J. Math. Anal. Appl. 455 (2017) 381–388

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the univalence of certain integral for harmonic mappings

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ARTICLE INFO

Article history:
Received 11 November 2016
Available online 30 May 2017
Submitted by L. Fialkow

Keywords:
Harmonic mapping
Univalent functions
Integral transform

ABSTRACT

We generalize the problem of univalence of the integral of $f'(z)^\alpha$ when f is univalent to the complex harmonic mappings. To do this, we extend the univalence criterion by Ahlfors in [1] to those mappings.

1. Introduction

Let S be the set of all holomorphic and univalent functions in the unit disc $\mathbb{D}$, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Let α be a complex number and define the function f_α as

$$f_\alpha(z) = \int_0^z [f'(\zeta)]^\alpha \, d\zeta \,. \tag{1}$$

The power in the mapping f_α is defined via the branch of $\log f'(\zeta)$ for which $\log f'(0) = 0$. It is trivial to check that given any f, if either $\alpha = 1$ or $\alpha = 0$, the corresponding function f_α belongs to S. Nevertheless, the question of how to determine the values of α for which the function f_α is in S when $f \in S$ is not so simple. W.C. Royster showed that, in general, f_α fails to be univalent for $|\alpha| > 1/3$ and $\alpha \neq 1$, see [13]. J. Pfaltzgraff in [11] proved that for any given function f in the unit disc normalized as above in a linearly invariant family $\mathcal{F}$ with order λ, defined by the supremum of the modulus of the second Taylor coefficient, then f_α belongs to S if $|\alpha| \leq 1/(2\lambda)$. When $\mathcal{F} = S$, it is well known that $\lambda = 2$, hence if $|\alpha| \leq 1/4$, then

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http://dx.doi.org/10.1016/j.jmaa.2017.05.056

$f_\alpha \in S$. However, it is an open problem to determine if f_α is univalent when $1/4 < |\alpha| \leq 1/3$. Moreover, for real α and some subclasses of univalent functions in the unit disc, the problem was completely resolved, for instance when the original function f is convex or close-to-convex mapping. The reader can find further explanation in [5, p. 153].

Let f now be a complex value univalent harmonic mapping in the unit disc. We assume that f is sense preserving, this is, $f = h + \overline{g}$ where h and g are analytic functions in $\mathbb{D}$ such that h is locally univalent and the dilatation $\omega = g'/h'$ is an analytic function, mapping the unit disc into itself. The main purpose of this note is to extend the J. Pfaltzgraff's result to these harmonic mappings and the tools used in that work. In Theorem 4 we prove concretely that, if α is a real number such that

$$|\alpha| \leq \frac{1}{2\lambda + (\|\omega\|_\infty + 4)\|\omega^*\|},$$

then f_α is univalent in $\mathbb{D}$.

2. Background

One of the most used univalence criterion for locally univalent holomorphic mappings was showed by J. Becker in [2], which asserts that, if $(1-|z|^2)|zf''(z)/f'(z)| \leq 1$ then f is univalent in $\mathbb{D}$. L. Ahlfors in [1] gave a generalization of these results which became the main tools used by J. Pfaltzgraff. This criterion can be formulated as:

Theorem A. *Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function with $f' \neq 0$. Let $c \in \mathbb{C}$ with $|c| < 1$, $c \neq -1$, and assume that*

$$\left|(1-|z|^2)z\frac{f''}{f'}(z) + c|z|^2\right| \leq 1, \qquad \forall z \in \mathbb{D}.$$

Then f is univalent on $\mathbb{D}$.

2.1. Complex value harmonic mappings

A complex-valued harmonic function in a simply connected domain Ω has a representation $f = h + \overline{g}$, where h and g are analytic functions in Ω, that is unique up to an additive constant. Notice that when $\Omega = \mathbb{D}$ it is convenient to choose the additive constant so that $g(0) = 0$. The representation $f = h + \overline{g}$ is therefore unique and is called the canonical representation of f. Lewy in [10] proved that f is locally univalent if and only if the Jacobian satisfies $J_f = |h'|^2 - |g'|^2 \neq 0$. Thus, harmonic mappings are either sense-preserving or sense-reversing depending on the conditions $J_f > 0$ and $J_f < 0$ respectively throughout the domain Ω where f is locally univalent. Since $J_f > 0$ if and only if $J_{\overline{f}} < 0$ we will consider sense-preserving mappings in $\mathbb{D}$ throughout all of this work. In this case the analytic part h is locally univalent in $\mathbb{D}$ since $h' \neq 0$, and the second complex dilatation of f, $\omega = g'/h'$, is an analytic function in $\mathbb{D}$ with $|\omega| < 1$, see [3].

Hernández and Martín, [6], defined the harmonic Schwarzian and pre-Schwarzian derivative for locally univalent sense-preserving mappings $f = h + \overline{g}$. Using this definition, they generalized different results regarding analytic functions to the harmonic case [6–9]. The pre-Schwarzian derivative of a sense-preserving harmonic function f coincides with

$$P_f = \frac{h''}{h'} - \frac{\bar{\omega}\omega'}{1-|\omega|^2} = \frac{\partial}{\partial z}\log(J_f). \tag{2}$$

It's easy to see that if f is analytic ($f = h$ and $\omega = 0$) then $P_f = h''/h'$ and recovers the classical definition of this operator. In [8] the authors proved that for any $a \in \mathbb{D}$, $P_{f+\overline{af}} = P_f$. Observe that $f+\overline{af} = h+\overline{a}g+\overline{g+ah}$. In addition, they showed the extension of the Becker's criterion of univalence, this is, if

$$(1-|z|^2)|zP_f(z)| + \frac{|z\omega'(z)|(1-|z|^2)}{1-|\omega(z)|^2} \leq 1, \tag{3}$$

then f is univalent in $\mathbb{D}$. Note that f is analytic, then $\omega \equiv 0$ we recover the classical result by J. Becker [2].

2.2. Affine and linearly invariant families

T. Sheil-Small in [14], gives a generalization of affine and linearly invariant families, introduced by Pommerenke in [12], for sense-preserving harmonic mappings in $\mathbb{D}$. We say that $\mathcal{F}$ is a linearly invariant family of sense-preserving harmonic mappings, $f = h + \overline{g}$, normalized to $h(0) = 0$ and $h'(0) = 1$, if for any $f \in \mathcal{F}$ we have that the *Koebe transform of* f belongs to $\mathcal{F}$, this is, for any $a \in \mathbb{D}$ we have that:

$$K_a(f)(z) = \frac{f \circ \varphi_a(z) - f(a)}{(1-|a|^2)h'(a)} =\in \mathcal{F}, \tag{4}$$

where $\varphi_a : \mathbb{D} \longrightarrow \mathbb{D}$ is the automorphism for any $a \in \mathbb{D}$ defined by

$$\varphi_a(z) = \frac{a+z}{1+\overline{a}z}. \tag{5}$$

The order of this family is defined by

$$\operatorname{Ord}\mathcal{F} = \sup_{f\in\mathcal{F}} |a_2(f)| = \frac{1}{2}\sup_{f\in\mathcal{F}} |h''(0)|.$$

It's not difficult to see that $\operatorname{Ord}\mathcal{F} = \lambda > 1$ when $\mathcal{F}$ is not contained in the analytic space. In addition, we say that $\mathcal{F}$ is affine invariant family if for any $f \in \mathcal{F}$ and $\epsilon \in \mathbb{D}$, we have that

$$A_\varepsilon(f)(z) = \frac{f(z) - \overline{\epsilon f(z)}}{1-\overline{\epsilon}g'(0)} \in \mathcal{F}. \tag{6}$$

The reader can find more details in [14]. The family $\mathcal{S}_H$ of normalized harmonic univalent functions in $\mathbb{D}$ is an affine linearly invariant family where its order is unknown. One of the most important references for this topic is [4]. Another affine and linearly invariant family is given by all stable harmonic convex mappings (SHC) which are the mappings $f = h + \overline{g}$ such that $h + \varepsilon g$ are convex for all $|\varepsilon| = 1$ introduced in [6]. In that paper, the authors showed that if $f = h + \overline{g} \in SHC$ then $h + \varepsilon g$ is convex mapping for all $|\varepsilon| \leq 1$, in particular h is convex mapping, thus its order is equal to 1. It's a well known result that the family of convex mappings, $\mathcal{C}$, satisfies that its order is 1 and is the minimum order of linearly invariant families of holomorphic mappings, moreover if $\operatorname{Ord}\mathcal{F} = 1$, then $\mathcal{F}$ is a subfamily of $\mathcal{C}$. Now, we will show the harmonic version of this result:

Lemma 1. *Let $\mathcal{F}$ be an affine and linearly invariant family of sense-preserving harmonic mappings, then $\operatorname{Ord}\mathcal{F} \geq 1$ and the equality holds if and only if $\mathcal{F}$ is a subfamily of SHC.*

Proof. Since $\mathcal{H}_\mathcal{F} = \{h : h + \overline{g} \in \mathcal{F}\}$ is a linearly invariant family of holomorphic mappings it follows that $\operatorname{Ord}\mathcal{H}_\mathcal{F} \geq 1$, therefore $\operatorname{Ord}\mathcal{F} \geq 1$. If $\operatorname{Ord}\mathcal{F} = 1$ then $\operatorname{Ord}\mathcal{H}_\mathcal{F} = 1$ thus $\mathcal{H}_\mathcal{F} \subseteq \mathcal{C}$. Let $f \in \mathcal{F}$, but $h + ag$ is the analytic part of $f + \overline{af}$ in $\mathcal{F}$ for all $a \in \mathbb{D}$, then $h + ag$ is convex for all $a \in \mathbb{D}$ and $h + ag$ is convex for all $a \in \partial\mathbb{D}$ therefore $f \in SHC$. $\square$

3. Main results

The next proposition is a generalization of Theorem A, given previously, to complex value harmonic sense preserving mappings.

Proposition 1. *Let $f = h + \overline{g}$ be a sense-preserving harmonic function in the unit disc $\mathbb{D}$ with dilatation ω, $|\omega| < 1$, $c \in \mathbb{C}$ such that $|c| \leq 1$ and $c \neq -1$. If*

$$\left|(1-|z|^2)zP_f(z) + c|z|^2\right| + \frac{|z\omega'(z)|(1-|z|^2)}{1-|\omega(z)|^2} \leq 1, \qquad z \in \mathbb{D}, \tag{7}$$

then f is univalent.

Proof. Let $f = h + \overline{g}$ which satisfies (7). We note that

$$\begin{aligned}\left|z\frac{h''}{h'} + \frac{c|z|^2}{1-|z|^2}\right| &= \left|z\frac{h''}{h'} - \frac{z\overline{\omega}\omega'}{1-|\omega|^2} + \frac{c|z|^2}{1-|z|^2} + \frac{z\overline{\omega}\omega'}{1-|\omega|^2}\right| \\ &= \left|zP_f + \frac{c|z|^2}{1-|z|^2} + \frac{z\overline{\omega}\omega'}{1-|\omega|^2}\right| \\ &\leq \left|zP_f + \frac{c|z|^2}{1-|z|^2}\right| + \frac{|z\overline{\omega}\omega'|}{1-|\omega|^2} \\ &\leq \left|zP_f + \frac{c|z|^2}{1-|z|^2}\right| + \frac{|z\omega'|}{1-|\omega|^2} \\ &\leq \frac{1}{1-|z|^2}.\end{aligned}$$

Hence, using Theorem A we obtain that h is univalent. Now, give $a \in \mathbb{D}$, the function $f + \overline{af} = h + \overline{a}g + \overline{g + ah}$, using (5), satisfies that its dilatation is

$$\omega_a = \frac{g_a'}{h_a'} = \frac{g' + ah'}{h' + \bar{a}g'} = \frac{a+\omega}{1+\bar{a}\omega} = \varphi_a \circ \omega.$$

Since $P_{f+\overline{af}} = P_f$ (see equation (2)) and $\varphi_a'(z) = (1-|a|^2)/(1+\overline{a}z)^2$ we can see that

$$\frac{|z\omega_a'|}{1-|\omega_a|^2} = \frac{|z\varphi_a'(\omega)\omega'|}{1-|\varphi_a(\omega)|^2} = \frac{|z\omega'|}{1-|\omega|^2}\cdot\frac{|\varphi_a'(\omega)|(1-|\omega|^2)}{1-|\varphi_a(\omega)|^2} = \frac{|z\omega'|}{1-|\omega|^2},$$

therefore $f + \overline{af}$ satisfies the hypothesis of the theorem, then for any $a \in \mathbb{D}$, function $h + \overline{a}g$ is univalent. By Hurwitz's theorem the functions $h + \lambda g$ are univalent for all $|\lambda| = 1$ which implies that $h + \lambda\overline{g}$ is univalent (see [6]), in particular we have that f is univalent. □

The hyperbolic derivative of $w : \mathbb{D} \to \mathbb{D}$ is given by

$$w^* = \frac{|w'(z)|(1-|z|^2)}{1-|w(z)|^2}, \tag{8}$$

and the norm is defined by $\|\omega^*\| = \sup\{z \in \mathbb{D} : \omega^*(z)\}$.

Lemma 2. *Let $\mathcal{F}$ be an affine and linearly invariant family with* $\operatorname{Ord}\mathcal{F} = \lambda < \infty$, $\lambda > 1$ *and* $f = h + \overline{g} \in \mathcal{F}$ *with dilatation* ω, *then*

$$\left|(1-|z|^2)\frac{h''}{h'}(z) - 2\overline{z}\right| \leq 2\lambda + \|\omega\|\|\omega^*\|.$$

Proof. Let $f = h + \bar{g} \in \mathcal{F}$ with $h(z) = z + a_2 z^2 + \cdots$, $g(z) = b_1 z + b_2 z^2 + \cdots$ and using (6) $A_\varepsilon(f)(z) = H + \overline{G}$ belongs to $\mathcal{F}$ and satisfies that

$$H = \frac{h - \overline{b_1} g}{1 - |b_1|^2}, \qquad G = \frac{g - b_1 h}{1 - |b_1|^2}.$$

As $H'' = (h'' - \overline{b_1} g'')/(1 - |b_1|^2)$, we have that

$$\left|\frac{1}{2}H'(0)\right| = \frac{1}{2}\left|\frac{h''(0) - \overline{b_1} g''(0)}{1 - |b_1|^2}\right| \leq \frac{1}{2}\sup_{F \in \mathcal{F}} |H''(0)| \leq \lambda.$$

Considering $\omega(0) = b_1$, the last inequality and $g''(0) = \omega'(0) + \omega(0)h''(0) = \omega'(0) + 2a_2 b_1$, by Schwarz–Pick lemma we can conclude that

$$|a_2| \leq \lambda + \frac{|b_1 \omega'(0)|}{2(1 - |b_1|^2)}. \tag{9}$$

Now, give $a \in \mathbb{D}$, the function $K_a(f)$ defined by (4) can be expressed as

$$K_a(f)(z) = H_a(z) + \overline{G_a(z)} = H(\varphi_a(z)) + \overline{G(\varphi_a(z))} =$$

$$= \sum_{n=1}^{\infty} c_n(a) z^n + \overline{\sum_{n=1}^{\infty} d_n(a) z^n} \in \mathcal{F}.$$

In addition, noting that

$$H_a'(z) = \frac{h'\left(\dfrac{a+z}{1+\overline{a}z}\right)\left(\dfrac{1-|a|^2}{(1+\overline{a}z)^2}\right)}{(1-|a|^2)h'(a)} = \frac{1}{h'(a)}\left[h'\left(\frac{a+z}{1+\overline{a}z}\right)(1+\overline{a}z)^{-2}\right],$$

and

$$H_a''(z) = \frac{1}{h'(a)}\left[h''\left(\frac{a+z}{1+\overline{a}z}\right)\frac{1-|a|^2}{(1+\overline{a}z)^2}\frac{1}{(1+\overline{a}z)^2} + \frac{-2\overline{a}}{1+\overline{a}z}h'\left(\frac{a+z}{1+\overline{a}z}\right)\right],$$

we obtain

$$c_2(a) = \frac{H_a''(0)}{2!} = \frac{1}{2}\left(\frac{h''(a)}{h'(a)}(1-|a|^2) - 2\overline{a}\right).$$

Finally, replacing the last equality in (9) we have

$$\left|\frac{h''}{h'}(1-|a|^2) - 2\overline{a}\right| \leq 2\lambda + \frac{|d_1 \omega_{F_a}'(0)|}{1 - |d_1|^2}$$

$$= 2\lambda + |\omega_{F_a}(0)|\frac{|\omega'(a)(1-|a|^2)|}{1 - |\omega(a)|^2}$$

$$\leq 2\lambda + \sup_{a\in\mathbb{D}} |\omega(a)| \frac{|\omega'(a)(1-|a|^2)|}{1-|\omega(a)|^2}$$
$$\leq 2\lambda + \|\omega\|\|\omega^*\|. \quad \square$$

3.1. Extension of f_α for harmonic mappings

Let $f = h + \overline{g}$ be a locally univalent sense-preserving mapping with dilatation ω. For any complex value α and $z \in \mathbb{D}$, we define the function f_α as

$$f_\alpha(z) = h_\alpha(z) + \overline{(h+g)_\alpha(z) - h_\alpha(z)}. \tag{10}$$

Since $h' \neq 0$, h_α is well defined. In addition, if $h' + g' = 0$, then $|g'/h'| = 1$, which cannot be, that is, $(h+g)_\alpha$ is well defined, thus f_α is too. Assuming that f is normalized to $h'(0) = 1$ and $g'(0) = 0$, f_α satisfies that f_0 is the identity while $f_1 = f$. The dilatation w_α of f_α defined in (10) is given by $w_\alpha = (1+w)^\alpha - 1$. Indeed,

$$w_\alpha = \frac{(h'+g')^\alpha - (h')^\alpha}{(h')^\alpha} = \left(1 + \frac{g'}{h'}\right)^\alpha - 1 = (1+w)^\alpha - 1. \tag{11}$$

Since $|e^z - 1| \leq e^{|z|} - 1$, for all $z \in \mathbb{C}$ we have that $|\omega_\alpha| < 1$. Let $\alpha = a + ib$ be a complex number in $\mathbb{D}$ and $\theta = Arg\{1+\omega\}$. In this case therefore $|\omega_\alpha| < 1$ if and only if $|1+\omega|^{2\alpha} < 2Re\{(1+\omega)^\alpha\}$ which is equivalent to

$$\frac{|1+\omega|^a}{e^{b\theta}} < 2\cos(b\log|1+\omega| + a\theta).$$

Considering $\theta = 0$ we have that

$$|1+\omega|^a < 2\cos(b\log|1+\omega|), \tag{12}$$

since when $|1+\omega|$ tends to 0, (for instance when w is the Identity) the right side of equation (12) can be negative for some $z \in \mathbb{D}$, which is a contradiction up until $b = 0$, therefore we need to assume that α is a real value.

On the other hand, it's not difficult to see that in the definition of f_α by equation (10) the co-analytic part can be replaced by $(g+\mu h)_\alpha - (\mu h)_\alpha$ where μ is any complex value in $\partial\mathbb{D}$. The corresponding dilatation is $\mu^\alpha((\overline{\mu}\omega+1)^\alpha - 1)$ which is a rotation of $(\overline{\mu}\omega)_\alpha$ defined by (11). Moreover the next results can be proved for those functions.

Lemma 3. *$\|\omega_\alpha^*\| \leq 2\alpha\|\omega^*\|$ with $\alpha \in (0,1)$.*

Proof. We define $\varphi_\alpha(z) = (1+z)^\alpha - 1$ therefore $\omega_\alpha = \varphi_\alpha \circ \omega$. Thus,

$$\omega_\alpha^* = \frac{|\varphi_\alpha'(\omega)||\omega'|(1-|z|^2)}{1-|\varphi_\alpha(\omega)|^2} = \frac{|\varphi_\alpha'(\omega)|(1-|\omega|^2)}{1-|\varphi_\alpha(\omega)|^2} \cdot \frac{|\omega'|(1-|z|^2)}{1-|\omega|^2},$$

then $\|\omega_\alpha^*\| \leq \|\varphi_\alpha^*\|\|\omega^*\|$. In consequence, it suffices to prove that $\|\varphi_\alpha^*\| \leq 2\alpha$. In fact, a direct calculation shows that

$$\varphi_\alpha^* = \frac{|\varphi_\alpha'(z)|(1-|z|^2)}{1-|\varphi_\alpha(z)|^2} = \frac{\alpha|1+z|^{-1}(1-|z|^2)}{2Re\left\{((1+z)/(|1+z|))^\alpha\right\} - |1+z|^\alpha}.$$

For fixed $z \in \mathbb{D}$, we consider $1+z=\rho e^{i\theta}$ and $\delta(\alpha)=2Re\left\{((1+z)/(|1+z|))^{\alpha}\right\}-|1+z|^{\alpha}=2\cos(\theta\alpha)-\rho^{\alpha}$. Moreover, as $\delta'(\alpha)=-2\sin(\theta\alpha)\theta-\rho^{\alpha}\ln(\rho)$ then $\delta'(0)=-\ln(\rho)$, and since $\theta\in(-\pi/2,\pi/2)$, we have that $\delta''(\alpha)=-2\cos(\theta\alpha)\theta^2-\rho^{\alpha}\ln^2(\rho)<0$ hence δ' is decreasing. If $\rho\geq 1$ then $\delta'(0)\leq 0$ therefore $\delta'(\alpha)\leq\delta'(0)\leq 0$ and in consequence δ is decreasing, which implies that $\delta(1)\leq\delta(\alpha)\leq\delta(0)=1$.

In the case that $\rho<1$ we have $\delta'(0)>0$. If there exists $\alpha_0\in(0,1)$ since δ' is decreasing it follows that this critical point is unique and is a maximum, thus $\delta(\alpha)\geq\min\{\delta(0),\delta(1)\}$, or $\delta'\neq 0$ hence δ is an increasing function therefore $\delta(\alpha)\geq\delta(0)$. Thus, for $\alpha\in(0,1)$ we have

$$\delta(\alpha)\geq\min\{\delta(0),\delta(1)\}. \tag{13}$$

Now, we note

- If $\delta(0)\leq\delta(1)$, using (13) we obtain

$$\begin{aligned}\frac{\alpha|1+z|^{\alpha-1}(1-|z|^2)}{1-|(1+z)^{\alpha}-1|}&\leq\frac{\alpha|1+z|^{-1}(1-|z|^2)}{\delta(0)}\\&=\alpha|1+z|^{-1}(1-|z|^2)\\&\leq\alpha(1+|z|).\end{aligned}$$

- If $\delta(1)\leq\delta(0)$, using (13) we obtain

$$\begin{aligned}\frac{\alpha|1+z|^{\alpha-1}(1-|z|^2)}{1-|(1+z)^{\alpha}-1|}&\leq\frac{\alpha|1+z|^{-1}(1-|z|^2)}{\delta(1)}\\&=\frac{\alpha|1+z|^{-1}(1-|z|^2)}{2Re\left\{\dfrac{1+z}{|1+z|}\right\}-|1+z|}\\&=\frac{\alpha(1-|z|^2)}{2Re\{(1+z)\}-|1+z|^2}\\&=\alpha(1+|z|).\end{aligned}$$

Therefore, for $\alpha\in(0,1)$ and fixed $z\in\mathbb{D}$ it follows that $\varphi_{\alpha}^{*}\leq\alpha(1+|z|)$, then the proof is complete. □

Theorem 4. *Let $\mathcal{F}$ be an affine and linearly invariant family with* $\operatorname{Ord}\mathcal{F}=\lambda<\infty$ *and $f=h+\overline{g}\in\mathcal{F}$ with dilatation ω. Then the function $f_{\alpha}(z)$ defined by (10) will be univalent when*

$$\alpha\leq\frac{1}{2\lambda+(\|\omega\|_{\infty}+4)\|\omega^{*}\|}.$$

Proof. Let $z\in\mathbb{D}$. We note that

$$\begin{aligned}&\left|z(1-|z|^2)P_{f_{\alpha}}+c|z|^2\right|+\frac{|z\omega_{\alpha}'|}{1-|\omega_{\alpha}|^2}(1-|z|^2)=\\&=\left|(\alpha\frac{h''}{h'}-\frac{\overline{\omega_{\alpha}}\omega_{\alpha}'}{1-|\omega_{\alpha}|^2})z(1-|z|^2)+c|z|^2\right|+\frac{|z\omega_{\alpha}'|}{1-|\omega_{\alpha}|^2}(1-|z|^2)\\&\leq\left|\alpha z\frac{h''}{h'}(1-|z|^2)+c|z|^2\right|+2|\omega_{\alpha}^{*}|\\&\leq\alpha\left|\frac{h''}{h'}(1-|z|^2)-2\overline{z}\right|+|(2\alpha+c)\overline{z}|+2|\omega_{\alpha}^{*}|.\end{aligned}$$

Hence, using Lemma 3 and Lemma 2 we obtain

$$\begin{aligned}\left|z(1-|z|^2)P_{f_\alpha}+c|z|^2\right| + \frac{|z\omega_\alpha'|}{1-|\omega_\alpha|^2}(1-|z|^2) \leq \\ \leq \alpha\left(2\lambda+\|\omega\|_\infty\|\omega^*\|\right)+|2\alpha+c|+4\alpha\|\omega^*\| \\ = \alpha(2\lambda+(\|\omega\|_\infty+4)\|\omega^*\|)+|2\alpha+c|.\end{aligned}$$

As $\lambda > 1$, the last inequality shows that $\alpha < 1/2$, considering $c = -2\alpha \neq -1$ and using Proposition 1, we get f_α is univalent for all α such that

$$\alpha \leq \frac{1}{2\lambda+(\|\omega\|_\infty+4)\|\omega^*\|}. \quad \square$$

Acknowledgments

This work was supported by Grant FONDECYT # 1150284.

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