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ON CONVEX COMBINATIONS OF CONVEX HARMONIC MAPPINGS

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Abstract

The family $\mathcal{F}_\lambda$ of orientation-preserving harmonic functions $f = h + \overline{g}$ in the unit disc $\mathbb{D}$ (normalised in the standard way) satisfying

$$h'(z) + g'(z) = \frac{1}{(1+\lambda z)(1+\overline{\lambda} z)}, \quad z \in \mathbb{D},$$

for some $\lambda \in \partial\mathbb{D}$, along with their rotations, play an important role among those functions that are harmonic and orientation-preserving and map the unit disc onto a convex domain. The main theorem in this paper generalises results in recent literature by showing that convex combinations of functions in $\mathcal{F}_\lambda$ are convex.

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1. Introduction

A complex-valued harmonic function f in the unit disc $\mathbb{D}$ can be represented as $f = h + \overline{g}$, where both h and g are analytic in $\mathbb{D}$. This representation is unique up to an additive constant which is usually determined by imposing the condition that the function g fixes the origin. The representation $f = h + \overline{g}$ is then unique and is called the *canonical representation* of f.

It is a consequence of the inverse mapping theorem that if the Jacobian of a C^1 mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ is different from zero, the function is locally univalent. Lewy [8] showed that when the function is harmonic, the converse also holds. Hence, a harmonic mapping $f = h + \overline{g}$ is locally univalent if and only if its Jacobian $J_f = |h'|^2 - |g'|^2 \neq 0$. Thus, locally univalent harmonic mappings in the unit disc can be classified as *orientation-preserving* mappings (if $J_f > 0$ in $\mathbb{D}$) or *orientation-reversing* (if $J_f < 0$ in $\mathbb{D}$).

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It is obvious that f is orientation-preserving if and only if $\overline{f}$ is orientation-reversing. Note that $f = h + \overline{g}$ is orientation-preserving if and only if $h' \neq 0$ in $\mathbb{D}$ and the *dilatation* $\omega = g'/h'$ is an analytic function in the unit disc which maps $\mathbb{D}$ into itself.

For a comprehensive treatment of harmonic mappings in the unit disc, we refer the reader to [4].

1.1. Convex harmonic mappings. Let $\mathcal{A}$ denote the class of all analytic functions φ in the unit disc $\mathbb{D}$ normalised by the conditions $\varphi(0) = \varphi'(0) - 1 = 0$. Let $\mathcal{H}$ denote the family of complex-valued harmonic mappings $f = h + \overline{g}$ in $\mathbb{D}$ that preserve the orientation and are normalised by the conditions $h(0) = g(0) = 0$ and $h'(0) = 1$. The class $\mathcal{H}_0$ consists of those functions $f \in \mathcal{H}$ with $g'(0) = 0$.

We will consider particular properties of functions $f \in \mathcal{H}_0$ that map the unit disc onto a convex domain. The family of such mappings is, as usual, denoted by K_H^0. It is known that every function $f \in K_H^0$ is univalent in $\mathbb{D}$ (see [2, Theorem 5.7] and [7, Corollary 2.2]).

According to [2, Theorem 5.7], a harmonic mapping $f = h + \overline{g}$ belongs to K_H^0 if and only if, for each $\theta \in (-\pi/2, \pi/2]$, the analytic function $\psi_\theta = h + e^{2i\theta} g$ belongs to $\mathcal{A}$ and is convex in the direction $(\theta + \pi/2)$, meaning that the intersection of $\psi_\theta(\mathbb{D})$ with any line parallel to the line through 0 and $ie^{i\theta}$ is an interval or the empty set.

It is obvious that ψ_θ is convex in the $(\theta + \pi/2)$-direction if and only if the function $\varphi_\theta = e^{-i\theta}\psi_\theta$ is convex in the direction of the imaginary axis (that is, the $\pi/2$-direction). The following characterisation, due to Royster and Ziegler [10], of analytic functions in the unit disc that map $\mathbb{D}$ onto a domain convex in the vertical direction will be used later in this paper.

Theorem A. Let φ be a locally univalent analytic function in the unit disc. Then φ maps $\mathbb{D}$ onto a domain convex in the direction of the imaginary axis if and only if there are numbers $\mu \in [0, 2\pi)$ and $\nu \in [0, \pi]$ such that

$$\mathrm{Re}\{-ie^{i\mu}(1 - 2ze^{-i\mu}\cos\nu + e^{-2i\mu}z^2)\,\varphi'(z)\} \geq 0, \quad z \in \mathbb{D}.$$

1.2. Some special convex harmonic mappings. By applying different transformations to some of the harmonic mappings considered by Hengartner and Schober in [6], Dorff [3] showed that if the harmonic function $f = h + \overline{g} \in \mathcal{H}_0$ maps the unit disc onto a vertical strip

$$\Omega_\alpha = \left\{ w \in \mathbb{C} : \frac{\alpha - \pi}{2\sin\alpha} < \mathrm{Re}\{w\} < \frac{\alpha}{2\sin\alpha} \right\},$$

where $\pi/2 \leq \alpha < \pi$, then

$$h(z) + g(z) = \frac{1}{2i\sin\alpha} \log\left(\frac{1 + e^{i\alpha}z}{1 + e^{-i\alpha}z}\right), \quad z \in \mathbb{D}. \tag{1.1}$$

Some nice consequences are obtained from this result. Thus, Dorff [3] shows that if $f = h + \overline{g} \in K_H^0$ maps the unit disc onto a half-plane of the form $\{\mathrm{Re}\{w\} > a\}$, where

a is any negative real number, then $a = -1/2$. Moreover, such a harmonic mapping f from the unit disc onto the half-plane $\Omega = \{w \in \mathbb{C} : \mathrm{Re}\{w\} > -1/2\}$ satisfies

$$h(z) + g(z) = \frac{z}{1-z}, \quad z \in \mathbb{D}. \tag{1.2}$$

Note that if a function $f = h + \overline{g} \in \mathcal{H}_0$ satisfies either (1.1) or (1.2), then there exists $\lambda \in \partial\mathbb{D}$ such that

$$h'(z) + g'(z) = \frac{1}{(1+\lambda z)(1+\overline{\lambda} z)}. \tag{1.3}$$

In what follows, given $\lambda \in \partial\mathbb{D}$, we denote by $\mathcal{F}_\lambda$ the family of harmonic mappings $f = h + \overline{g} \in K_H^0$ for which (1.3) holds.

1.3. Rotations. It is obvious that if $f = h + \overline{g} \in K_H^0$ and $\mu \in \partial\mathbb{D}$, then the rotation f_μ defined in the unit disc by the formula

$$f_\mu(z) = \overline{\mu} f(\mu z)$$

also belongs to K_H^0. Moreover, a straightforward calculation shows that the functions h_μ and g_μ in the canonical decomposition of $f_\mu = h_\mu + \overline{g_\mu}$ are, respectively,

$$h_\mu(z) = \overline{\mu} h(\mu z) \quad \text{and} \quad g_\mu(z) = \mu g(\mu z), \quad z \in \mathbb{D}.$$

Therefore, $f = h + \overline{g} \in \mathcal{F}_\lambda$ if and only if for all $z \in \mathbb{D}$ and λ and μ as above,

$$h_\mu'(z) + \overline{\mu^2} g_\mu'(z) = \frac{1}{(1+\lambda\mu z)(1+\overline{\lambda}\mu z)}.$$

In other words, we have proved the following result.

Proposition 1.1. *A harmonic mapping $F = H + \overline{G} \in \mathcal{H}_0$ satisfying*

$$H'(z) + \overline{\mu^2} G'(z) = \frac{1}{(1+\lambda\mu z)(1+\overline{\lambda}\mu z)} \tag{1.4}$$

for certain λ and μ in $\partial\mathbb{D}$ and all $|z| < 1$ has a rotation $f = h + \overline{g}$ which satisfies (1.3).

Some particular cases of harmonic functions $F = H + \overline{G} \in \mathcal{H}_0$ for which (1.4) holds have been considered in the literature. For example, harmonic mappings $F = H + \overline{G}$ satisfying (1.4) for the particular values $\lambda = \mu = i$ were considered in [5] and [12]. The functions H and G in the canonical decomposition of such functions F satisfy

$$H(z) - G(z) = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right), \quad z \in \mathbb{D}. \tag{1.5}$$

It was proved in [5] (and independently in [12]) that any normalised harmonic mapping $F = H + \overline{G} \in \mathcal{H}_0$ for which (1.5) holds is convex. It is shown in [13] that the convex combination of such harmonic mappings is convex in the horizontal direction. The following stronger result is given in [12]: convex combinations of functions satisfying (1.5) are convex (see [12, Theorem 4]).

A related result appears in [11]. Take $\lambda = -1$ and an arbitrary $\mu \in \partial\mathbb{D}$ in (1.4). Following the terminology in [9], these harmonic functions are called *slanted half-plane harmonic mappings with parameter* $\overline{\mu}$. The main result in [11] shows that convex combinations of slanted half-plane harmonic mappings are convex.

The goal in this paper is to show that there is no need to specialise the parameters λ and μ in (1.4) to get the results cited from [11] and [12]. More specifically, our main theorem is as follows.

THEOREM 1.2. *Let λ and μ be fixed but arbitrary complex numbers in $\partial\mathbb{D}$. Assume that the harmonic mappings $f_j \in \mathcal{H}_0$, $j = 1, 2, \ldots, n$, satisfy (1.4) for these values of λ and μ. Then, any convex combination of the f_j is a convex harmonic mapping.*

2. Two key results

The following lemma was proved in [11]. We include the proof here for the sake of completeness.

LEMMA 2.1. *Let ω_1 and ω_2 be two analytic functions in the unit disc that map $\mathbb{D}$ to itself. Then, for any real number θ and all $z \in \mathbb{D}$,*

$$\mathrm{Re}\left\{\frac{1-\omega_1(z)\overline{\omega_2(z)}}{(1+e^{-2i\theta}\omega_1(z))(1+e^{2i\theta}\overline{\omega_2(z)})}\right\} > 0. \tag{2.1}$$

PROOF. The analytic functions $\varphi_1(z) = 1/(1+z)$ and $\varphi_2(z) = z/(1-z)$ in $\mathbb{D}$ map the unit disc onto the half-planes $\{\mathrm{Re}\{w\} > 1/2\}$ and $\{\mathrm{Re}\{w\} > -1/2\}$, respectively. Hence, for any given ζ with $|\zeta| = 1$, any analytic function ω in $\mathbb{D}$ for which the inclusion $\omega(\mathbb{D}) \subset \mathbb{D}$ holds, and all $|z| < 1$,

$$\mathrm{Re}\left\{\frac{1}{1+\overline{\zeta\omega(z)}}\right\} > \frac{1}{2} \quad \text{and} \quad \mathrm{Re}\left\{\frac{-\zeta\omega(z)}{1+\zeta\omega(z)}\right\} > -\frac{1}{2}.$$

Using the identity

$$\frac{1-\omega_1\overline{\omega_2}}{(1+e^{-2i\theta}\omega_1)(1+e^{2i\theta}\overline{\omega_2})} = \frac{1}{1+\overline{e^{-2i\theta}\omega_2}} - \frac{e^{-2i\theta}\omega_1}{1+e^{-2i\theta}\omega_1},$$

we obtain (2.1). This ends the proof. □

A modification of the arguments used in [1] gives the following fundamental result, which will be used to prove Theorem 1.2.

THEOREM 2.2. *Let $f = h + \overline{g}$ belong to $\mathcal{F}_\lambda$ for some $\lambda \in \partial\mathbb{D}$. Then f is convex.*

PROOF. If $f = h + \overline{g} \in \mathcal{F}_\lambda$, we can write

$$h'(z) + g'(z) = \frac{1}{(1+\lambda z)(1+\overline{\lambda} z)} = \frac{1}{1+2z\cos\alpha + z^2}, \tag{2.2}$$

where $\alpha \in [0, \pi]$ is such that $\cos\alpha = \mathrm{Re}\{\lambda\}$. Also, since the dilatation $\omega = g'/h'$ of f maps the unit disc to itself, the function $(h' - g')/(h' + g')$ has a positive real part.

As explained in the introduction, according to both [2, Theorem 5.7] and Theorem A, in order to check that f is convex we need to show that for all values of $\theta \in (-\pi/2, \pi/2]$ there are real numbers μ and ν with $0 \le \mu < 2\pi$ and $\nu \in [0, \pi]$, possibly depending on θ, such that for all $z \in \mathbb{D}$,

$$\mathrm{Re}\{-ie^{i\mu}(1 - 2\,ze^{-i\mu}\cos\nu + e^{-2i\mu}z^2)\varphi_\theta'(z)\} \ge 0, \tag{2.3}$$

where $\varphi_\theta = e^{-i\theta}(h + e^{2i\theta}g)$.

Assume first that $\theta \in (-\pi/2, 0]$ and set $\mu = 0$ and $\nu = \pi - \alpha$ (so that $\cos\nu = -\cos\alpha$). From (2.2), for $z \in \mathbb{D}$, the function that appears in (2.3) satisfies

$$\begin{aligned}
&\mathrm{Re}\{-i(1 + 2z\cos\alpha + z^2)(e^{-i\theta}h'(z) + e^{i\theta}\,g'(z))\}\\
&\quad = \mathrm{Im}\{(1 + 2z\cos\alpha + z^2)(e^{-i\theta}h'(z) + e^{i\theta}\,g'(z))\}\\
&\quad = \mathrm{Im}\{(1 + 2z\cos\alpha + z^2)[\cos\theta(h'(z) + g'(z)) - i\sin\theta(h'(z) - g'(z))]\}\\
&\quad = \mathrm{Im}\Big\{\cos\theta - i\sin\theta\Big(\frac{h'(z) - g'(z)}{h'(z) + g'(z)}\Big)\Big\}\\
&\quad = -\sin\theta\,\mathrm{Re}\Big\{\frac{h'(z) - g'(z)}{h'(z) + g'(z)}\Big\} \ge 0.
\end{aligned}$$

For the remaining case when $\theta \in (0, \pi/2]$, we set $\mu = \pi$ and $\nu = \alpha$ and proceed in exactly the same way. □

3. Proof of Theorem 1.2

By Proposition 1.1, if $f_j = h_j + \overline{g_j}$, $j = 1, 2, \ldots, n$, satisfy (1.4), we can consider appropriate rotations (denoted again by f_j) such that $f_j \in \mathcal{F}_\lambda$. If we can show that

$$f = \sum_{j=1}^{n} t_j f_j \in \mathcal{F}_\lambda,$$

where $t_1, t_2, \ldots, t_n$ are nonnegative real numbers with $\sum_{j=1}^{n} t_j = 1$, then by Theorem 2.2 f is convex and we will be done.

Clearly, the function f is harmonic in the unit disc and its canonical decomposition is given by $f = h + \overline{g}$, where

$$h(z) = \sum_{j=1}^{n} t_j h_j(z) \quad \text{and} \quad g(z) = \sum_{j=1}^{n} t_j g_j(z), \qquad z \in \mathbb{D}.$$

Therefore, $h(0) = g(0) = 0$ and $h'(0) - 1 = g'(0) = 0$. Moreover, since $f_j = h_j + \overline{g_j}$ belongs to $\mathcal{F}_\lambda$ for $j = 1, 2, \ldots, n$,

$$h'(z) + g'(z) = \frac{1}{(1 + \lambda z)(1 + \overline{\lambda} z)}.$$

The only remaining step required to show that $f \in \mathcal{F}_\lambda$ is that f preserves the orientation.

Let ω_j denote the dilatation of f_j, so that $g_j' = \omega_j h_j'$. Since $f_j \in \mathcal{F}_\lambda$, for all z in the unit disc,

$$h_j'(z) = \frac{1}{(1+\lambda z)(1+\overline{\lambda} z)(1+\omega_j(z))}.$$

This gives

$$h'(z) = \frac{1}{(1+\lambda z)(1+\overline{\lambda} z)} \sum_{j=1}^{n} \frac{t_j}{1+\omega_j(z)}.$$

On the other hand,

$$g'(z) = \sum_{j=1}^{n} t_j g_j'(z) = \sum_{j=1}^{n} t_j \omega_j(z) h_j'(z) = \frac{1}{(1+\lambda z)(1+\overline{\lambda} z)} \sum_{j=1}^{n} \frac{t_j \omega_j(z)}{1+\omega_j(z)}.$$

Consider the function

$$\Phi(z) = \left| \sum_{j=1}^{n} \frac{t_j}{1+\omega_j(z)} \right|^2 - \left| \sum_{j=1}^{n} \frac{t_j \omega_j(z)}{1+\omega_j(z)} \right|^2, \quad z \in \mathbb{D}.$$

Since

$$J_f(z) = \frac{\Phi(z)}{|(1+\lambda z)(1+\overline{\lambda} z)|^2},$$

it is obvious that f preserves the orientation if $\Phi > 0$ in the unit disc. Now, a straightforward calculation shows that

$$\begin{aligned}
\Phi &= \left(\sum_{j=1}^{n} \frac{t_j}{1+\omega_j} \right)\left(\sum_{j=1}^{n} \frac{t_j}{1+\overline{\omega_j}} \right) - \left(\sum_{j=1}^{n} \frac{t_j \omega_j}{1+\omega_j} \right)\left(\sum_{j=1}^{n} \frac{t_j \overline{\omega_j}}{1+\overline{\omega_j}} \right) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{t_j t_k}{(1+\omega_j)(1+\overline{\omega_k})} - \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{t_j t_k \omega_j \overline{\omega_k}}{(1+\omega_j)(1+\overline{\omega_k})} \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{t_j t_k (1-\omega_j \overline{\omega_k})}{(1+\omega_j)(1+\overline{\omega_k})} \\
&= 2 \sum_{j=1}^{n} \sum_{k<j} \mathrm{Re}\left\{ \frac{t_j t_k (1-\omega_j \overline{\omega_k})}{(1+\omega_j)(1+\overline{\omega_k})} \right\} + \sum_{j=1}^{n} \frac{t_j^2 (1-|\omega_j|^2)}{|1+\omega_j|^2}.
\end{aligned}$$

Since ω_j are analytic and $\omega_j(\mathbb{D}) \subset \mathbb{D}$ for all $j = 1, 2, \ldots, n$, we see by Lemma 2.1 that $\Phi > 0$ in $\mathbb{D}$. This completes the proof of Theorem 1.2.

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