(t_i/\sqrt{2})$ , and  $N \gg 1$  as the initial time approaches to 0.

The features of these inflationary models with and without matter and radiation, such as e-foldings, passage from deceleration to acceleration, existence or not of fine-tuning problems, etc., will be explored further in forthcoming works.

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## References

1. Ryden, B.: Introduction to Cosmology. Addison Wesley, San Francisco (2003)
2. Tsujikawa, S.: Introductory review of cosmic inflation (2003). [arXiv:hep-ph/0304257](https://arxiv.org/abs/hep-ph/0304257)
3. Guth, A.: Inflationary universe: a possible solution to the horizon and flatness problems. *Phys. Rev. D* **23**, 347 (1981)
4. Liddle, A. R.: An introduction to cosmological inflation (1999). [arXiv:astro-ph/9901124](https://arxiv.org/abs/astro-ph/9901124)
5. Watson, S.: An Exposition on Inflationary Cosmology (2000). [arXiv:astro-ph/0005003](https://arxiv.org/abs/astro-ph/0005003)
6. Brandenberger, R.H.: Inflationary cosmology: progress and problems. *Astrophys. Space Sci. Libr.* **247**, 169 (2000)
7. Linde, A.: Particle Physics and Inflationary Cosmology. Harwood Academic Publishers, Chur (1990)
8. de la Macorra, A., Piccinelli, G.: Cosmological evolution of general scalar fields and quintessence. *Phys. Rev. D* **61**, 123503 (2000)
9. Barreiro, T., Copeland, E.J., Nunes, N.J.: Quintessence arising from exponential potentials. *Phys. Rev. D* **61**, 127301 (2000)
10. Rubano, C., Scudellaro, P.: On some exponential potentials for a cosmological scalar field as quintessence. *Gen. Relativ. Gravit.* **34**, 307 (2002)
11. Bartolo, N., Pietroni, M.: Scalar-tensor gravity and quintessence. *Phys. Rev. D* **61**, 023518 (1999)
12. Banerjee, N., Pavón, D.: A quintessence scalar field in Brans-Dicke theory. *Class. Quantum Gravity* **18**, 593 (2001)
13. Matos, T., Ureña-López, L.A.: Quintessence and scalar dark matter in the Universe. *Class. Quantum Gravity* **17**, L75 (2000)
14. Matos, T., Siddhartha Guzmán, F., Ureña-López, L.A.: Scalar field as dark matter in the universe. *Class. Quantum Gravity* **17**, 1707 (2000)
15. Sahni, V., Wang, L.: New cosmological model of quintessence and dark matter. *Phys. Rev. D* **62**, 103517 (2000)
16. Gu, J.-A., Hwang, W.-Y.P.: Can the quintessence be a complex scalar field? *Phys. Lett. B* **517**, 1 (2001)

17. Paliathanasis, A., Tsamparlis, M., Basilakos, S., Barrow, J.D.: Dynamical analysis in scalar field cosmology. *Phys. Rev. D* **91**, 123535 (2015)
18. Ureña-López, L.A., Matos, T.: New cosmological tracker solution for quintessence. *Phys. Rev. D* **62**, 081302 (2000)
19. Sahni, V., Starobinsky, A.A.: The case for a positive cosmological  $\Lambda$ -term. *Int. J. Mod. Phys. D* **9**, 373 (2000)
20. Chimento, L.P., Jakubi, A.S.: Scalar field cosmologies with perfect fluid in Robertson-Walker metric. *Int. J. Mod. Phys. D* **5**, 71 (1996)
21. Rubano, C., Barrow, J.D.: Scaling solutions and reconstruction of scalar field potentials. *Phys. Rev. D* **64**, 127301 (2001)
22. Russo, J.: Exact solution of scalar field cosmology with exponential potentials and transient acceleration. *Phys. Lett. B* **600**, 185 (2004)
23. Fré, P., Sagnotti, A., Sorin, A.S.: Integrable scalar cosmologies I. Foundations and links with string theory. *Nucl. Phys. B* **877**, 1028 (2013)
24. Muslimov, A.G.: On the scalar field dynamics in a spatially flat Friedman universe. *Class. Quantum Gravity* **7**, 231 (1990)
25. Ellis, G.F., Madsen, M.S.: Exact scalar field cosmologies. *Class. Quantum Gravity* **8**, 667 (1991)
26. Barrow, J.D., Saich, P.: Scalar-field cosmologies. *Class. Quantum Gravity* **10**, 279 (1993)
27. Basilakos, S., Tsamparlis, M., Paliathanasis, A.: Using the Noether symmetry approach to probe the nature of dark energy. *Phys. Rev. D* **83**, 103512 (2011)
28. Paliathanasis, A., Tsamparlis, M., Basilakos, S.: Dynamical symmetries and observational constraints in scalar field cosmology. *Phys. Rev. D* **90**, 103524 (2014)
29. Kamenshchik, A.Y., Pozdeeva, E.O., Alessandro, T., Venturi, G., Vernov, S.Y.: Interdependence between integrable cosmological models with minimal and non-minimal coupling. *Class. Quantum Gravity* **33**, 015004 (2016)
30. Kamenshchik, A.Y., Pozdeeva, E.O., Alessandro, T., Venturi, G., Vernov, S.Y.: Integrable cosmological models with non-minimally coupled scalar fields. *Class. Quantum Gravity* **31**, 105003 (2014)
31. Easther, R.: Exact superstring motivated cosmological models. *Class. Quantum Gravity* **10**, 2203 (1993)
32. Gumjudpai, B.: Scalar field exact solutions for non-flat FLRW cosmology: a technique from non-linear Schrödinger-type formulation. *Gen. Relative. Grav.* **41**, 249 (2009)
33. Steinhardt, P.J.: A quintessential introduction to dark energy. *Philos. Trans. R. Soc. Lond. A* **361**, 2497 (2003)
34. Hojman, S., Rosenbaum, M., Ryan, M.P., Shepley, L.C.: Gauge invariance, minimal coupling, and torsion. *Phys. Rev. D* **17**, 3141 (1978)
35. Tsujikawa, S.: Quintessence: a review. *Class. Quantum Gravity* **30**, 214003 (2013)
36. Copeland, E.J., Sami, M., Tsujikawa, S.: Dynamics of dark energy. *Int. J. Mod. Phys. D* **15**, 1753 (2006)
37. Capozziello, S., Roshan, M.: Exact cosmological solutions from Hojman conservation quantities. *Phys. Lett. B* **726**, 471 (2013)
38. Chamseddinea, A.H., Mukhanov, V.: Mimetic dark matter. *JHEP* **11**, 135 (2013)
39. Salopek, D.S., Bond, J.R.: Nonlinear evolution of long-wavelength metric fluctuations in inflationary models. *Phys. Rev. D* **42**, 3936 (1990)
40. Neupane, I.P.: Accelerating cosmologies from exponential potentials. *Class. Quantum Gravity* **21**, 4383 (2004)
41. Elizalde, E., Nojiri, S., Odintsov, S.D.: Late-time cosmology in a (phantom) scalar-tensor theory: dark energy and the cosmic speed-up. *Phys. Rev. D* **70**, 043539 (2004)
42. Bazeia, D., Gomes, C.B., Losano, L., Menezes, R.: First-order formalism and dark energy. *Phys. Lett. B* **633**, 415 (2006)
43. Harko, T., Lobo, F.S.N., Mak, M.K.: Arbitrary scalar-field and quintessence cosmological models. *Eur. Phys. J. C* **74**, 2784 (2014)
44. Hojman, S.A., Asenjo, F.A.: Supersymmetric Majorana quantum cosmologies. *Phys. Rev. D* **92**, 083518 (2015)
45. Kamenshchik, A.Y., Manti, S.: Scalar field potentials for closed and open cosmological models. *Gen. Relativ. Gravit.* **44**, 2205 (2012)