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Feet in orthogonal-Buekenhout–Metz unitals

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Abstract: Given an orthogonal-Buekenhout–Metz unital $U_{\alpha,\beta}$, embedded in $\mathrm{PG}(2, q^2)$, and a point $P \notin U_{\alpha,\beta}$, we study the set $\tau_P(U_{\alpha,\beta})$ of feet of P in $U_{\alpha,\beta}$. We characterize geometrically each of these sets as either $q + 1$ collinear points or as $q + 1$ points partitioned into two arcs. Other results about the geometry of these sets are also given.

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1 Preliminaries

Most definitions and theorems in this section may be found in [5]. We direct the reader to this source for more information, and details, about projective planes.

Let $\mathrm{GF}(q^2)$ be the field with q^2 elements, where $q = p^n$ with p prime (we always consider p to be odd in this article), and $n \in \mathbb{N}$. Throughout this article we use

$$\mathrm{GF}(q^2) = \{a + \varepsilon b;\, a, b \in \mathrm{GF}(q),\ \text{and } \varepsilon^2 = w \in \mathrm{GF}(q)\},$$

and we write $\overline{x} = x^q$, $T(x) = x + \overline{x}$ and $N(x) = x\overline{x}$, for all $x \in \mathrm{GF}(q^2)$.

We let V be a 3-dimensional vector space over $\mathrm{GF}(q^2)$ and we consider the projective plane $\Pi = \mathrm{PG}(2, q^2)$: its points are the 1-dimensional subspaces of V and its lines are the 2-dimensional subspaces of V. A point P of Π is denoted by $P = [a, b, c]$, where (a, b, c) is a vector generating the subspace defining P. If l is a line of Π then it is denoted by

$$l = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [x, y, z]^t,$$

where (x, y, z) is a vector that is orthogonal (using the standard dot product) to the 2-dimensional subspace defining l. The incidence in Π is given by natural set-theoretic containment. Thus,

$$P \in l \quad \Longleftrightarrow \quad \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \Longleftrightarrow \quad ax + by + cz = 0.$$

It is known that Π is a projective plane of order q^2. Hence every line of Π contains exactly $q^2 + 1$ points, every point of Π is on exactly $q^2 + 1$ lines, and the number of points, and the number of lines, in Π is $q^4 + q^2 + 1$.

Definition 1. *A* unital *in Π is a set U of $q^3 + 1$ points of Π such that every line of Π intersects U in exactly* 1 *or* $q + 1$ *points. Lines of Π will be called* tangent *or* secant *to U depending on whether they intersect U in* 1 *or* $q + 1$ *points, respectively.*

Remark 1. Unitals may be defined in a much more general way but here we focus on unitals embedded in Π. So, our definition has been written with this purpose in mind. We refer the reader to [3] for a detailed exposition about unitals and for the concepts we use in this article that we may fail to explain in detail.

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Two standard examples of unitals are

1. The set $H = \{[x, y, z] \in \Pi; N(x) + N(y) + N(z) = 0\}$ of absolute points of a non-degenerate unitary polarity of Π is a unital in Π, called *classical*.
2. Buekenhout [4] proved that, for $\alpha, \beta \in GF(q^2)$ such that $4N(\alpha) + (\overline{\beta} - \beta)^2$ is a non-square in $GF(q)$, the set
$$U_{\alpha,\beta} = \{[x, \alpha x^2 + \beta N(x) + r, 1]; x \in GF(q^2), r \in GF(q)\} \cup \{P_\infty\}$$
is a unital (said to be an *orthogonal-Buekenhout–Metz unital*) in Π, where $P_\infty = [0, 1, 0]$. Moreover, $\alpha = 0$ if and only if the unital $U_{\alpha,\beta}$ is classical, and $\beta = \overline{\beta}$ if and only if the unital $U_{\alpha,\beta}$ is a union of conics (see [2] or [8], and [6]).

From now on we focus on non-classical orthogonal-Buekenhout–Metz unitals $U_{\alpha,\beta}$. So, for the rest of this article we assume $\alpha \neq 0$.

Elementary counting shows that if $U_{\alpha,\beta}$ is an orthogonal-Buekenhout–Metz unital in Π and $P \in U_{\alpha,\beta}$ then there is exactly one tangent line to $U_{\alpha,\beta}$ through P and there are exactly q^2 secant lines to $U_{\alpha,\beta}$ through P. Similarly, if $P \notin U_{\alpha,\beta}$ then there are exactly $q + 1$ lines tangent to $U_{\alpha,\beta}$ through P and there are exactly $q^2 - q$ secant lines to $U_{\alpha,\beta}$ through P.

Definition 2. *Let $U_{\alpha,\beta}$ be an orthogonal-Buekenhout–Metz unital and P a point not in $U_{\alpha,\beta}$. Each of the $q + 1$ points of $U_{\alpha,\beta}$ that are on a tangent line to $U_{\alpha,\beta}$ through P is said to be a foot of P. We denote the set of feet of P by $\tau_P(U_{\alpha,\beta})$ and we call it the* pedal *of P.*

It is known that $\tau_P(U_{\alpha,\beta})$ has the following properties:

1. $\tau_P(U_{\alpha,\beta})$ is contained in a line of Π for all $P \in \Pi \setminus U_{\alpha,\beta}$ if, and only if, $U_{\alpha,\beta}$ is classical; see Thas [10]. The conditions for this result have been relaxed after Thas's work, see [1].
2. If $U_{\alpha,\beta}$ is non-classical then $\tau_P(U_{\alpha,\beta})$ is contained in a line of Π if and only if $P \in \ell_\infty$. Note that $\ell_\infty \cap U_{\alpha,\beta} = P_\infty$ (see e.g. [3]). This implies that every line, different from ℓ_∞, through P_∞ contains a pedal.

Finally, there is a group $G \leq P\Gamma L(3, q^2)$ leaving $U_{\alpha,\beta}$ invariant and fixing P_∞ such that

1. G is transitive on the set of points of $U_{\alpha,\beta} \setminus \ell_\infty$.
2. G is transitive on the points of $l_\infty \setminus \{P_\infty\}$.
3. G has either one or two orbits on the points of $\Pi \setminus (U_{\alpha,\beta} \cup l_\infty)$. Moreover, these orbits are those of $P_\lambda = [0, \lambda\varepsilon, 1]$, with $\lambda = 1$ or $\lambda = w = \varepsilon^2$.

2 Intersections of lines and pedals

Our objective is to find geometric properties that describe the pedals of points $P \notin \ell_\infty$ in unitals $U_{\alpha,\beta}$ where $\alpha \neq 0$. In particular, we care about how lines of Π intersect these sets. Not much is known about pedals in non-classical unitals, albeit the work by Krčadinac and Smoljak is pertinent; in [9] they study all possible configurations for pedals in unitals that are embedded in (not necessarily Desarguesian) projective planes of order 9 and 16.

Because of the results listed above about the group G we now study $\tau_P(U_{\alpha,\beta})$ only for $P = P_\lambda = [0, \lambda\varepsilon, 1]$, with $\lambda = 1$ or $\lambda = w$; this is justified by the following lemma.

Lemma 1. *If $\sigma \in G$, $A, C \in \tau_P(U_{\alpha,\beta})$, $\sigma(A) = B$, $\sigma(C) = D$, $Q = \sigma(P)$, then $\sigma(\tau_P(U_{\alpha,\beta})) = \tau_Q(U_{\alpha,\beta})$ and*
$$|AC \cap \tau_P(U_{\alpha,\beta})| = |BD \cap \tau_Q(U_{\alpha,\beta})|.$$

Proof. It is easy to see that σ preserves the number of points of intersection between lines and $U_{\alpha,\beta}$, and so σ maps tangent lines into tangent lines. It follows that $\sigma(\tau_P(U_{\alpha,\beta})) = \tau_Q(U_{\alpha,\beta})$.

Note that $\sigma(AC) = BD$ and that $B, D \in \tau_Q(U_{\alpha,\beta})$. So, if we repeat this argument with A and any other point $E \in AC \cap \tau_P(U_{\alpha,\beta})$ we would get another point in $BD \cap \tau_Q(U_{\alpha,\beta})$. Hence, since σ is injective we get one direction of the desired inequality. We obtain the other direction by repeating the argument using σ^{-1} instead of σ. □

We first look at the lines through P_∞. It was mentioned earlier that there is a bijection between the set of pedals containing P_∞ and the set of lines, different from ℓ_∞, through this point. We now want to look at how these q^2 lines intersect $\tau_{P_\lambda}(U_{\alpha,\beta})$. The following remark gives enough information for us to address this issue in the subsequent lemma.

Remark 2. Two distinct pedals can intersect in at most one point, as for every point $A \in \tau_P(U_{\alpha,\beta}) \cap \tau_Q(U_{\alpha,\beta})$ we always get that A, P and Q are collinear. Moreover, two distinct pedals intersect if and only if they are the pedals of two points on a line tangent to $U_{\alpha,\beta}$; their intersection is the tangency point.

The following lemma is immediate.

Lemma 2. *Let $\ell \neq \ell_\infty$ be a line such that $P_\infty \in \ell$. Then ℓ is either tangent or exterior to all the pedals not contained in ℓ.*

The generalization of this lemma to lines intersecting pedals of points not on ℓ_∞ is not true (see Section 3). However, Lemma 2 implies that the line $\ell \neq \ell_\infty$ can be partitioned into singletons, all of them in distinct pedals. We are able to prove that result for all other pedals as well.

Lemma 3. *Let ℓ be a line that is not tangent to $U_{\alpha,\beta}$. Then there are $q+1$ distinct pedals intersecting ℓ in singletons, creating a partition of the points in $\ell \cap U_{\alpha,\beta}$.*

Proof. The case when $P_\infty \in \ell$ follows immediately from Lemma 2. For $P_\infty \notin \ell$ we use that every point in $U_{\alpha,\beta}$ is in q^2 pedals, and ℓ contains $q+1$ points of $U_{\alpha,\beta}$, thus for each point in $\ell \cap U_{\alpha,\beta}$ there are at least $q^2-(q+1)$ pedals containing no other point of $\ell \cap U_{\alpha,\beta}$. Hence, using that $q \geq 3$ implies that $q^2-(q+1) \geq q+1$, we can choose the pedals to create the desired partition. □

Now we consider the intersections of lines, not through P_∞, with pedals of points not on ℓ_∞.

3 Lines not containing P_∞

In this section we study lines that do not pass through P_∞. We consider the orthogonal-Buekenhout–Metz unital

$$U_{\alpha,\beta} = \{[x, \alpha x^2 + \beta N(x) + r, 1];\, x \in \mathrm{GF}(q^2),\, r \in \mathrm{GF}(q)\} \cup \{P_\infty\}.$$

in Π. The tangent line to $U_{\alpha,\beta}$ through $[x, \alpha x^2 + \beta N(x) + r, 1]$ is $[-2\alpha x + (\overline{\beta} - \beta)\overline{x}, 1, \alpha x^2 - \overline{\beta} N(x) - r]^t$. In order to study $\tau_{P_\lambda}(U_{\alpha,\beta})$ we need to determine all $x \in \mathrm{GF}(q^2)$ and $r \in \mathrm{GF}(q)$ such that

$$[0, \lambda\varepsilon, 1] \in [-2\alpha x + (\overline{\beta} - \beta)\overline{x}, 1, \alpha x^2 - \overline{\beta} N(x) - r]^t$$

which means that $\lambda\varepsilon + \alpha x^2 - \overline{\beta} N(x) - r = 0$, or equivalently $r = \lambda\varepsilon + \alpha x^2 - \overline{\beta} N(x)$. Since $r \in \mathrm{GF}(q)$ we get $\overline{r} = r$. Hence $\overline{\lambda\varepsilon + \alpha x^2 - \overline{\beta} N(x)} = \lambda\varepsilon + \alpha x^2 - \overline{\beta} N(x)$ and thus

$$2\lambda\varepsilon + \alpha x^2 - \overline{\alpha}\,\overline{x}^2 + (\beta - \overline{\beta}) N(x) = 0. \tag{1}$$

We let

$$M_{\alpha,\beta} = \begin{bmatrix} \alpha & \frac{1}{2}(\beta - \overline{\beta}) \\ \frac{1}{2}(\beta - \overline{\beta}) & -\overline{\alpha} \end{bmatrix},$$

and note that

$$2\lambda\varepsilon + \alpha x^2 - \overline{\alpha}\,\overline{x}^2 + (\beta - \overline{\beta}) N(x) = 0 \quad \Longleftrightarrow \quad 2\lambda\varepsilon + \begin{bmatrix} x & \overline{x} \end{bmatrix} M_{\alpha,\beta} \begin{bmatrix} x \\ \overline{x} \end{bmatrix} = 0.$$

Hence $\tau_{P_\lambda}(U_{\alpha,\beta})$ is the set of all the points of the form $[x, 2\alpha x^2 + (\beta - \overline{\beta})N(x) + \lambda\varepsilon, 1]$, where $x \in \mathrm{GF}(q^2)$ and

$$2\lambda\varepsilon + \begin{bmatrix} x & \overline{x} \end{bmatrix} M_{\alpha,\beta} \begin{bmatrix} x \\ \overline{x} \end{bmatrix} = 0.$$

We can now use Equation (1) to find a different way to represent points in $\tau_{P_\lambda}(U_{\alpha,\beta})$. Note that

$$T(\alpha x^2) - \lambda\varepsilon = \alpha x^2 + \overline{\alpha}\,\overline{x}^2 - \lambda\varepsilon = \alpha x^2 + (2\lambda\varepsilon + \alpha x^2 + (\beta - \overline{\beta})N(x)) - \lambda\varepsilon = 2\alpha x^2 + \lambda\varepsilon + (\beta - \overline{\beta})N(x).$$

Hence, letting $Q_x = [x, T(\alpha x^2) - \lambda\varepsilon, 1]$ we get

$$\tau_{P_\lambda}(U_{\alpha,\beta}) = \left\{Q_x; x \in \mathrm{GF}(q^2), 2\lambda\varepsilon + \begin{bmatrix} x & \overline{x} \end{bmatrix} M_{\alpha,\beta} \begin{bmatrix} x \\ \overline{x} \end{bmatrix} = 0\right\}.$$

Remark 3. Equation (1) can be re-written as $2\lambda\varepsilon + 2\varepsilon\,\mathrm{Im}(\alpha x^2) + (\beta - \overline{\beta})N(x) = 0$. It follows that for two points $Q_x, Q_y \in \tau_{P_\lambda}(U_{\alpha,\beta})$ we get $N(x) = N(y)$ if and only if $\mathrm{Im}(\alpha x^2) = \mathrm{Im}(\alpha y^2)$.

We now introduce some notation. Let

$$T_\lambda = \{x \in \mathrm{GF}(q^2); Q_x \in \tau_{P_\lambda}(U_{\alpha,\beta})\} = \left\{x \in \mathrm{GF}(q^2); 2\lambda\varepsilon + \begin{bmatrix} x & \overline{x} \end{bmatrix} M_{\alpha,\beta} \begin{bmatrix} x \\ \overline{x} \end{bmatrix} = 0\right\}.$$

It is easy to see that $x \in T_\lambda$ if and only if $-x \in T_\lambda$. Moreover, if $x, -x \in T_\lambda$ then they have the same value of r associated to them (in the representation of Q_x and Q_{-x} as points in $U_{\alpha,\beta}$).

Lemma 4. *Let $l_{x,-x}$ be the line through Q_x and Q_{-x}, where $\pm x \in T_\lambda$. If $Q_y \in l_{x,-x}$ for some $y \in T_\lambda$, then $Q_{-y} \in l_{x,-x}$.*

Proof. The line passing through Q_x and Q_{-x} is given by $l_{x,-x} = [0, -1, T(\alpha x^2) - \lambda\varepsilon]^t$. If $Q_y \in l_{x,-x}$ then

$$[y, T(\alpha y^2) - \lambda\varepsilon, 1] \begin{bmatrix} 0 \\ -1 \\ T(\alpha x^2) - \lambda\varepsilon \end{bmatrix} = 0,$$

which can be simplified to $T(\alpha x^2) - T(\alpha y^2) = 0$. On the other hand,

$$[-y, T(\alpha(-y)^2) - \lambda\varepsilon, 1] \begin{bmatrix} 0 \\ -1 \\ T(\alpha x^2) - \lambda\varepsilon \end{bmatrix} = T(\alpha x^2) - T(\alpha y^2),$$

which is equal to zero. Hence $Q_{-y} \in l_{x,-x}$. □

Remark 4. All lines of the form $l_{x,-x}$ pass through $[1, 0, 0]$.

We want to learn about the conditions under which the line through Q_x and Q_y, for $x, y \in T_\lambda$, contains more points of $U_{\alpha,\beta}$ besides Q_x and Q_y.

Lemma 5. *Let $l_{x,y}$ be the line through Q_x and Q_y, for $x \neq y$ in T_λ. If $Q_z \in l_{x,y}$, $z \in T_\lambda$, and $T(\alpha x^2) = T(\alpha y^2)$, then $Q_{-z} \in l_{x,y}$.*

Proof. The line $l_{x,y}$ is given by $l_{x,y} = [T(\alpha x^2) - T(\alpha y^2), y - x, xT(\alpha y^2) - yT(\alpha x^2) + (y - x)\lambda\varepsilon]^t$. But $T(\alpha x^2) = T(\alpha y^2)$ and $x \neq y$, so $l_{x,y} = [0, -1, T(\alpha x^2) - \lambda\varepsilon]^t$. If $Q_z \in l_{x,y}$ then, after routine simplifications, we get $-T(\alpha z^2) + T(\alpha x^2) = 0$. It follows that $l_{x,y} = [0, -1, T(\alpha z^2) - \lambda\varepsilon]^t$, which is the line $l_{z,-z}$. □

Theorem 1. *Let $U_{\alpha,\beta}$ be an orthogonal-Buekenhout–Metz unital with $\alpha \neq 0$. Let Q_x and Q_y be two distinct points in $\tau_{P_\lambda}(U_{\alpha,\beta})$, and let $l_{x,y}$ be the line through them. If $T(\alpha x^2) \neq T(\alpha y^2)$ then $l_{x,y} \cap \tau_{P_\lambda}(U_{\alpha,\beta}) = \{Q_x, Q_y\}$.*

Proof. Suppose that $l_{x,y}$ contains a point $Q_z \in \tau_{P_\lambda}(U_{\alpha,\beta})$, different from Q_x and Q_y. Then there exists a $\mu \in \mathrm{GF}(q^2) \setminus \{0\}$ such that $Q_z = Q_x + \mu Q_y$. Note that $1 + \mu \neq 0$, otherwise $Q_x + \mu Q_y \notin U_{\alpha,\beta}$. Thus

$$[z, T(\alpha z^2) - \lambda\varepsilon, 1] = Q_x + \mu Q_y = \left[\frac{x + \mu y}{1 + \mu}, \frac{T(\alpha x^2) + \mu T(\alpha y^2)}{1 + \mu} - \lambda\varepsilon, 1\right].$$

This implies that

$$T\Big(\alpha\Big(\frac{x+\mu y}{1+\mu}\Big)^2\Big) = \frac{T(\alpha x^2)+\mu T(\alpha y^2)}{1+\mu},$$

which we re-write as:

$$(1+\mu)T\Big(\alpha\Big(\frac{x+\mu y}{1+\mu}\Big)^2\Big) = T(\alpha x^2)+\mu T(\alpha y^2).$$

It follows that

$$T\Big(\alpha\Big(\frac{x+\mu y}{1+\mu}\Big)^2 - \alpha x^2\Big) = \mu T\Big(\alpha y^2 - \alpha\Big(\frac{x+\mu y}{1+\mu}\Big)^2\Big). \tag{2}$$

If

$$T\Big(\alpha\Big(\frac{x+\mu y}{1+\mu}\Big)^2 - \alpha x^2\Big) = T\Big(\alpha y^2 - \alpha\Big(\frac{x+\mu y}{1+\mu}\Big)^2\Big) = 0$$

then

$$T\Big(\alpha\Big(\frac{x+\mu y}{1+\mu}\Big)^2\Big) = T(\alpha x^2) \quad \text{and} \quad T(\alpha y^2) = T\Big(\alpha\Big(\frac{x+\mu y}{1+\mu}\Big)^2\Big)$$

and thus $T(\alpha x^2) = T(\alpha y^2)$, which contradicts our hypothesis. It follows that Equation (2) implies $\mu \in \mathrm{GF}(q)$, and thus $1+\mu \in \mathrm{GF}(q)$.

Since $z \in T_\lambda$ and $z = \frac{x+\mu y}{1+\mu}$ we get

$$2\lambda\varepsilon + \alpha\Big(\frac{x+\mu y}{1+\mu}\Big)^2 - \overline{\alpha}\,\overline{\Big(\frac{x+\mu y}{1+\mu}\Big)}^2 + (\beta-\overline{\beta})\Big(\frac{x+\mu y}{1+\mu}\Big)^{q+1} = 0,$$

which is equivalent to $2\lambda\varepsilon(1+\mu)^2+\alpha(x+\mu y)^2-\overline{\alpha}\,\overline{(x+\mu y)}^2+(\beta-\overline{\beta})(x+\mu y)^{q+1} = 0$. After some simplifications we get

$$\begin{aligned}(2\lambda\varepsilon + \alpha x^2 - \overline{\alpha}\,\overline{x}^2 + (\beta-\overline{\beta})N(x)) + \mu^2(2\lambda\varepsilon + \alpha y^2 - \overline{\alpha}\,\overline{y}^2 + (\beta-\overline{\beta})N(y))&\\ + \mu(4\lambda\varepsilon + 2\alpha xy - 2\overline{\alpha}\,\overline{x}\,\overline{y} + (\beta-\overline{\beta})(\overline{y}x+\overline{x}y)) = 0.&\end{aligned} \tag{3}$$

Since $x, y \in T_\lambda$, we know that $2\lambda\varepsilon + \alpha x^2 - \overline{\alpha}\,\overline{x}^2 + (\beta-\overline{\beta})N(x) = 2\lambda\varepsilon + \alpha y^2 - \overline{\alpha}\,\overline{y}^2 + (\beta-\overline{\beta})N(y) = 0$ and thus, Equation (3) implies $4\lambda\varepsilon + 2\alpha xy - 2\overline{\alpha}\,\overline{x}\,\overline{y} + (\beta-\overline{\beta})(\overline{y}x+\overline{x}y) = 0$, as $\mu \neq 0$.

Since the expression above is zero, independent of the value of μ, every point $Q_x+\mu Q_y$, for $\mu \in \mathrm{GF}(q)\backslash\{-1\}$, is in $\tau_{P_\lambda}(U_{\alpha,\beta})$. Hence there are exactly q feet of P_λ in $U_{\alpha,\beta}$ lying on the same line ℓ. But since $x \in T_\lambda$ implies $-x \in T_\lambda$, there is a $y \in T_\lambda$ such that $\ell = l_{y,-y}$. However, by Lemma 4 the number of points on $\ell \cap \tau_{P_\lambda}(U_{\alpha,\beta})$ must be even, which contradicts q being odd. □

As of now we have that a secant line cannot intersect $\tau_{P_\lambda}(U_{\alpha,\beta})$ in 3 points, and that if the line is not of the form $l_{x,-x}$ then this intersection contains at most 2 points. Next we obtain a bound for the maximum number of collinear points on $\tau_{P_\lambda}(U_{\alpha,\beta})$.

Theorem 2. *Let $U_{\alpha,\beta}$ be an orthogonal-Buekenhout–Metz unital with $\alpha \neq 0$ and let ℓ be the line through two distinct points in $\tau_{P_\lambda}(U_{\alpha,\beta})$. Then ℓ intersects $\tau_{P_\lambda}(U_{\alpha,\beta})$ in at most four points.*

Proof. Because of the previous results, the only case to consider is when ℓ is of the form $l_{x,-x}$, for some $x \in T_\lambda$. Hence, the conditions for $Q_z \in \ell \cap \tau_{P_\lambda}(U_{\alpha,\beta})$ are

$$T(\alpha x^2) - T(\alpha z^2) = 0 \qquad \text{and} \qquad 2\lambda\varepsilon + 2\varepsilon\,\mathrm{Im}(\alpha z^2) + (\beta-\overline{\beta})N(z) = 0. \tag{4}$$

We let $z = z_1 + z_2\varepsilon$, $\alpha = \alpha_1 + \alpha_2\varepsilon$, and $\beta = \beta_1 + \beta_2\varepsilon$, where $z_1, z_2, \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathrm{GF}(q)$. Using these variables we can re-write Equations (4) as the system

$$\begin{aligned}2^{-1}T(\alpha x^2) &= \alpha_1 z_1^2 + \alpha_1 w z_2^2 + 2\alpha_2 w z_1 z_2\\ -\lambda &= (\alpha_2+\beta_2)z_1^2 + (\alpha_2 - w\beta_2)z_2^2 + 2\alpha_1 z_1 z_2\end{aligned} \tag{5}$$

We define the following elements in GF(q).

$$A = \alpha_1 \quad B = \alpha_1 w \quad C = 2\alpha_2 w \quad D = -2^{-1}T(\alpha x^2)$$
$$E = \alpha_2 + \beta_2 \quad F = \alpha_2 - w\beta_2 \quad G = 2\alpha_1 \quad H = \lambda.$$

These elements allow to re-write System (5) as the following system of equations with coefficients in GF(q):

$$Az_1^2 + Bz_2^2 + Cz_1z_2 + D = 0 \quad \text{and} \quad Ez_1^2 + Fz_2^2 + Gz_1z_2 + H = 0 \tag{6}$$

If these equations have a common linear factor then we get three linear equations equal to zero, which is three intersecting lines. This yields one solution or a triplet of coinciding lines, which would imply that each equation in System (6) is a multiple of the other. But we know that the equation $T(\alpha x^2) - T(\alpha z^2) = 0$ has exactly $2(q+1)$ solutions, implying that System (6) has $2(q+1)$ solutions, which is more than the maximum number of points on $\ell \cap \tau_{P_\lambda}(U_{\alpha,\beta})$, which is $q+1$.

If the equations in System (6) do not have common factors we can use Bézout's Theorem for the curves given by

$$p(z_1, z_2) = Az_1^2 + Bz_2^2 + Cz_1z_2 + D \quad \text{and} \quad q(z_1, z_2) = Ez_1^2 + Fz_2^2 + Gz_1z_2 + H$$

and since both are polynomials in two variables with coefficients in GF(q), and both have degree two, we get that System (5) has at most $4 = \deg(p) \cdot \deg(q)$ solutions. □

We summarize our results on the size of the intersections between lines and pedals in the following theorem.

Theorem 3. *Let $P \notin \ell_\infty$ and $\alpha \neq 0$. Then,*

1. *lines in Π intersect $\tau_P(U_{\alpha,\beta})$ in exactly 0, 1, 2, or 4 points.*
2. *the points of $\tau_P(U_{\alpha,\beta})$ may be partitioned into two arcs.*

Proof. The first part of the theorem follows from Theorems 1 and 2, and Lemmas 1, 4 and 5.

For the second part, we use Lemma 1 to allow ourselves to consider the particular case $P = P_\lambda$. Since we know that only the lines of the form $l_{x,-x}$ can intersect $\tau_{P_\lambda}(U_{\alpha,\beta})$ in four points, we look at these lines first.

Assume that the lines of the form $l_{x,-x}$ intersecting $\tau_{P_\lambda}(U_{\alpha,\beta})$ are partitioned as follows: $\ell_1, \ell_2, \dots, \ell_n$ intersect $\tau_{P_\lambda}(U_{\alpha,\beta})$ in exactly two points and $\ell_{n+1}, \ell_{n+2}, \dots, \ell_t$ intersect $\tau_{P_\lambda}(U_{\alpha,\beta})$ in four. We label the points of $\tau_{P_\lambda}(U_{\alpha,\beta})$ by Q_x, where x is one of the following

$$x_{11}, x_{12}, \dots, x_{n1}, x_{n2}, x_{(n+1)1}, x_{(n+1)2}, x_{(n+1)3}, x_{(n+1)4}, \dots, x_{t1}, x_{t2}, x_{t3}, x_{t4}$$

where $Q_{x_{ij}} \in \ell_i$, and $x_{i2} = -x_{i1}$ and $x_{i3} = -x_{i4}$, for all i. Then the points of $\tau_{P_\lambda}(U_{\alpha,\beta})$ can be partitioned into the following two arcs

$$A_1 = \{Q_x; x = x_{ij}, i = 1, \dots, t \text{ and } j = 1, 2\} \cup \{P_\lambda\} \quad \text{and} \quad A_2 = \{Q_x; x = x_{ij}, i = n+1, \dots, t \text{ and } j = 3, 4\}.$$

Note that the partition given is just one of many possible ones. □

In the particular case when $\beta = \overline{\beta}$ we get an even stronger result.

Corollary 1. *If $\alpha \neq 0$ and $\beta = \overline{\beta}$, then the points of $\tau_P(U_{\alpha,\beta})$ are contained in lines or arcs.*

Proof. We already know that $\tau_P(U_{\alpha,\beta})$ is contained in a line when $P \in \ell_\infty$. For when $P \notin \ell_\infty$ we use Lemma 1 to restrict ourselves to study the structure of $\tau_{P_\lambda}(U_{\alpha,\beta})$.

Let $x \neq y$ and let $l_{x,y}$ be the line through $Q_x, Q_y \in \tau_{P_\lambda}(U_{\alpha,\beta})$. Since $\beta = \overline{\beta}$ we get that $T(\alpha x^2) = 2\alpha x^2 + 2\lambda\varepsilon$, for all $x \in T_\lambda$ (this follows from the argument before Remark 3). Hence, a point $Q_z \in \tau_{P_\lambda}(U_{\alpha,\beta})$ now looks like $Q_z = [z, 2\alpha z^2 + \lambda\varepsilon, 1]$, and the line $l_{x,y}$ is given by $l_{x,y} = [-2\alpha(x+y), 1, 2\alpha xy - \lambda\varepsilon]^t$.

Thus $Q_z \in l_{x,y}$ if and only if $2\alpha[(x^2 - y^2)z - (x-y)z^2 - xy(x-y)] = 0$. Since $\alpha \neq 0$ and $x \neq y$, this equation reduces to $z^2 - (x+y)z + xy = 0$, which can be re-written as $(z-x)(z-y) = 0$. The result follows. □

4 The elation group of $U_{\alpha,\beta}$

We consider the collineation group of $U_{\alpha,\beta}$ given by $\mathcal{E} = \{E_t : (x, y, z) \mapsto (x, y + tz, z); t \in \mathrm{GF}(q)\}$. Note that $\mathcal{E}$ is an elation group with center P_∞ and axis ℓ_∞. It is easy to show that lines of the form $AE_t(A)$ pass through P_∞, for all $A \notin \ell_\infty$ and $E_t \in \mathcal{E}$. Also, since $\mathcal{E}$ acts semi-regularly on points not on its axis, nothing but the identity in $\mathcal{E}$ stabilizes a $\tau_P(U_{\alpha,\beta})$, and if $Q = E_t(P)$, for some $E_t \in \mathcal{E}$, then $\tau_P(U_{\alpha,\beta})$ and $\tau_Q(U_{\alpha,\beta})$ are disjoint.

It has been mentioned before that every line through P_∞, except from ℓ_∞, contains a pedal (of a point on ℓ_∞). Moreover, it is easy to see that the q points, different from P_∞, on each of these pedals form an orbit under the group $\mathcal{E}$. We take this observation as a 'suggestion' to take a closer look at the orbits of pedals under $\mathcal{E}$ and to study how lines intersect these sets.

From now on, we use $\mathcal{O}(X)$ to denote the orbit of a set X under the group $\mathcal{E}$.

Lemma 6. *Given a pedal $\tau_P(U_{\alpha,\beta})$, there is a point $Q \in \ell_\infty$ and q lines through Q that partition $\mathcal{O}(\tau_P(U_{\alpha,\beta}))$. That is, the intersection of each of these lines with $U_{\alpha,\beta}$ is completely contained in $\mathcal{O}(\tau_P(U_{\alpha,\beta}))$.*

Proof. Because of Lemma 1, it is enough to look at how lines intersect $\tau_{P_\lambda}(U_{\alpha,\beta})$. We consider the point $[1, 0, 0]$ and the lines through it. We know that lines, different from ℓ_∞, through $[1, 0, 0]$ look like $l_\gamma = [0, -1, \gamma]^t$.

It is easy to see that the orbit of P_λ is contained in a line through P_∞. So, we let $\mathcal{O}(P_\lambda) = \{P_1, P_2, \dots, P_q\}$, where $P_t = E_t(P_\lambda)$, for all $t \in \mathrm{GF}(q)$. Moreover, since $E_t(x, y, z) = (x, y + tz, z)$ for all $t \in \mathrm{GF}(q)$, and $P_\lambda = [0, \lambda\varepsilon, 1]$, with $\lambda = 1$ or $\lambda = w$, we obtain $P_t = [0, \lambda\varepsilon + t, 1]$

Using the arguments at the beginning of Section 3, we get that

$$\tau_{P_t}(U_{\alpha,\beta}) = \left\{R_y; y \in \mathrm{GF}(q^2), 2\lambda\varepsilon + \begin{bmatrix} y & \bar{y} \end{bmatrix} M_{\alpha,\beta} \begin{bmatrix} y \\ \bar{y} \end{bmatrix} = 0\right\}$$

where $R_y = [y, T(\alpha y^2) - \lambda\varepsilon + t, 1]$.

Now, the points of intersection (if any) of $\tau_{P_t}(U_{\alpha,\beta})$ with l_γ are given by

$$0 = [y, T(\alpha y^2) - \lambda\varepsilon + t, 1] \begin{bmatrix} 0 \\ -1 \\ \gamma \end{bmatrix} = -(T(\alpha y^2) - \lambda\varepsilon + t) + \gamma$$

which means

$$t = \gamma + \lambda\varepsilon - T(\alpha y^2). \tag{7}$$

It follows that, if y and γ are given, and $\gamma = s - \lambda\varepsilon$ for some $s \in \mathrm{GF}(q)$, then we can always find a $t \in \mathrm{GF}(q)$ that satisfies Equation (7). In this case, given a line $l_{s-\lambda\varepsilon}$, for every $y \in \mathrm{GF}(q^2)$ such that $R_y \in \tau_{P_t}(U_{\alpha,\beta})$ there is a point of intersection between $l_{s-\lambda\varepsilon}$ and $\mathcal{O}(\tau_{P_\lambda}(U_{\alpha,\beta}))$.

Hence, for $\gamma \neq s - \lambda\varepsilon$, for all $s \in \mathrm{GF}(q)$ the intersection is empty, and for $\gamma = ks - \lambda\varepsilon$ the intersection contains $q + 1$ points. Note that for every $s \in \mathrm{GF}(q)$ we are able to choose such a γ, thus we get q lines through $[1, 0, 0]$ intersecting $\mathcal{O}(\tau_{P_\lambda}(U_{\alpha,\beta}))$ in $q + 1$ points each. □

We would like to close this paper stating a few open problems.

1. Do lines intersecting pedals in at least four points exist if and only if $\beta \neq \overline{\beta}$? Corollary 1 gives us one direction of this conjecture.
2. What geometric properties determine when a line of the form $l_{x,-x}$ intersects a given $\tau_P(U_{\alpha,\beta})$ in four points?
3. When a $\tau_P(U_{\alpha,\beta})$ is partitioned into two arcs (or contained in one arc for the case $\beta = \overline{\beta}$), is any of these arcs contained in a conic?
4. In how many points does a line intersect the set of points in the orbit of any given $\tau_P(U_{\alpha,\beta})$ under $\mathcal{E}$? Lemma 6 gives a partial answer, but there are several other lines that are not considered in this result.
5. Is there a combinatorial characterization for the structure formed by the lines of Π and the points on $\mathcal{O}(\tau_{P_\lambda}(U_{\alpha,\beta}))$?

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References

[1] A. Aguglia, G. L. Ebert, A combinatorial characterization of classical unitals. *Arch. Math. (Basel)* **78** (2002), 166–172. MR1888419 Zbl 1006.51004

[2] R. D. Baker, G. L. Ebert, Intersection of unitals in the Desarguesian plane. In: *Proceedings of the Twentieth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1989)*, volume 70, 87–94, 1990. MR1041587 Zbl 0695.51009

[3] S. Barwick, G. Ebert, *Unitals in projective planes*. Springer 2008. MR2440325 Zbl 1156.51006

[4] F. Buekenhout, Existence of unitals in finite translation planes of order q^2 with a kernel of order q. *Geometriae Dedicata* **5** (1976), 189–194. MR0448236 Zbl 0336.50014

[5] P. Dembowski, *Finite geometries*. Springer 1997. MR1434062 Zbl 0865.51004

[6] N. Durante, A. Siciliano, Unitals of $\mathrm{PG}(2, q^2)$ containing conics. *J. Combin. Des.* **21** (2013), 101–111. MR3011984 Zbl 1273.05025

[7] J. W. P. Hirschfeld, *Projective geometries over finite fields*. Oxford Univ. Press 1998. MR1612570 Zbl 0899.51002

[8] J. W. P. Hirschfeld, T. Szőnyi, Sets in a finite plane with few intersection numbers and a distinguished point. *Discrete Math.* **97** (1991), 229–242. MR1140805 Zbl 0748.51011

[9] V. Krčadinac, K. Smoljak, Pedal sets of unitals in projective planes of order 9 and 16. *Sarajevo J. Math.* **7(20)** (2011), 255–264. MR2906536 Zbl 1277.51010

[10] J. A. Thas, A combinatorial characterization of Hermitian curves. *J. Algebraic Combin.* **1** (1992), 97–102. MR1162643 Zbl 0784.51023