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Nuclear Physics B 934 (2018) 240–264

NUCLEAR PHYSICS B

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Generalized Chern–Simons higher-spin gravity theories in three dimensions

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Received 31 December 2017; received in revised form 22 June 2018; accepted 6 July 2018
Available online 12 July 2018
Editor: Stephan Stieberger

Abstract

The coupling of spin-3 gauge fields to three-dimensional Maxwell and AdS-Lorentz gravity theories is presented. After showing how the usual spin-3 extensions of the AdS and the Poincaré algebras in three dimensions can be obtained as expansions of $\mathfrak{sl}(3,\mathbb{R})$ algebra, the procedure is generalized so as to define new higher-spin symmetries. Remarkably, the spin-3 extension of the Maxwell symmetry allows one to introduce a novel gravity model coupled to higher-spin topological matter with vanishing cosmological constant, which in turn corresponds to a flat limit of the AdS-Lorentz case. We extend our results to define two different families of higher-spin extensions of three-dimensional Einstein gravity.

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https://doi.org/10.1016/j.nuclphysb.2018.07.005

1. Introduction

Higher-spin (HS) fields have received renewed interest over the last years due to their appearance in the spectrum of string theory, as well as in simplified models of the AdS/CFT correspondence [1–4] (for more recent developments see [5–9]). The covariant analysis of free massless HS fields in four dimensions was described long ago by Fronsdal [10]. However, it was realized that in space–times of dimension greater than three, the coupling of HS fields to gravity displays inconsistencies [11–18], particularly due to the non-invariance of the action under HS gauge transformations. Such obstructions restrict the analysis to gravity coupled to spin-3/2 fields, which corresponds to supergravity. These no-go theorems were later surpassed by allowing non-minimal couplings and non-local interactions within the framework of Vasiliev theory, which requires to relax the flat background condition and to introduce an infinite tower of massless HS fields [19–22].

In three dimensions, on the other hand, a consistent coupling of massless HS fields to AdS gravity can be described by means of a Chern–Simons (CS) action whose gauge group is given by two copies of $SL(n,\mathbb{R})$ [23–25], where a finite number of interacting HS fields can be considered for finite n [26]. Even though the absence of local degrees of freedom, CS theories are known to possess a rich structure that makes them worth to be studied. In fact, similarly to what happens in the pure gravity case, the $SL(n,\mathbb{R}) \times SL(n,\mathbb{R})$ CS theory has interesting solutions, such as HS black holes [27–34] and conical singularities [35,36]. Moreover, the asymptotic symmetry of the theory realizes two copies of the $\mathcal{W}_n$ algebra [26,37,38], which led to conjecture the duality between three-dimensional HS theories and a $\mathcal{W}_n$ minimal model CFT in a large-N limit [39,40]. Remarkably, the no-go results can even be avoided in locally flat three-dimensional spacetimes and the main aforementioned results can be generalized to the case of vanishing cosmological constant [41–46].

In the present paper we further generalize the coupling of spin-3 fields to gravity theories in three dimensions. With the aim of extending the previous results beyond the AdS and the Poincaré cases, we address the problem of constructing HS extensions of the Maxwell and AdS-Lorentz algebras and their generalizations, given by the $\mathfrak{B}_m$ and the $\mathfrak{C}_m$ algebras. The motivation to do this lies in the fact that these symmetries, have been of recent interest in the context of (super)gravity [47–61].

Initially, the Maxwell symmetry has been introduced in [47,48] to describe a Minkowski spacetime in presence of a constant electromagnetic field background. Later it was presented in [50] an alternative geometric scenario to introduce a generalized cosmological constant term. Further generalizations of the Maxwell algebras and their supersymmetric extensions have been then successfully developed by diverse authors in [62–67]. Recently, the Maxwell algebra and its generalizations $\mathfrak{B}_m$ have been useful to recover General Relativity from CS and Born–Infeld gravity theories in an appropriate limit [68–72]. In particular, the $\mathfrak{B}_m$ algebras can be obtained as a flat limit of the so-called $\mathfrak{C}_m$ algebras. The latter have been used to relate diverse (pure) Lovelock gravity theories [73–76]. More recently, there has been a particular interest in the three-dimensional Maxwell CS gravity theory [52,54]. In particular, it has been shown in [77] that the gravitational Maxwell field appearing on the Maxwell CS action modifies not only the vacuum of the theory but also its asymptotic sector. Furthermore, it has been pointed out that the asymptotic symmetries of the Maxwell and AdS-Lorentz gravity theories are given by an enlarged deformation of the $\mathfrak{bms}_3$ algebra and three copies of the Virasoro algebra, respectively [77,78], making the study of their HS generalization even more appealing.

A key ingredient in the construction of these algebras is the semigroup expansion method [79–84], which combines the structure a given Lie algebra with an abelian semigroup to form a Lie algebra of greater dimension. By applying this method to the $\mathfrak{sl}(3,\mathbb{R})$ algebra we will construct spin-3 extensions of the Maxwell and the AdS-Lorentz algebras in three dimensions and the associated CS actions. After studying how the known results of HS gravity in three-dimensions fit in the framework of the expansion method, we extend the three-dimensional Maxwell [52] and AdS-Lorentz [49,51] CS gravity theories to a more general setup that include spin-3 gauge fields. We also show that Maxwell gravity coupled to spin-3 fields can be recovered as a flat limit of the HS extension of the AdS-Lorentz case. Finally, present two different families of gravity theories coupled to spin-3 fields that correspond to HS extensions of the $\mathfrak{B}_m$ and the $\mathfrak{C}_m$ algebras and generalize the AdS and the AdS-Lorentz cases as well as their respective flat limits. The corresponding CS actions provide novel HS topological matter actions coupled to three-dimensional gravity.

The paper is organized as follows. In the next section we briefly review the $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ CS gravity. Subsequently, in section 3, we show that this HS gravity theory, together with its flat limit, can be alternatively recovered from one copy of $\mathfrak{sl}(3,\mathbb{R})$. In section 4, we construct the HS extension of Maxwell CS gravity as well as the AdS-Lorentz case, and show that they are related by a flat limit procedure. Section 5 is devoted to the construction of HS extensions of the $\mathfrak{B}_m$ and $\mathfrak{C}_m$ gravities. We conclude our work with some comments and possible future developments.

2. Review of $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ Chern–Simons gravity

The spin-3 extension of three-dimensional AdS gravity can be formulated as a CS theory for the group $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ [26]. As pure gravity corresponds to the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ sector of the theory, the field content of the full theory is determined by the embedding of the $\mathfrak{sl}(2,\mathbb{R})$ algebra in $\mathfrak{sl}(3,\mathbb{R})$. The $\mathfrak{sl}(3,\mathbb{R})$ algebra is defined by the commutation relations

$$[L_i, L_j] = (i-j)L_{i+j}\,, \tag{2.1}$$
$$[L_i, W_m] = (2i-m)W_{i+m}\,, \tag{2.2}$$
$$[W_m, W_n] = \frac{\sigma}{3}(m-n)(2m^2+2n^2-mn-8)L_{m+n}\,, \tag{2.3}$$

where $i, j = -1, 0, 1$ and $m, n = -2, -1, 0, 1, 2$. Here we consider $\sigma < 0$[1] and the corresponding Killing form in the fundamental representation is normalized such that

$$\begin{aligned}
\langle L_0 L_0 \rangle &= \frac{1}{2}\,, & \langle W_0 W_0 \rangle &= -\frac{2}{3}\sigma\,,\\
\langle L_1 L_{-1} \rangle &= -1\,, & \langle W_2 W_{-2} \rangle &= -4\sigma\,.\\
\langle W_1 W_{-1} \rangle &= \sigma\,, & &
\end{aligned} \tag{2.4}$$

The algebra $\mathfrak{sl}(2,\mathbb{R})$ can be non-trivially embedded in $\mathfrak{sl}(3,\mathbb{R})$ in two inequivalent ways: the principal embedding $\{L_0, L_{\pm1}\}$, which gives rise to an interacting theory of massless spin-2 and spin-3 fields; and the diagonal embedding $\left\{\frac{1}{2}L_0, \frac{1}{4}W_{\pm2}\right\}$, leading to a theory for a spin-2 field, two spin-3/2 fields and a spin-1 current [30,85]. As we want to describe gravity coupled to spin-3

[1] As explained in [26], the $\mathfrak{sl}(3,\mathbb{R})$ corresponds to $\sigma < 0$ while $\sigma > 0$ reproduces the $\mathfrak{su}(1,2)$ algebra. For completeness, we shall consider an arbitrary σ, keeping in mind that we are interested in the negative value of σ.

matter fields, the principal embedding of the $\mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{sl}(3,\mathbb{R})$ will be considered throughout this article. For our purposes it will be convenient to write $\mathfrak{sl}(3,\mathbb{R})$ in the form

$$[J_a, J_b] = \epsilon_{abc} J^c , \tag{2.5}$$

$$[J_a, T_{bc}] = \epsilon^m_{\ a(b} T_{c)m} , \tag{2.6}$$

$$[T_{ab}, T_{cd}] = \sigma \left(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\epsilon_{d)am} \right) J^m , \tag{2.7}$$

where the generators $\{J_a, T_{ab}\}$ are related to those of (2.2) by

$$\begin{aligned}
J_0 &= \tfrac{1}{2}(L_{-1} + L_1) , & J_1 &= \tfrac{1}{2}(L_{-1} - L_1) , & J_2 &= L_0 , \\
T_{00} &= \tfrac{1}{4}(W_2 + W_{-2} + 2W_0) , & T_{11} &= \tfrac{1}{4}(W_2 + W_{-2} - 2W_0) , & T_{22} &= W_0 , \\
T_{01} &= \tfrac{1}{4}(W_2 - W_{-2}) , & T_{02} &= \tfrac{1}{2}(W_1 + W_{-1}) , & T_{12} &= \tfrac{1}{2}(W_1 - W_{-1}) .
\end{aligned} \tag{2.8}$$

In this case, instead of the 8 generators of the fundamental representation, there are 9 generators $\{J_a, T_{ab} = T_{ba}\}$; $a, b = 1, 2, 3$, plus the constraint $T^a_a = 0$, were indices are lowered and raised with the metric $\eta_{ab} = diag(-1, 1, 1)$.

The action of the system is given by

$$S = S_{CS}[A] - S_{CS}[\bar{A}] , \tag{2.9}$$

where $S_{CS}[A]$ corresponds to the CS action

$$S_{CS}[A] = \kappa \int \left\langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\rangle . \tag{2.10}$$

The components of the invariant tensor follow from (2.8) and (2.4), giving

$$\langle J_a J_b \rangle = \frac{1}{2} \eta_{ab} , \tag{2.11}$$

$$\langle J_a T_{bc} \rangle = 0 , \tag{2.12}$$

$$\langle T_{ab} T_{bc} \rangle = -\frac{\sigma}{2} \left(\eta_{a(c}\eta_{d)b} - \frac{2}{3} \eta_{ab}\eta_{bc} \right) , \tag{2.13}$$

and the $\mathfrak{sl}(3,\mathbb{R})$ valued connection one-forms A and $\bar{A}$ have the form

$$A = \left(\omega^a + \frac{1}{\ell} e^a \right) J_a + \left(\omega^{ab} + \frac{1}{\ell} e^{ab} \right) T_{ab} , \quad \bar{A} = \left(\omega^a - \frac{1}{\ell} e^a \right) \bar{J}_a + \left(\omega^{ab} - \frac{1}{\ell} e^{ab} \right) \bar{T}_{ab} .$$

The field equations are naturally given by the vanishing of the curvatures associated to A and $\bar{A}$

$$dA + A \wedge A = 0 \quad d\bar{A} + \bar{A} \wedge \bar{A} = 0 .$$

As each subset of $\mathfrak{sl}(3,\mathbb{R})$ generators satisfies (2.5)–(2.7) and (2.11)–(2.13), the action (2.9) takes the form

$$\begin{aligned}
S = \frac{k}{2\pi\ell} \int \Bigg[& e^a \left(d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b\omega^c + 2\sigma\epsilon_{aec}\omega^{cd}\omega^e_{\ d} \right) \\
& -2\sigma e^{ab} \left(d\omega_{ab} + 2\epsilon_{acd}\omega^c\omega^d_{\ b} \right) + \frac{1}{6\ell^2}\epsilon_{abc}e^a e^b e^c + \frac{2\sigma}{\ell^2}\epsilon_{aec}e^a e^{cd} e^e_{\ d} \Bigg] ,
\end{aligned} \tag{2.14}$$

where, in this case, $\kappa = \frac{k}{4\pi}$ and the CS level k is related to the Newton constant by

$$k=\frac{\ell}{4G}\,.$$

The field equations coming from this action can be expanded around a vacuum solution and, after using the torsion constraints to express ω^a and ω^{ab} in terms of e^a and e^{ab}, they reduce to the Fronsdal equations [10] for the space–time metric and a spin-3 field.

3. $SL(3,\mathbb{R})\times SL(3,\mathbb{R})$ gravity from $SL(3,\mathbb{R})$

In this section, we construct the most general CS action for the gauge group $SL(3,\mathbb{R})\times SL(3,\mathbb{R})$ by means of the semigroup expansion method [79], which allows one to obtain the $\mathfrak{sl}(3,\mathbb{R})\times\mathfrak{sl}(3,\mathbb{R})$ algebra from $\mathfrak{sl}(3,\mathbb{R})$. Furthermore, this method will provide us with the non-vanishing components of the invariant tensor for the expanded algebra. Note that this kind of construction has already been considered in the context of HS [86].

Let us consider the $\mathbb{Z}_2=\{\lambda_0,\lambda_1\}$ (semi)group, whose elements satisfy the following multiplication law

$$\lambda_\alpha\lambda_\beta=\begin{cases}\lambda_{\alpha+\beta}, & \text{if } \alpha+\beta\le 1\\ \lambda_{\alpha+\beta-2}, & \text{if } \alpha+\beta>1\end{cases}\,. \tag{3.1}$$

Hence, the $\mathbb{Z}_2$-expanded algebra is spanned by the set of generators

$$\mathbb{Z}_2\times\mathfrak{sl}(3,\mathbb{R})=\{M_a,P_a,M_{ab},P_{ab}\}\,, \tag{3.2}$$

which can be written in terms of the original ones as follows

$$M_a=\lambda_0 J_a\,,\qquad \ell P_a=\lambda_1 J_a\,, \tag{3.3}$$

$$M_{ab}=\lambda_0 T_{ab}\,,\qquad \ell P_{ab}=\lambda_1 T_{ab}\,. \tag{3.4}$$

Using the semigroup multiplication law (3.1) and the commutation relations of the original algebra (2.5)–(2.7), it can be shown that the generators of the expanded algebra satisfy the following commutation relations:

$$[M_a,M_b]=\epsilon_{abc}M^c\,,\quad [M_a,P_b]=\epsilon_{abc}P^c\,,\quad [P_a,P_b]=\frac{1}{\ell^2}\epsilon_{abc}M^c\,, \tag{3.5}$$

$$[M_a,M_{bc}]=\epsilon^m_{\ a(b}M_{c)m}\,,\quad [M_a,P_{bc}]=\epsilon^m_{\ a(b}P_{c)m}\,, \tag{3.6}$$

$$[P_a,M_{bc}]=\epsilon^m_{\ a(b}P_{c)m}\,,\quad [P_a,P_{bc}]=\frac{1}{\ell^2}\epsilon^m_{\ a(b}M_{c)m}\,, \tag{3.7}$$

$$[M_{ab},M_{cd}]=\sigma\left(\eta_{a(c}\epsilon_{d)bm}+\eta_{b(c}\epsilon_{d)am}\right)M^m\,, \tag{3.8}$$

$$[M_{ab},P_{cd}]=\sigma\left(\eta_{a(c}\epsilon_{d)bm}+\eta_{b(c}\epsilon_{d)am}\right)P^m\,, \tag{3.9}$$

$$[P_{ab},P_{cd}]=\frac{\sigma}{\ell^2}\left(\eta_{a(c}\epsilon_{d)bm}+\eta_{b(c}\epsilon_{d)am}\right)M^m\,. \tag{3.10}$$

Note that the spin-2 generators $\{M_a,P_a\}$ (which satisfy the $\mathfrak{so}(2,2)$ algebra) and the spin-3 generators $\{M_{ab},P_{ab}\}$ satisfy the $\mathfrak{sl}(3,\mathbb{R})\times\mathfrak{sl}(3,\mathbb{R})$ algebra. This can be explicitly seen by redefining the generators in the form

$$M_a=J_a+\bar{J}_a\,,\qquad P_a=\frac{1}{\ell}\left(J_a-\bar{J}_a\right)\,, \tag{3.11}$$

$$M_{ab}=T_{ab}+\bar{T}_{ab}\,,\quad P_{ab}=\frac{1}{\ell}\left(T_{ab}-\bar{T}_{ab}\right)\,. \tag{3.12}$$

Remarkably, the semigroup expansion procedure can also be used to recover the quadratic Casimir of the expanded algebra from the original Casimir operator. In particular, the $\mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{sl}(3,\mathbb{R})$ algebra has the following quadratic Casimir,

$$C = \mu_0 \left(M_a M^a + \frac{1}{\ell^2} P_a P^a - \frac{1}{2\sigma} \left[M_{ab} M^{ab} + \frac{1}{\ell^2} P_{ab} P^{ab} \right] \right) \\ +2\frac{\mu_1}{\ell} \left(P_a M^a - \frac{1}{2\sigma} P_{ab} M^{ab} \right) , \tag{3.13}$$

where μ_0 and μ_1 are arbitrary constants.

In order to write down a CS action for this algebra, we define the one-form gauge connection,

$$A = \omega^a M_a + e^a P_a + \omega^{ab} M_{ab} + e^{ab} P_{ab} . \tag{3.14}$$

Following Theorem VII.2 of ref. [79], the invariant tensor for the $\mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{sl}(3,\mathbb{R})$ algebra can be obtained from (2.11)–(2.13) using the expansion method, which yields

$$\langle M_a M_b \rangle = \mu_0 \langle J_a J_b \rangle = \frac{\mu_0}{2} \eta_{ab} , \tag{3.15}$$

$$\langle P_a P_b \rangle = \frac{\mu_0}{\ell^2} \langle J_a J_b \rangle = \frac{\mu_0}{2\ell^2} \eta_{ab} , \tag{3.16}$$

$$\langle M_a P_b \rangle = \frac{\mu_1}{\ell} \langle J_a J_b \rangle = \frac{\mu_1}{2\ell} \eta_{ab} , \tag{3.17}$$

$$\langle M_{ab} M_{bc} \rangle = \mu_0 \langle T_{ab} T_{bc} \rangle = -\frac{\sigma \mu_0}{2} \left(\eta_{a(c} \eta_{d)b} - \frac{2}{3} \eta_{ab} \eta_{dc} \right) , \tag{3.18}$$

$$\langle P_{ab} P_{bc} \rangle = \frac{\mu_0}{\ell^2} \langle T_{ab} T_{bc} \rangle = -\frac{\sigma \mu_0}{2\ell^2} \left(\eta_{a(c} \eta_{d)b} - \frac{2}{3} \eta_{ab} \eta_{dc} \right) , \tag{3.19}$$

$$\langle M_{ab} P_{bc} \rangle = \frac{\mu_1}{\ell} \langle T_{ab} T_{bc} \rangle = -\frac{\sigma \mu_1}{2\ell} \left(\eta_{a(c} \eta_{d)b} - \frac{2}{3} \eta_{ab} \eta_{dc} \right) . \tag{3.20}$$

Then, the $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ CS action (2.10) can be written, modulo boundary terms, as

$$S_{CS} = \kappa \mu_0 \int \left[\frac{1}{2} \left(\omega^a d\omega_a + \frac{1}{3} \epsilon_{abc} \omega^a \omega^b \omega^c \right) \right. \\ + \frac{1}{2\ell^2} e^a \left(de_a + \epsilon_{abc} \omega^b e^c \right) - \sigma \left(\omega^{ab} d\omega_{ab} + 2\epsilon_{abc} \omega^a \omega^{be} \omega^c_{\ e} \right) \\ \left. - \frac{\sigma}{\ell^2} e^{ab} \left(de_{ab} + 2\epsilon_{acd} \omega^c e^d_{\ b} + 4\epsilon_{acd} e^c \omega^d_{\ b} \right) \right] \\ + \kappa \frac{\mu_1}{\ell} \int \left[e^a \left(d\omega_a + \frac{1}{2} \epsilon_{abc} \omega^b \omega^c - 2\sigma \epsilon_{abc} \omega^{bd} \omega^c_{\ d} \right) \right. \\ \left. -2\sigma e^{ab} \left(d\omega_{ab} + 2\epsilon_{acd} \omega^c \omega^d_{\ b} \right) + \frac{1}{6\ell^2} \epsilon_{abc} \left(e^a e^b e^c - 12\sigma \epsilon_{abc} e^a e^{bd} e^c_{\ d} \right) \right] . \tag{3.21}$$

Note that the term proportional to μ_0 contains the exotic Lagrangian [87] plus contributions coming from the presence of the spin-3 fields. The term proportional to μ_1, on the other hand, corresponds to the Lagrangian presented in [26], (2.14), up to a factor 2. Therefore, the above action is the most general CS action describing the coupling of a spin-3 gauge field to AdS gravity in three dimensions.

The equations of motion for the gravitational fields are

$$\mathcal{T}^a \equiv de^a + \epsilon^{abc}\omega_b e_c - 4\sigma\epsilon^{abc}e_{bd}\omega_c^{\ d} = 0\,, \tag{3.22}$$

$$\mathcal{R}^a \equiv d\omega^a + \frac{1}{2}\epsilon^{abc}\left(\omega_b\omega_c + \frac{1}{\ell^2}e_b e_c\right) - 2\sigma\epsilon^{abc}\left(\omega_{bd}\omega_c^{\ d} + \frac{1}{\ell^2}\epsilon^{abc}e_{bd}e_c^{\ d}\right) = 0\,, \tag{3.23}$$

while the field equations of the spin-3 fields read

$$\mathcal{T}^{ab} \equiv de^{ab} + \epsilon^{cd(a|}\omega_c e_d^{\ |b)} + \epsilon^{cd(a|}e_c\omega_d^{\ |b)} = 0\,, \tag{3.24}$$

$$\mathcal{R}^{ab} \equiv d\omega^{ab} + \epsilon^{cd(a|}\omega_c\omega_d^{\ |b)} + \frac{1}{\ell^2}\epsilon^{cd(a|}e_c e_d^{\ |b)} = 0\,. \tag{3.25}$$

As shown in ref. [26], the gauge transformations of the theory $\delta A = D\lambda = d\lambda + [A, \lambda]$, with gauge parameter

$$\lambda = \Lambda^a M_a + \xi^a P_a + \Lambda^{ab}M_{ab} + \xi^{ab}P_{ab} \tag{3.26}$$

lead to the following relations for the spin-2 fields:

$$\delta\omega^a = D_\omega\Lambda^a - \frac{1}{\ell^2}\epsilon^{abc}\xi_b e_c - 4\sigma\epsilon^{abc}\omega_{bd}\Lambda_c^{\ d} - 4\frac{\sigma}{\ell^2}\epsilon^{abc}e_{bd}\xi_c^{\ d}\,, \tag{3.27}$$

$$\delta e^a = D_\omega\xi^a - \epsilon^{abc}\Lambda_b e_c - 4\sigma\epsilon^{abc}\omega_{bd}\xi_c^{\ d} - 4\sigma\epsilon^{abc}e_{bd}\Lambda_c^{\ d}\,, \tag{3.28}$$

where, apart from the usual gauge transformations, there are new ones along the spin-3 gauge parameters ξ^{ab} and Λ^{ab}. Analogously, the spin-3 fields transform as

$$\delta\omega^{ab} = d\Lambda^{ab} + \epsilon^{cd(a|}\omega_c\Lambda_d^{\ |b)} + \frac{1}{\ell^2}\epsilon^{cd(a|}e_c\xi_d^{\ |b)} + \epsilon^{cd(a}\omega_c^{\ b)}\Lambda_d + \frac{1}{\ell^2}\epsilon^{cd(a}e_c^{\ b)}\xi_d\,, \tag{3.29}$$

$$\delta e^{ab} = d\xi^{ab} + \epsilon^{cd(a|}\omega_c\xi_d^{\ |b)} + \epsilon^{cd(a|}e_c\Lambda_d^{\ |b)} + \epsilon^{cd(a}e_c^{\ b)}\Lambda_d + \epsilon^{cd(a}\omega_c^{\ b)}\xi_d\,. \tag{3.30}$$

3.1. Poincaré limit

It is well known that Poincaré (super)gravity can be recovered from the (super) AdS case through a flat limit, which can be generalized to the case of $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ HS gravity [26,41–43,45]. Here we present a novel procedure to obtain the spin-3 extension of Poincaré algebra through the semigroup expansion method. In fact, such HS symmetry is obtained by considering an expansion of the $\mathfrak{sl}(3,\mathbb{R})$ algebra with $S_E^{(1)} = \{\lambda_0, \lambda_1, \lambda_2\}$ as the relevant semigroup:

$$\lambda_\alpha\lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{if } \alpha+\beta \leq 1 \\ \lambda_2\,, & \text{if } \alpha+\beta > 1 \end{cases}, \tag{3.31}$$

where λ_2 is the zero element of the semigroup. Then, the $S_E^{(1)}$-expanded algebra is spanned by the set of generators

$$S_E^{(1)} \times \mathfrak{sl}(3,\mathbb{R}) = \{M_a, P_a, M_{ab}, P_{ab}\}\,, \tag{3.32}$$

where

$$M_a = \lambda_0 J_a\,, \qquad \ell P_a = \lambda_1 J_a\,, \tag{3.33}$$

$$M_{ab} = \lambda_0 T_{ab}\,, \quad \ell P_{ab} = \lambda_1 T_{ab}\,. \tag{3.34}$$

Using the semigroup multiplication law (3.31) and the commutation relations of the original algebra (2.5)–(2.7), it can be shown that the generators of the expanded algebra satisfy:

$$[M_a, M_b] = \epsilon_{abc} M^c\,, \quad [M_a, P_b] = \epsilon_{abc} P^c\,, \quad [P_a, P_b] = 0\,, \tag{3.35}$$

$$[M_a, M_{bc}] = \epsilon^{m}_{\ a(b}\, M_{c)m}\,, \quad [M_a, P_{bc}] = \epsilon^{m}_{\ a(b}\, P_{c)m}\,, \tag{3.36}$$

$$[P_a, M_{bc}] = \epsilon^{m}_{\ a(b}\, P_{c)m}\,, \quad [P_a, P_{bc}] = 0\,, \tag{3.37}$$

$$[M_{ab}, M_{cd}] = \sigma \left(\eta_{a(c}\,\epsilon_{d)bm} + \eta_{b(c}\,\epsilon_{d)am}\right) M^m\,, \tag{3.38}$$

$$[M_{ab}, P_{cd}] = \sigma \left(\eta_{a(c}\,\epsilon_{d)bm} + \eta_{b(c}\,\epsilon_{d)am}\right) P^m\,, \tag{3.39}$$

$$[P_{ab}, P_{cd}] = 0\,, \tag{3.40}$$

which corresponds to the spin-3 extension of the Poincaré algebra. As it is known, this symmetry can also be recovered as a flat limit $\ell \to \infty$ of the $\mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{sl}(3,\mathbb{R})$ algebra. However, as pointed out in ref. [88], in order to apply the flat limit at the level of the action (3.21) it is necessary to consider an appropriate redefinition of the constant appearing in the invariant tensor (3.15)–(3.20),

$$\mu_1 \to \ell \mu_1\,. \tag{3.41}$$

This redefinition is also required to obtain appropriately the quadratic Casimir of this algebra in the flat limit

$$C = \mu_0 \left(M_a M^a - \frac{1}{2\sigma} M_{ab} M^{ab} \right) + 2\mu_1 \left(P_a M^a - \frac{1}{2\sigma} P_{ab} M^{ab} \right). \tag{3.42}$$

Note that the invariant tensor for the Poincaré gravity coupled to spin-3 fields is recovered in this limit

$$\langle M_a M_b \rangle = \frac{\mu_0}{2} \eta_{ab}\,, \quad \langle M_a P_b \rangle = \frac{\mu_1}{2} \eta_{ab}\,, \tag{3.43}$$

$$\langle M_{ab} M_{bc} \rangle = -\mu_0 \frac{\sigma}{2} \left(\eta_{a(c}\eta_{d)b} - \frac{2}{3}\eta_{ab}\eta_{dc} \right), \tag{3.44}$$

$$\langle M_{ab} P_{bc} \rangle = -\mu_1 \frac{\sigma}{2} \left(\eta_{a(c}\eta_{d)b} - \frac{2}{3}\eta_{ab}\eta_{dc} \right). \tag{3.45}$$

It is important to note that this invariant tensor can alternatively be obtained by means of the semigroup expansion procedure. In the same way, the corresponding CS action can then be either constructed by considering the connection one-form (3.14) or taking the limit $\ell \to \infty$ in (3.21) after implementing (3.41). This leads to

$$\begin{aligned} S = \kappa\mu_0 \int \Bigg[\frac{1}{2} \left(\omega^a d\omega_a + \frac{1}{3}\epsilon_{abc}\omega^a\omega^b\omega^c \right) - \sigma \left(\omega^{ab} d\omega_{ab} + 2\epsilon_{abc}\omega^a\omega^{be}\omega^{c}_{\ e} \right) \Bigg] \\ +\kappa\mu_1 \int \Bigg[e^a \left(d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b\omega^c - 2\sigma\epsilon_{abc}\omega^{bd}\omega^{c}_{\ d} \right) \\ -2\sigma e^{ab} \left(d\omega_{ab} + 2\epsilon_{acd}\omega^c\omega^{d}_{\ b} \right) \Bigg]. \end{aligned} \tag{3.46}$$

The CS action (3.46) describes the most general coupling of spin-3 fields to the Poincaré gravity. Here an exotic term is present besides the usual Poincaré CS action coupled to spin-3 fields introduced in [41–43,45]. Interestingly, the flat limit can also be applied to the equations of motion and the gauge transformations. In fact, considering $\ell \to \infty$ in (3.22)–(3.25), we obtain

$$\mathcal{T}^a \equiv de^a + \epsilon^{abc}\omega_b e_c - 4\sigma\epsilon^{abc} e_{bd}\omega_c^{\ d} = 0\,, \tag{3.47}$$

$$\mathcal{R}^a \equiv d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b\omega_c - 2\sigma\epsilon^{abc}\omega_{bd}\omega_c^{\ d} = 0\,, \tag{3.48}$$

$$\mathcal{T}^{ab} \equiv de^{ab} + \epsilon^{cd(a|}\omega_c e_d^{\ |b)} + \epsilon^{cd(a|} e_c \omega_d^{\ |b)} = 0\,, \tag{3.49}$$
$$\mathcal{R}^{ab} \equiv d\omega^{ab} + \epsilon^{cd(a|}\omega_c \omega_d^{\ |b)} = 0\,, \tag{3.50}$$

and in (3.27)–(3.30) we find

$$\delta\omega^a = D_\omega \Lambda^a - 4\sigma\epsilon^{abc}\omega_{bd}\Lambda_c^{\ d}\,, \tag{3.51}$$
$$\delta e^a = D_\omega \xi^a - \epsilon^{abc}\Lambda_b e_c - 4\sigma\epsilon^{abc}\omega_{bd}\xi_c^{\ d} - 4\sigma\epsilon^{abc}e_{bd}\Lambda_c^{\ d}\,, \tag{3.52}$$
$$\delta\omega^{ab} = d\Lambda^{ab} + \epsilon^{cd(a|}\omega_c\Lambda_d^{\ |b)} + \epsilon^{cd(a}\omega_c^{\ b)}\Lambda_d\,, \tag{3.53}$$
$$\delta e^{ab} = d\xi^{ab} + \epsilon^{cd(a|}\omega_c\xi_d^{\ |b)} + \epsilon^{cd(a|}e_c\Lambda_d^{\ |b)} + \epsilon^{cd(a}e_c^{\ b)}\Lambda_d + \epsilon^{cd(a}\omega_c^{\ b)}\xi_d\,. \tag{3.54}$$

4. Coupling spin-3 fields to extended CS gravity

One way to generalize three-dimensional CS gravity theories consists in introducing additional fields into the gauge connection. The minimal extension can be constructed by adding an extra generator to the former space–time symmetry. Of recent interest are the Maxwell and AdS-Lorentz algebras, which have led to interesting results. In particular, diverse gravity theories can be recovered from CS and Born–Infeld gravity models based on those symmetries [68–76].

In this section, using the semigroup expansion mechanism [79], we present the coupling of spin-3 fields to the three-dimensional CS gravity based on the Maxwell and AdS-Lorentz symmetries. In particular, the Maxwell gravity coupled to HS fields can be alternatively obtained as a flat limit of the HS AdS-Lorentz gravity.

4.1. Maxwell gravity coupled to spin-3 fields

In this section, we will describe the coupling of spin-3 gauge fields to Maxwell gravity in three dimensions. For this purpose we will consider $S_E^{(2)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ as the abelian semigroup, whose elements satisfy the following multiplication law

$$\begin{array}{c|cccc} \lambda_3 & \lambda_3 & \lambda_3 & \lambda_3 & \lambda_3 \\ \lambda_2 & \lambda_2 & \lambda_3 & \lambda_3 & \lambda_3 \\ \lambda_1 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_3 \\ \lambda_0 & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\ \hline & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \end{array} \tag{4.1}$$

and $\lambda_3 = 0_S$ represents the zero element of the semigroup. Hence, the $S_E^{(2)}$-expanded algebra is spanned by the set of generators

$$S_E^{(2)} \times \mathfrak{sl}(3,\mathbb{R}) = \{M_a, P_a, Z_a, M_{ab}, P_{ab}, Z_{ab}\}\,, \tag{4.2}$$

where

$$\begin{aligned} M_a &= \lambda_0 J_a, & \ell P_a &= \lambda_1 J_a, & \ell^2 Z_a &= \lambda_2 J_a, \\ M_{ab} &= \lambda_0 T_{ab}, & \ell P_{ab} &= \lambda_1 T_{ab}, & \ell^2 Z_{ab} &= \lambda_2 T_{ab}. \end{aligned}$$

Using the semigroup multiplication law (4.1) and the commutation relations of the original algebra $\mathfrak{sl}(3,\mathbb{R})$, it can be shown that the generators of the expanded algebra satisfy the following commutation relations:

$$[M_a, M_b] = \epsilon_{abc}M^c\,, \quad [M_a, P_b] = \epsilon_{abc}P^c\,, \quad [P_a, P_b] = \epsilon_{abc}Z^c\,, \tag{4.3}$$
$$[M_a, Z_b] = \epsilon_{abc}Z^c\,, \tag{4.4}$$

$$[M_a, M_{bc}] = \epsilon^m{}_{a(b} M_{c)m} , \qquad [M_a, P_{bc}] = \epsilon^m{}_{a(b} P_{c)m} , \tag{4.5}$$

$$[P_a, M_{bc}] = \epsilon^m{}_{a(b} P_{c)m} , \qquad [P_a, P_{bc}] = \epsilon^m{}_{a(b} Z_{c)m} , \tag{4.6}$$

$$[Z_a, M_{bc}] = \epsilon^m{}_{a(b} Z_{c)m} , \qquad [M_a, Z_{bc}] = \epsilon^m{}_{a(b} Z_{c)m}, \tag{4.7}$$

$$[M_{ab}, M_{cd}] = \sigma \left(\eta_{a(c}\, \epsilon_{d)bm} + \eta_{b(c}\, \epsilon_{d)am}\right) M^m , \tag{4.8}$$

$$[M_{ab}, P_{cd}] = \sigma \left(\eta_{a(c}\, \epsilon_{d)bm} + \eta_{b(c}\, \epsilon_{d)am}\right) P^m , \tag{4.9}$$

$$[M_{ab}, Z_{cd}] = \sigma \left(\eta_{a(c}\, \epsilon_{d)bm} + \eta_{b(c}\, \epsilon_{d)am}\right) Z^m , \tag{4.10}$$

$$[P_{ab}, P_{cd}] = \sigma \left(\eta_{a(c}\, \epsilon_{d)bm} + \eta_{b(c}\, \epsilon_{d)am}\right) Z^m , \tag{4.11}$$

$$\text{others} = 0 . \tag{4.12}$$

This algebra describes the coupling of the spin-3 generators $\{M_{ab}, P_{ab}, Z_{ab}\}$ to the Maxwell symmetry (also known in the literature as $\mathfrak{B}_4$) generated by $\{M_a, P_a, Z_a\}$. Note that, similarly to what happens with the translation generators in the pure gravity case, the generators P_{ab} are no longer abelian due to the presence of the new generators Z_a and Z_{ab}. The introduction of the additional spin-3 generator Z_{ab} is therefore required in order to consistently couple spin-3 fields to the Maxwell algebra. The new HS algebra obtained here has the quadratic Casimir

$$\begin{aligned} C = {} & \mu_0 \left(M_a M^a - \frac{1}{2\sigma} M_{ab} M^{ab} \right) + 2\mu_1 \left(P_a M^a - \frac{1}{2\sigma} P_{ab} M^{ab} \right) \\ & + \mu_2 \left[P_a P^a + 2 Z_a M^a - \frac{1}{2\sigma} \left(P_{ab} P^{ab} + 2 Z_{ab} M^{ab} \right) \right] , \end{aligned} \tag{4.13}$$

allowing us to define a non-degenerate bilinear form.

In order to write down the CS action invariant under the HS extension of the Maxwell algebra, we consider the one-form gauge connection

$$A = \omega^a M_a + e^a P_a + k^a Z_a + \omega^{ab} M_{ab} + e^{ab} P_{ab} + k^{ab} Z_{ab}, \tag{4.14}$$

and the corresponding invariant tensor

$$\langle M_a M_b \rangle = \mu_0 \, \langle J_a J_b \rangle_{\mathfrak{sl}(3,\mathbb{R})} = \frac{\mu_0}{2} \eta_{ab} , \tag{4.15}$$

$$\langle P_a P_b \rangle = \frac{\tilde{\mu}_2}{\ell^2} \, \langle J_a J_b \rangle_{\mathfrak{sl}(3,\mathbb{R})} = \frac{\mu_2}{2} \eta_{ab} , \tag{4.16}$$

$$\langle M_a P_b \rangle = \frac{\tilde{\mu}_1}{\ell} \, \langle J_a J_b \rangle_{\mathfrak{sl}(3,\mathbb{R})} = \frac{\mu_1}{2} \eta_{ab} , \tag{4.17}$$

$$\langle M_a Z_b \rangle = \frac{\tilde{\mu}_2}{\ell^2} \, \langle J_a J_b \rangle_{\mathfrak{sl}(3,\mathbb{R})} = \frac{\mu_2}{2} \eta_{ab} , \tag{4.18}$$

$$\langle M_{ab} M_{bc} \rangle = \mu_0 \, \langle T_{ab} T_{bc} \rangle_{\mathfrak{sl}(3,\mathbb{R})} = -\mu_0 \frac{\sigma}{2} \left(\eta_{a(c} \eta_{d)b} - \frac{2}{3} \eta_{ab} \eta_{dc} \right) , \tag{4.19}$$

$$\langle P_{ab} P_{bc} \rangle = \frac{\tilde{\mu}_2}{\ell^2} \, \langle T_{ab} T_{bc} \rangle_{\mathfrak{sl}(3,\mathbb{R})} = -\mu_2 \frac{\sigma}{2} \left(\eta_{a(c} \eta_{d)b} - \frac{2}{3} \eta_{ab} \eta_{dc} \right) , \tag{4.20}$$

$$\langle M_{ab} P_{bc} \rangle = \frac{\tilde{\mu}_1}{\ell} \, \langle T_{ab} T_{bc} \rangle_{\mathfrak{sl}(3,\mathbb{R})} = -\mu_1 \frac{\sigma}{2} \left(\eta_{a(c} \eta_{d)b} - \frac{2}{3} \eta_{ab} \eta_{dc} \right) , \tag{4.21}$$

$$\langle M_{ab} Z_{bc} \rangle = \frac{\tilde{\mu}_2}{\ell^2} \, \langle T_{ab} T_{bc} \rangle_{\mathfrak{sl}(3,\mathbb{R})} = -\mu_2 \frac{\sigma}{2} \left(\eta_{a(c} \eta_{d)b} - \frac{2}{3} \eta_{ab} \eta_{dc} \right) , \tag{4.22}$$

where we have defined $\mu_1 = \tilde{\mu}_1/\ell$ and $\mu_2 = \tilde{\mu}_2/\ell^2$. Then, considering the invariant tensor and the one-form connection (4.14) the CS action reads

$$\begin{aligned} S = \kappa \int \mu_0 \Big[& \frac{1}{2}\left(\omega^a d\omega_a + \frac{1}{3}\epsilon_{abc}\omega^a\omega^b\omega^c\right) - \sigma\left(\omega^a_{\ b} d\omega^b_{\ a} + 2\epsilon_{abc}\omega^a\omega^{bd}\omega^c_{\ d}\right)\Big] \\ & +\mu_1\left[e^a\left(d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b\omega^c - 2\sigma\epsilon_{abc}\omega^{bd}\omega^c_{\ d}\right) - 2\sigma e^{ab}\left(d\omega_{ab} + 2\epsilon_{acd}\omega^c\omega^d_{\ b}\right)\right] \\ & +\mu_2\Big[\frac{1}{2}e^a\left(de_a + \epsilon_{abc}\omega^b e^c\right) + k^a\left(d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b\omega^c\right) \\ & -\sigma e^{ab}\left(de_{ab} + 2\epsilon_{acd}\omega^c e^d_{\ b} + 4\epsilon_{acd}e^c\omega^d_{\ b}\right) \\ & -2\sigma\left(\omega^{ab}dk_{ab} + \epsilon_{abc}k^a\omega^{be}\omega^c_{\ e} + 2\epsilon_{abc}\omega^a k^{be}\omega^c_{\ e}\right)\Big]. \end{aligned} \tag{4.23}$$

This action describes the coupling of spin-3 gauge fields to CS Maxwell gravity, which splits in three different sectors proportional to μ_0, μ_1 and μ_2. One can see that the term proportional to μ_1 corresponds to an Euler type CS form while the term proportional to μ_0 and μ_2 are Pontryagin type CS forms. Interestingly, as in the Poincaré case, the action (4.23) does not contain the cosmological term. The equations of motion for the spin-2 fields are given by

$$\mathcal{T}^a \equiv de^a + \epsilon^{abc}\omega_b e_c - 4\sigma\epsilon^{abc}e^{bd}\omega_c^{\ d} = 0\,, \tag{4.24}$$

$$\mathcal{R}^a \equiv d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b\omega_c - 2\sigma\epsilon^{abc}\omega_{bd}\omega_c^{\ d} = 0\,, \tag{4.25}$$

$$\mathcal{F}^a \equiv dk^a + \epsilon^{abc}\omega_b k_c + \frac{1}{2}\epsilon^{abc}e_b e_c - 2\sigma\epsilon^{abc}\left(2\omega_{bd}k_c^{\ d} + e_{bd}e_c^{\ d}\right) = 0\,, \tag{4.26}$$

while the corresponding field equations for the spin-3 fields are

$$\mathcal{T}^{ab} \equiv de^{ab} + \epsilon^{cd(a|}\omega_c e_d^{\ |b)} + \epsilon^{cd(a|}e_c\omega_d^{\ |b)} = 0\,, \tag{4.27}$$

$$\mathcal{R}^{ab} \equiv d\omega^{ab} + \epsilon^{cd(a|}\omega_c\omega_d^{\ |b)} = 0\,, \tag{4.28}$$

$$\mathcal{F}^{ab} \equiv dk^{ab} + \epsilon^{cd(a|}\omega_c k_d^{\ |b)} + \epsilon^{cd(a|}k_c\omega_d^{\ |b)} + \epsilon^{cd(a|}e_c e_d^{\ |b)} = 0\,. \tag{4.29}$$

Let us note that the terms $\epsilon^{abc}e_b e_c$ and $\epsilon^{abc}e_{bd}e_c^{\ d}$ do not contribute anymore to the Lorentz curvature equation in the very same way as it happens in the case of the HS flat gravity. Nevertheless, these terms appear explicitly in the new curvature equation $\mathcal{F}^a = 0$. Analogously, the term $\epsilon^{cd(a|}e_c e_d^{\ |b)}$ is now in the spin-3 curvature equation $\mathcal{F}^{ab} = 0$. Thus, the present theory does not only generalize the HS extension of the Poincaré gravity introducing extra spin-2 and spin-3 fields, but also modifies the dynamics. The CS action (4.23) is invariant under gauge transformations $\delta A = D\lambda$, where the gauge parameter is given by

$$\lambda = \Lambda^a M_a + \xi^a P_a + \chi^a Z_a + \Lambda^{ab}M_{ab} + \xi^{ab}P_{ab} + \chi^{ab}Z_{ab}\,. \tag{4.30}$$

For the spin-2 fields we get

$$\delta\omega^a = D_\omega\Lambda^a - 4\sigma\epsilon^{abc}\omega_{bd}\Lambda_c^{\ d}\,, \tag{4.31}$$

$$\delta e^a = D_\omega\xi^a - \epsilon^{abc}\Lambda_b e_c - 4\sigma\epsilon^{abc}\omega_{bd}\xi_c^{\ d} - 4\sigma\epsilon^{abc}e_{bd}\Lambda_c^{\ d}\,, \tag{4.32}$$

$$\begin{aligned}\delta k^a = D_\omega\chi^a &- \epsilon^{abc}\xi_b e_c - \epsilon^{abc}\Lambda_b k_c - 4\sigma\epsilon^{abc}e_{bd}\xi_c^{\ d} \\ &-4\sigma\epsilon^{abc}\omega_{bd}\chi_c^{\ d} - 4\sigma\epsilon^{abc}k_{bd}\Lambda_c^{\ d}\,,\end{aligned} \tag{4.33}$$

where, besides the usual gauge transformations of CS Maxwell gravity, there are new terms proportional to the spin-3 gauge parameters ξ^{ab}, Λ^{ab} and χ^{ab}. The spin-3 fields, on the other hand, transform as

$$\delta\omega^{ab} = d\Lambda^{ab} + \epsilon^{cd(a|}\omega_c\Lambda_d{}^{|b)} + \epsilon^{cd(a}\omega_c^{b)}\Lambda_d\,, \tag{4.34}$$

$$\delta e^{ab} = d\xi^{ab} + \epsilon^{cd(a|}\omega_c\xi_d{}^{|b)} + \epsilon^{cd(a|}e_c\Lambda_d{}^{|b)} + \epsilon^{cd(a}e_c^{b)}\Lambda_d + \epsilon^{cd(a}\omega_c^{b)}\xi_d\,, \tag{4.35}$$

$$\begin{aligned}\delta k^{ab} = {} & d\chi^{ab} + \epsilon^{cd(a|}k_c\Lambda_d{}^{|b)} + \epsilon^{cd(a|}\omega_c\chi_d{}^{|b)} + \epsilon^{cd(a|}e_c\xi_d{}^{|b)} \\ & + \epsilon^{cd(a}\omega_c^{b)}\chi_d + \epsilon^{cd(a}k_c^{b)}\Lambda_d + \epsilon^{cd(a}e_c^{b)}\xi_d\,. \end{aligned} \tag{4.36}$$

It is worth to note that the field equation for the field k^{ab} resembles the structure of the HS matter introduced in [89,90].

4.2. AdS-Lorentz gravity coupled to spin-3 fields

Let us consider now the coupling of spin-3 gauge fields to three-dimensional AdS-Lorentz gravity. The abelian semigroup to be considered in this case is $S_{\mathcal{M}}^{(2)} = \{\lambda_0, \lambda_1, \lambda_2\}$, whose elements satisfy

$$\begin{array}{c|ccc} \lambda_2 & \lambda_2 & \lambda_1 & \lambda_2 \\ \lambda_1 & \lambda_1 & \lambda_2 & \lambda_1 \\ \lambda_0 & \lambda_0 & \lambda_1 & \lambda_2 \\ \hline & \lambda_0 & \lambda_1 & \lambda_2 \end{array} \tag{4.37}$$

Note that the $S_{\mathcal{M}}^{(2)}$ semigroup does not have a zero element as in the $S_E^{(2)}$ semigroup used in the previous case. The new generators are expressed in term of the original ones as

$$\begin{array}{lll} M_a = \lambda_0 J_a, & \ell P_a = \lambda_1 J_a, & \ell^2 Z_a = \lambda_2 J_a, \\ M_{ab} = \lambda_0 T_{ab}, & \ell P_{ab} = \lambda_1 T_{ab}, & \ell^2 Z_{ab} = \lambda_2 T_{ab}. \end{array}$$

Using the commutation relations of the $\mathfrak{sl}(3,\mathbb{R})$ algebra and the semigroup multiplication law (4.37) one can show that the generators of the expanded algebra satisfy

$$[M_a, M_b] = \epsilon_{abc}M^c\,, \qquad [M_a, P_b] = \epsilon_{abc}P^c\,, \qquad [P_a, P_b] = \epsilon_{abc}Z^c\,, \tag{4.38}$$

$$[M_a, Z_b] = \epsilon_{abc}Z^c, \qquad [P_a, Z_b] = \frac{1}{\ell^2}\epsilon_{abc}P^c, \qquad [Z_a, Z_b] = \frac{1}{\ell^2}\epsilon_{abc}Z^c\,, \tag{4.39}$$

$$[M_a, M_{bc}] = \epsilon^m{}_{a(b}M_{c)m}\,, \qquad [M_a, P_{bc}] = \epsilon^m{}_{a(b}P_{c)m}\,, \tag{4.40}$$

$$[P_a, M_{bc}] = \epsilon^m{}_{a(b}P_{c)m}\,, \qquad [P_a, P_{bc}] = \epsilon^m{}_{a(b}Z_{c)m}\,, \tag{4.41}$$

$$[Z_a, M_{bc}] = \epsilon^m{}_{a(b}Z_{c)m}\,, \qquad [Z_a, P_{bc}] = \frac{1}{\ell^2}\epsilon^m{}_{a(b}P_{c)m}\,, \tag{4.42}$$

$$[M_a, Z_{bc}] = \epsilon^m{}_{a(b}Z_{c)m}\,, \qquad [P_a, Z_{bc}] = \frac{1}{\ell^2}\epsilon^m{}_{a(b}P_{c)m}\,, \tag{4.43}$$

$$[Z_a, Z_{bc}] = \frac{1}{\ell^2}\epsilon^m{}_{a(b}Z_{c)m}\,, \tag{4.44}$$

$$[M_{ab}, M_{cd}] = \sigma\left(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\epsilon_{d)am}\right)M^m\,, \tag{4.45}$$

$$[M_{ab}, P_{cd}] = \sigma\left(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\epsilon_{d)am}\right)P^m\,, \tag{4.46}$$

$$[M_{ab}, Z_{cd}] = \sigma\left(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\epsilon_{d)am}\right)Z^m\,, \tag{4.47}$$

$$[P_{ab}, P_{cd}] = \sigma \left(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\epsilon_{d)am}\right) Z^m \,, \tag{4.48}$$

$$[P_{ab}, Z_{cd}] = \frac{\sigma}{\ell^2} \left(\eta_{a(c}\epsilon_{d)mb} + \eta_{b(c}\epsilon_{d)ma}\right) P^m \,, \tag{4.49}$$

$$[Z_{ab}, Z_{cd}] = \frac{\sigma}{\ell^2} \left(\eta_{a(c}\epsilon_{d)mb} + \eta_{b(c}\epsilon_{d)ma}\right) Z^m \,. \tag{4.50}$$

This algebra describes the coupling of the spin-3 generators $\{M_{ab}, P_{ab}, Z_{ab}\}$ to the AdS-Lorentz algebra [49,55] (also known as $\mathfrak{C}_4$ [73]) spanned by $\{M_a, P_a, Z_a\}$. It can be written as three copies of the $\mathfrak{sl}(3, \mathbb{R})$ when the generators are redefined as

$$J_a = \frac{1}{2}\left(\ell^2 Z_a + \ell P_a\right) , \qquad \bar{J}_a = \frac{1}{2}\left(\ell^2 Z_a - \ell P_a\right) , \qquad \tilde{J}_a = M_a - \ell^2 Z_a \,, \tag{4.51}$$

$$T_{ab} = \frac{1}{2}\left(\ell^2 Z_{ab} + \ell P_{ab}\right) , \quad \bar{T}_{ab} = \frac{1}{2}\left(\ell^2 Z_{ab} - \ell P_{ab}\right) , \quad \tilde{T}_{ab} = M_{ab} - \ell^2 Z_{ab} \,. \tag{4.52}$$

This new spin-3 symmetry is quite different from the Maxwell one although having the same number of spin-2 and spin-3 generators. In particular, there are no abelian generators and the parameter ℓ appears explicitly in some commutation relations. Interestingly, the spin-3 Maxwell algebra is recovered in the limit $\ell \to \infty$. Something similar happens with the quadratic Casimir associated to this algebra, which can be deduced using the expansion method and reads

$$\begin{aligned} C = {} & \mu_0 \left(M_a M^a - \frac{1}{2\sigma} M_{ab} M^{ab} \right) \\ & + 2\frac{\mu_1}{\ell} \left[P_a M^a + \frac{1}{\ell} Z_a P^a - \frac{1}{2\sigma} \left(P_{ab} M^{ab} + \frac{1}{\ell} P_{ab} Z^{ab} \right) \right] \\ & + \frac{\mu_2}{\ell^2} \left[P_a P^a + 2 Z_a M^a + \frac{1}{\ell^2} Z_a Z^a - \frac{1}{2\sigma} \left(P_{ab} P^{ab} + 2 Z_{ab} M^{ab} + \frac{1}{\ell^2} Z_{ab} Z^{ab} \right) \right] . \end{aligned} \tag{4.53}$$

In this case, however, in order to define a flat limit it is necessary to redefine the constants μ_1 and μ_2 in the form

$$\mu_1 \to \ell \mu_1 \,, \quad \mu_2 \to \ell^2 \mu_2 \,. \tag{4.54}$$

Then, the Casimir operator of the spin-3 extension of the Maxwell algebra (4.13) is recovered in the limit $\ell \to \infty$.

Let us consider now the one-form gauge connection

$$A = \omega^a M_a + e^a P_a + k^a Z_a + \omega^{ab} M_{ab} + e^{ab} P_{ab} + k^{ab} Z_{ab} \,, \tag{4.55}$$

and the invariant tensor that follows from the expansion

$$\langle M_a M_b \rangle = \frac{\mu_0}{2} \eta_{ab} \,, \quad \langle M_a P_b \rangle = \frac{\mu_1}{2\ell} \eta_{ab} \,, \qquad \langle P_a P_b \rangle = \frac{\mu_2}{2\ell^2} \eta_{ab} \,, \tag{4.56}$$

$$\langle M_a Z_b \rangle = \frac{\mu_2}{2\ell^2} \eta_{ab} \,, \quad \langle P_a Z_b \rangle = \frac{\mu_1}{2\ell^3} \eta_{ab} \,, \qquad \langle Z_a Z_b \rangle = \frac{\mu_2}{2\ell^4} \eta_{ab} \,, \tag{4.57}$$

$$\langle M_{ab} M_{bc} \rangle = -\mu_0 \frac{\sigma}{2} \left(\eta_{a(c} \eta_{d)b} - \frac{2}{3} \eta_{ab} \eta_{dc} \right) , \tag{4.58}$$

$$\langle P_{ab} P_{bc} \rangle = -\frac{\mu_2}{\ell^2} \frac{\sigma}{2} \left(\eta_{a(c} \eta_{d)b} - \frac{2}{3} \eta_{ab} \eta_{dc} \right) , \tag{4.59}$$

$$\langle M_{ab}P_{bc}\rangle = -\frac{\mu_1}{\ell}\frac{\sigma}{2}\left(\eta_{a(c}\eta_{d)b} - \frac{2}{3}\eta_{ab}\eta_{dc}\right), \tag{4.60}$$

$$\langle M_{ab}Z_{bc}\rangle = -\frac{\mu_2}{\ell^2}\frac{\sigma}{2}\left(\eta_{a(c}\eta_{d)b} - \frac{2}{3}\eta_{ab}\eta_{dc}\right), \tag{4.61}$$

$$\langle P_{ab}Z_{bc}\rangle = -\frac{\mu_1}{\ell^3}\frac{\sigma}{2}\left(\eta_{a(c}\eta_{d)b} - \frac{2}{3}\eta_{ab}\eta_{dc}\right), \tag{4.62}$$

$$\langle Z_{ab}Z_{bc}\rangle = -\frac{\mu_2}{\ell^4}\frac{\sigma}{2}\left(\eta_{a(c}\eta_{d)b} - \frac{2}{3}\eta_{ab}\eta_{dc}\right). \tag{4.63}$$

The CS action (2.10) in this case takes the form

$$\begin{aligned}
S = \kappa \int \mu_0 &\left[\frac{1}{2}\left(\omega^a d\omega_a + \frac{1}{3}\epsilon_{abc}\omega^a\omega^b\omega^c\right) - \sigma\left(\omega^a_{\ b} d\omega^b_{\ a} + 2\epsilon_{abc}\omega^a\omega^{bd}\omega^c_{\ d}\right)\right] \\
&+ \frac{\mu_1}{\ell}\left[e^a\left(d\omega_a + \frac{1}{2}\epsilon_{abc}\left\{\omega^b\omega^c + \frac{1}{2\ell^4}k^bk^c\right\} + \frac{1}{\ell^2}dk_a + \frac{1}{\ell^2}\epsilon_{abc}\left\{\omega^b k^c + \frac{1}{\ell^4}k^bk^c\right\}\right)\right. \\
&+ \frac{1}{6\ell^2}\epsilon_{abc}\left(e^ae^be^c - 12\sigma\epsilon_{abc}e^ae^{bd}e^c_{\ d}\right) - 2\sigma e^a\left(\epsilon_{abc}\omega^{bd}\omega^c_{\ d} + \frac{2}{\ell^2}\epsilon_{abc}\omega^{bd}k^c_{\ d}\right) \\
&-2\sigma e^{ab}\left(d\omega_{ab} + 2\epsilon_{acd}\left\{\omega^c\omega^d_{\ b} + \frac{1}{\ell^2}k^c\omega^d_{\ b} + \frac{1}{\ell^4}k^ck^d_{\ b}\right\} + \frac{1}{\ell^2}dk_{ab}\right. \\
&\left.\left.+ \frac{2}{\ell^2}\epsilon_{acd}\left\{k^c\omega^d_{\ b} + 2\omega^ck^d_{\ b} + \frac{2}{\ell^2}k^ck^d_{\ b}\right\}\right)\right] \\
&+ \frac{\mu_2}{\ell^2}\left[\frac{1}{2}e^a\left(de_a + \epsilon_{abc}\omega^be^c + \frac{1}{\ell^2}\epsilon_{abc}k^be^c\right)\right. \\
&+ k^a\left(d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b\omega^c + \frac{1}{\ell^2}\left\{dk_a + \frac{1}{2}\epsilon_{abc}\omega^bk^c + \frac{1}{3\ell^2}\epsilon_{abc}k^bk^c\right\}\right) \\
&-2\sigma\left(\omega^{ab}dk_{ab} + \frac{1}{2\ell^2}k^{ab}dk_{ab} + \epsilon_{abc}k^a\omega^{be}\omega^c_{\ e} + \frac{2}{\ell^2}\epsilon_{abc}k^a\omega^{be}k^c_{\ e}\right. \\
&\left.+ \frac{1}{\ell^4}\epsilon_{abc}k^ak^{be}k^c_{\ e} + 2\epsilon_{abc}\omega^ak^{be}\omega^c_{\ e} + \frac{1}{\ell^2}\epsilon_{abc}\omega^ak^{be}k^c_{\ e}\right) \\
&\left.-\sigma e^{ab}\left(de_{ab} + 2\epsilon_{acd}\omega^ce^d_{\ b} + \frac{2}{\ell^2}\epsilon_{acd}k^ce^d_{\ b} + 4\epsilon_{acd}e^c\omega^d_{\ b} + \frac{4}{\ell^2}\epsilon_{acd}e^ck^d_{\ b}\right)\right],
\end{aligned} \tag{4.64}$$

and describes the coupling of spin-3 fields to the AdS-Lorentz gravity. Note that the absence of abelian generators in this new HS symmetry gives terms proportional to μ_1 and μ_2 that are different to the ones that appear in the spin-3 Maxwell case. However, the CS action (4.23) describing the coupling of spin-3 fields to the Maxwell gravity can be recovered considering the redefinition (4.54) of the constants and applying the flat limit $\ell \to \infty$. The same procedure can be done at the level of the invariant tensor to obtain the one corresponding to the spin-3 Maxwell algebra.

The field equations in this case are given by

$$\mathcal{T}^a \equiv de^a + \epsilon^{abc}\left(\omega_be_c + \frac{1}{\ell^2}k_be_c\right) - 4\sigma\epsilon^{abc}\left(e^{bd}\omega_c^{\ d} + \frac{1}{\ell^2}e^{bd}k_c^{\ d}\right) = 0, \tag{4.65}$$

$$\mathcal{R}^a \equiv d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b\omega_c - 2\sigma\epsilon^{abc}\omega_{bd}\omega_c^{\ d} = 0, \tag{4.66}$$

$$\mathcal{F}^a \equiv dk^a + \epsilon^{abc}\left(\omega_b k_c + \frac{1}{\ell^2}k_b k_c + \frac{1}{2}e_b e_c\right)$$
$$-2\sigma\epsilon^{abc}\left(2\omega_{bd}k_c^{\ d} + \frac{1}{\ell^2}k_{bd}k_c^{\ d} + e_{bd}e_c^{\ d}\right) = 0\,, \tag{4.67}$$
$$\mathcal{T}^{ab} \equiv de^{ab} + \epsilon^{cd(a|}\left(\omega_c e_d^{\ |b)} + \frac{1}{\ell^2}k_c e_d^{\ |b)} + e_c\omega_d^{\ |b)} + \frac{1}{\ell^2}e_c k_d^{\ |b)}\right) = 0\,, \tag{4.68}$$
$$\mathcal{R}^{ab} \equiv d\omega^{ab} + \epsilon^{cd(a|}\omega_c\omega_d^{\ |b)} = 0\,, \tag{4.69}$$
$$\mathcal{F}^{ab} \equiv dk^{ab} + \epsilon^{cd(a|}\left(\omega_c k_d^{\ |b)} + \frac{1}{\ell^2}k_c k_d^{\ |b)} + k_c\omega_d^{\ |b)} + \frac{1}{\ell^2}k_c k_d^{\ |b)} + e_c e_d^{\ |b)}\right) = 0\,, \tag{4.70}$$

where presence of non-abelian generators also modifies the gauge transformations w.r.t the spin-3 Maxwell case. Indeed, the spin-2 gauge transformations have the form

$$\delta\omega^a = D_\omega\Lambda^a - 4\sigma\epsilon^{abc}\omega_{bd}\Lambda_c^{\ d}\,, \tag{4.71}$$
$$\delta e^a = D_\omega\xi^a + \frac{1}{\ell^2}\epsilon^{abc}k_b\xi_c - \epsilon^{abc}\Lambda_b e_c - \frac{1}{\ell^2}\epsilon^{abc}\chi_b e_c$$
$$-4\sigma\epsilon^{abc}\left(\omega_{bd}\xi_c^{\ d} + \frac{1}{\ell^2}k_{bd}\xi_c^{\ d} + e_{bd}\Lambda_c^{\ d} + \frac{1}{\ell^2}e_{bd}\chi_c^{\ d}\right), \tag{4.72}$$
$$\delta k^a = D_\omega\chi^a + \frac{1}{\ell^2}\epsilon^{abc}k_b\chi_c - \epsilon^{abc}\xi_b e_c - 4\sigma\epsilon^{abc}\left(e_{bd}\xi_c^{\ d} + k_{bd}\chi_c^{\ d}\right), \tag{4.73}$$

while the spin-3 gauge transformations are given by

$$\delta\omega^{ab} = d\Lambda^{ab} + \epsilon^{cd(a|}\omega_c\Lambda_d^{\ |b)} + \epsilon^{cd(a}\omega_c^{\ b)}\Lambda_d\,, \tag{4.74}$$
$$\delta e^{ab} = d\xi^{ab} + \epsilon^{cd(a|}\omega_c\xi_d^{\ |b)} + \epsilon^{cd(a}\omega_c^{\ b)}\xi_d + \frac{1}{\ell^2}\epsilon^{cd(a|}k_c\xi_d^{\ |b)} + \frac{1}{\ell^2}\epsilon^{cd(a}k_c^{\ b)}\xi_d$$
$$+\epsilon^{cd(a|}e_c\Lambda_d^{\ |b)} + \frac{1}{\ell^2}\epsilon^{cd(a|}e_c\chi_d^{\ |b)} + \epsilon^{cd(a}e_c^{\ b)}\Lambda_d + \frac{1}{\ell^2}\epsilon^{cd(a}e_c^{\ b)}\chi_d\,, \tag{4.75}$$
$$\delta k^{ab} = d\chi^{ab} + \epsilon^{cd(a|}\omega_c\chi_d^{\ |b)} + \epsilon^{cd(a}\omega_c^{\ b)}\chi_d + \frac{1}{\ell^2}\epsilon^{cd(a|}k_c\chi_d^{\ |b)} + \frac{1}{\ell^2}\epsilon^{cd(a}k_c^{\ b)}\chi_d$$
$$+\epsilon^{cd(a|}e_c\xi_d^{\ |b)} + \epsilon^{cd(a}e_c^{\ b)}\xi_d\,. \tag{4.76}$$

Note that the limit $\ell \to \infty$ properly reproduces the spin-3 Maxwell field equations and gauge transformations.

5. Generalizations to $\mathfrak{B}_m$ and $\mathfrak{C}_m$ gravity theories

A wide class of expanded Lie algebras have recently been introduced in the context of CS gravity in diverse dimensions. In particular, the $\mathfrak{B}_m$ family has been useful in order to recover standard General Relativity without cosmological constant as a particular limit of a CS and Born–Infeld like gravity theories [68–72]. Subsequently, the $\mathfrak{C}_m$ algebras were used to recover diverse Pure Lovelock gravity actions in a matter-free configuration limit [74–76]. Furthermore, it was shown in [73,91], that the AdS and Poincaré algebra, correspond to the simplest cases ($m = 3$) of the $\mathfrak{C}_m$ family and the $\mathfrak{B}_m$ family, respectively. Along the same line, the AdS-Lorentz and its Inönü–Wigner contraction, the Maxwell algebra, are given by the $\mathfrak{C}_4$ and the $\mathfrak{B}_4$ algebras, respectively. Generically, the $\mathfrak{B}_m$ symmetries can always be recovered as an Inönü–Wigner contraction of the $\mathfrak{C}_m$ symmetries.

In this section, we extend this construction to present new non-trivial Lie algebras that correspond to the spin-3 extensions of the $\mathfrak{B}_m$ and $\mathfrak{C}_m$ algebras. In every case, the new commutation relations can be obtained as an expansion of the $\mathfrak{sl}(3,\mathbb{R})$ algebra considering two families of semigroups. In addition, we study the three-dimensional CS action based on these new HS symmetries. Finally, we provide with the general limit relating such new symmetries.

5.1. $\mathfrak{B}_m$ gravity coupled to spin-3 fields

Let us consider the semigroup $S_E^{(m-2)} = \{\lambda_0, \lambda_1, \dots, \lambda_{m-1}\}$, whose multiplication law is given by

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{if } \alpha+\beta \le m-1 \\ \lambda_{m-1}, & \text{if } \alpha+\beta > m-1 \end{cases}, \tag{5.1}$$

where $\lambda_{m-1} = 0_S$ is the zero element of the semigroup. The $S_E^{(m-2)}$-expanded algebra is spanned by the set of generators

$$\mathfrak{B}_m^{s_3} = S_E^{(m-2)} \times \mathfrak{sl}(3,\mathbb{R}) = \left\{ M_a^{(i)}, P_a^{(\bar{\imath})}, M_{ab}^{(i)}, P_{ab}^{(\bar{\imath})} \right\}, \tag{5.2}$$

which are related to the original ones by

$$\ell^i M_a^{(i)} = J_{(a,i)} \; = \; \lambda_i J_a\,, \quad \ell^i M_{ab}^{(i)} \; = \; T_{(ab,i)} \; = \lambda_i T_{ab}\,, \tag{5.3}$$

$$\ell^{\bar{\imath}} P_a^{(\bar{\imath})} = J_{(a,\bar{\imath})} \; = \; \lambda_{\bar{\imath}} J_a\,, \quad \ell^{\bar{\imath}} P_{ab}^{(\bar{\imath})} \; = \; T_{(ab,\bar{\imath})} \; = \; \lambda_{\bar{\imath}} T_{ab}\,. \tag{5.4}$$

Here i takes even values and $\bar{\imath}$ takes odd values. The spin-3 and spin-2 generators of the expanded algebra satisfy

$$\left[M_a^{(i)}, M_b^{(j)}\right] = \epsilon_{abc} M^{c,(i+j)}, \quad \text{for} \quad i+j \le m-2\,, \tag{5.5}$$

$$\left[M_a^{(i)}, P_b^{(\bar{\imath})}\right] = \epsilon_{abc} P^{c,(i+\bar{\imath})}, \quad \text{for} \quad i+\bar{\imath} \le m-2\,, \tag{5.6}$$

$$\left[P_a^{(\bar{\imath})}, P_b^{(\bar{j})}\right] = \epsilon_{abc} M^{c,(\bar{\imath}+\bar{j})}, \quad \text{for} \quad \bar{\imath}+\bar{j} \le m-2\,, \tag{5.7}$$

$$\left[M_a^{(i)}, M_{bc}^{(j)}\right] = \epsilon^m{}_{a(b} M_{c)m}^{(i+j)}, \quad \text{for} \quad i+j \le m-2\,, \tag{5.8}$$

$$\left[M_a^{(i)}, P_{bc}^{(\bar{\imath})}\right] = \epsilon^m{}_{a(b} P_{c)m}^{(i+\bar{\imath})}, \quad \text{for} \quad i+\bar{\imath} \le m-2\,, \tag{5.9}$$

$$\left[P_a^{(\bar{\imath})}, M_{bc}^{(j)}\right] = \epsilon^m{}_{a(b} P_{c)m}^{(\bar{\imath}+j)}, \quad \text{for} \quad \bar{\imath}+j \le m-2\,, \tag{5.10}$$

$$\left[P_a^{(\bar{\imath})}, P_{bc}^{(\bar{j})}\right] = \epsilon^m{}_{a(b} M_{c)m}^{(\bar{\imath}+\bar{j})}, \quad \text{for} \quad \bar{\imath}+\bar{j} \le m-2\,, \tag{5.11}$$

$$\left[M_{ab}^{(i)}, M_{cd}^{(j)}\right] = \sigma \left(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\epsilon_{d)am}\right) M^{m,(i+j)}, \quad \text{for} \quad i+j \le m-2\,, \tag{5.12}$$

$$\left[M_{ab}^{(i)}, P_{cd}^{(\bar{\imath})}\right] = \sigma \left(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\epsilon_{d)am}\right) P^{m,(i+\bar{\imath})}, \quad \text{for} \quad i+\bar{\imath} \le m-2\,, \tag{5.13}$$

$$\left[P_{ab}^{(i)}, P_{cd}^{(\bar{j})}\right] = \sigma \left(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\epsilon_{d)am}\right) M^{m,(\bar{\imath}+\bar{j})}, \quad \text{for} \quad \bar{\imath}+\bar{j} \le m-2\,, \tag{5.14}$$

$$\text{others} = 0\,. \tag{5.15}$$

The new algebra, which we denote by $\mathfrak{B}_m^{s_3}$, describes the coupling of the spin-3 generators $\left\{M_{ab}^{(i)}, P_{ab}^{(\bar{\imath})}\right\}$ to the spin-2 generators $\left\{M_a^{(i)}, P_a^{(\bar{\imath})}\right\}$ of the algebra $\mathfrak{B}_m$. Let us remark that $m=3,4$ reproduce the HS extension of the Poincaré and Maxwell algebras discussed in the previous sections. The quadratic Casimir in this case is given by

$$
\begin{aligned}
C = {} & \mu_{i+j}\left(\sum_{i,j}^{i+j\leq m-2} M_a^{(i)}M^{a,(j)} - \frac{1}{2\sigma}M_{ab}^{(i)}M^{ab,(j)}\right) \\
& + \mu_{i+\bar{\imath}}\left(\sum_{i,\bar{\imath}}^{i+\bar{\imath}\leq m-2} M_a^{(i)}P^{a,(\bar{\imath})} - \frac{1}{2\sigma}M_{ab}^{(i)}P^{ab,(\bar{\imath})}\right) \\
& + \mu_{\bar{\imath}+\bar{\jmath}}\left(\sum_{\bar{\imath},\bar{\jmath}}^{\bar{\imath}+\bar{\jmath}\leq m-2} P_a^{(\bar{\imath})}P^{a,(\bar{\jmath})} - \frac{1}{2\sigma}P_{ab}^{(\bar{\imath})}P^{ab,(\bar{\jmath})}\right),
\end{aligned}
\tag{5.16}
$$

while the invariant tensor reads

$$\left\langle M_a^{(i)}M_b^{(j)}\right\rangle = \frac{1}{2}\mu_{i+j}\,\eta_{ab}\,, \quad \text{for} \quad i+j\leq m-2\,, \tag{5.17}$$

$$\left\langle M_a^{(i)}P_b^{(\bar{\imath})}\right\rangle = \frac{1}{2}\mu_{i+\bar{\imath}}\,\eta_{ab}\,, \quad \text{for} \quad i+\bar{\imath}\leq m-2\,, \tag{5.18}$$

$$\left\langle P_a^{(\bar{\imath})}P_b^{(\bar{\jmath})}\right\rangle = \frac{1}{2}\mu_{\bar{\imath}+\bar{\jmath}}\,\eta_{ab}\,, \quad \text{for} \quad \bar{\imath}+\bar{\jmath}\leq m-2\,, \tag{5.19}$$

$$\left\langle M_{ab}^{(i)}M_{cd}^{(j)}\right\rangle = -\frac{\sigma}{2}\mu_{i+j}\left(\eta_{a(c}\,\eta_{d)b} - \frac{2}{3}\eta_{ab}\eta_{cd}\right), \quad \text{for} \quad i+j\leq m-2\,, \tag{5.20}$$

$$\left\langle M_{ab}^{(i)}P_{cd}^{(\bar{\imath})}\right\rangle = -\frac{\sigma}{2}\mu_{i+\bar{\imath}}\left(\eta_{a(c}\,\eta_{d)b} - \frac{2}{3}\eta_{ab}\eta_{cd}\right), \quad \text{for} \quad i+\bar{\imath}\leq m-2\,, \tag{5.21}$$

$$\left\langle P_{ab}^{(\bar{\imath})}P_{cd}^{(\bar{\jmath})}\right\rangle = -\frac{\sigma}{2}\mu_{\bar{\imath}+\bar{\jmath}}\left(\eta_{a(c}\,\eta_{d)b} - \frac{2}{3}\eta_{ab}\eta_{cd}\right), \quad \text{for} \quad \bar{\imath}+\bar{\jmath}\leq m-2\,. \tag{5.22}$$

By introducing the $\mathfrak{B}_m^{s_3}$-valued connection one-form

$$A = \omega^{a,(i)}M_a^{(i)} + e^{a,(\bar{\imath})}P_a^{(\bar{\imath})} + \omega^{ab,(i)}M_{ab}^{(i)} + e^{ab,(\bar{\imath})}P_{ab}^{(\bar{\imath})}\,, \tag{5.23}$$

a CS action (2.10) can be constructed:

$$
\begin{aligned}
S = \kappa\int & \mu_{i+j}\left[\frac{1}{2}\left(\omega^{a,(i)}d\omega_a^{(j)} + \frac{1}{3}\epsilon_{abc}\omega^{a,(l)}\omega^{b,(m)}\omega^{c,(n)}\,\delta_{l+m+n}^{i+j}\right)\right. \\
& \left.-\sigma\left(\omega_b^{a,(i)}d\omega_a^{b,(j)} + 2\epsilon_{abc}\omega^{a,(l)}\omega^{bd,(m)}\omega_d^{c,(n)}\,\delta_{l+m+n}^{i+j}\right)\right] \\
& + \mu_{i+\bar{\imath}}\left[e^{a,(\bar{\imath})}\left(d\omega_a^{(i)} + \frac{1}{2}\epsilon_{abc}\omega^{b,(m)}\omega^{c,(n)}\,\delta_{m+n}^{i} - 2\sigma\,\epsilon_{abc}\omega^{bd,(m)}\omega_d^{c,(n)}\delta_{m+n}^{i}\right)\right. \\
& \left.-2\sigma\,e^{ab,(\bar{\imath})}\left(d\omega_a^{b,(i)} + 2\epsilon_{acd}\omega^{c,(m)}\omega_b^{d,(n)}\,\delta_{m+n}^{i}\right)\right] \\
& + \mu_{\bar{\imath}+\bar{\jmath}}\left[\frac{1}{2}e^{a,(\bar{\imath})}\left(de_a^{(\bar{\jmath})} + \epsilon_{abc}\omega^{b,(m)}e^{c,(\bar{n})}\delta_{m+\bar{n}}^{\bar{\jmath}}\right)\right. \\
& \left.-\sigma\,e^{ab,(\bar{\imath})}\left(de_{ab}^{(\bar{\jmath})} + 2\epsilon_{acd}\omega^{c,(m)}e_b^{d,(\bar{n})}\delta_{m+\bar{n}}^{\bar{\jmath}} + 4\epsilon_{acd}e^{c,(\bar{m})}\omega_b^{d,(n)}\delta_{\bar{m}+n}^{\bar{\jmath}}\right)\right],
\end{aligned}
\tag{5.24}
$$

and describes the coupling of spin-3 gauge fields to $\mathfrak{B}_m$ gravity. It is important to clarify that the constants μ_{p+q} are well-defined only for $p+q \leq m-2$. In particular, the Poincaré gravity action coupled to spin-3 gauge fields is recovered for $m=3$ (see eq. (3.46)). New gravity actions coupled to spin-3 fields appear for $m \geq 4$, where for the $m=4$ we recover the Maxwell case (see eq. (4.23)). It is important to emphasize that the procedure considered here allows us to build the most general CS action for $\mathfrak{B}_m$ gravity coupled to spin-3 fields. Indeed, besides the terms proportional to $\mu_{i+\bar{\imath}}$, which correspond to Euler type CS term, there are CS terms related to the Pontryagin type densities, which are proportional to the μ_{i+j} and $\mu_{\bar{\imath}+\bar{j}}$ constants. The action (5.24) is invariant under $\mathfrak{B}_m^{s3}$ gauge transformations with parameters of the form

$$\lambda = \Lambda^{a,(i)} M_a^{(i)} + \xi^{a,(\bar{\imath})} P_a^{(\bar{\imath})} + \Lambda^{ab,(i)} M_{ab}^{(i)} + \xi^{ab,(\bar{\imath})} P_{ab}^{(\bar{\imath})} , \tag{5.25}$$

which explicitly look like

$$\begin{aligned}\delta\omega^{a,(i)} &= d\Lambda^{a,(i)} + \delta^i_{j+k}\epsilon^{abc}\left(\omega_b^{(j)}\Lambda_c^{(k)} - 4\sigma\,\omega_{bd}^{(j)}\,\Lambda_c^{\ d,(k)}\right) \\ &\quad -\epsilon^{abc}\left(\xi_b^{(\bar{j})} e_c^{(\bar{k})} + 4\sigma\, e_{bd}^{(\bar{j})}\,\xi_c^{\ d,(\bar{k})}\right)\delta^i_{\bar{j}+\bar{k}} ,\end{aligned} \tag{5.26}$$

$$\begin{aligned}\delta e^{a,(\bar{\imath})} &= d\xi^{a,(\bar{\imath})} \\ &\quad +\delta^{\bar{\imath}}_{j+\bar{k}}\,\epsilon^{abc}\left(\omega_b^{(j)}\xi_c^{(\bar{k})} - \Lambda_b^{(j)} e_c^{(\bar{k})} - 4\sigma\,\omega_{bd}^{(j)}\,\xi_c^{\ d,(\bar{k})} - 4\sigma\, e_{bd}^{(\bar{k})}\,\Lambda_c^{\ d,(j)}\right) ,\end{aligned} \tag{5.27}$$

$$\begin{aligned}\delta\omega^{ab,(i)} &= d\Lambda^{ab,(i)} + \delta^i_{j+k}\epsilon^{cd(a|}\left(\omega_c^{(j)}\Lambda_d^{\ |b),(k)} + \omega_{\ c}^{b),(j)}\Lambda_d^{(k)}\right) \\ &\quad +\delta^i_{\bar{j}+\bar{k}}\epsilon^{cd(a|}\left(e_c^{(\bar{j})}\xi_d^{\ |b),(\bar{k})} + e_{\ c}^{b),(\bar{j})}\xi_d^{(\bar{k})}\right) ,\end{aligned} \tag{5.28}$$

$$\begin{aligned}\delta e^{ab,(\bar{\imath})} &= d\xi^{ab,(\bar{\imath})} \\ &\quad +\delta^{\bar{\imath}}_{j+\bar{k}}\epsilon^{cd(a|}\left(\omega_c^{(j)}\xi_d^{\ |b),(\bar{k})} + e_c^{(\bar{k})}\Lambda_d^{\ |b),(j)} + e_{\ c}^{b),(\bar{k})}\Lambda_d^{(j)} + \omega_{\ c}^{b),(j)}\xi_d^{(\bar{k})}\right) .\end{aligned} \tag{5.29}$$

The spin-3 gauge fields appear explicitly in the equation of motion for the spin-2 fields, which are given by

$$\mathcal{T}^{a,(\bar{\imath})} \equiv de^{a,(\bar{\imath})} + \delta^{\bar{\imath}}_{j+\bar{k}}\left(\epsilon^{abc}\omega_b^{(j)} e_c^{(\bar{k})} - 4\sigma\,\epsilon^{abc} e_{bd}^{(\bar{k})}\,\omega_c^{\ d,(j)}\right) = 0 , \tag{5.30}$$

$$\begin{aligned}\mathcal{R}^{a,(i)} &\equiv d\omega^{a,(i)} + \frac{\delta^i_{j+k}}{2}\epsilon^{abc}\left(\omega_b^{(j)}\omega_c^{(k)} - 4\sigma\,\omega_{bd}^{(j)}\omega_c^{\ d,(k)}\right) \\ &\quad + \frac{\delta^i_{\bar{j}+\bar{k}}}{2}\epsilon^{abc}\left(e_b^{(\bar{j})} e_c^{(\bar{k})} - 4\sigma\, e_{bd}^{(\bar{j})}\, e_c^{\ d,(\bar{k})}\right) = 0 .\end{aligned} \tag{5.31}$$

On the other hand, the equations of motion for the spin-3 gauge fields read

$$\mathcal{T}^{ab,(\bar{\imath})} \equiv de^{ab,(\bar{\imath})} + \delta^{\bar{\imath}}_{j+\bar{k}}\,\epsilon^{cd(a|}\left(\omega_c^{(j)} e_d^{\ |b),(\bar{k})} + e_c^{(\bar{k})}\,\omega_d^{\ |b),(j)}\right) = 0 , \tag{5.32}$$

$$\mathcal{R}^{ab,(i)} \equiv d\omega^{ab,(i)} + \delta^i_{j+k}\epsilon^{cd(a|}\left(\omega_c^{(j)}\omega_d^{\ |b),(k)}\right) + \delta^i_{\bar{j}+\bar{k}}\epsilon^{cd(a|}e_c^{(\bar{j})} e_d^{\ |b),(\bar{k})} = 0 . \tag{5.33}$$

Here the delta δ^i_{p+q} is restricted to $p+q \leq m-2$ due to the non-vanishing components of the invariant tensor for the $\mathfrak{B}_m^{s3}$ symmetry. It is interesting that, as in the HS Maxwell case, the field equations of the HS fields resemble the form of the ones of in [89,90] and could be a way to realize that kind of matter couplings from an algebraic point of view.

5.2. $\mathfrak{C}_m$ gravity coupled to spin-3 fields

The same procedure can be applied in order to obtain a $\mathfrak{C}_m$ algebra coupled to spin-3 generators. Indeed, let us consider now the semigroup $S_{\mathcal{M}}^{(m-2)} = \{\lambda_0, \lambda_1, \dots, \lambda_{m-2}\}$, whose elements satisfy

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta} & \text{if } \alpha+\beta \leq m-2 \\ \lambda_{\alpha+\beta-2\left[\frac{m-1}{2}\right]} & \text{if } \alpha+\beta > m-2 \end{cases} . \tag{5.34}$$

The $S_{\mathcal{M}}^{(m-2)}$-expanded algebra has the same number of generators as the $\mathfrak{B}_m^{s_3}$ algebra,

$$\mathfrak{C}_m^{s_3} = S_{\mathcal{M}}^{(m-2)} \times \mathfrak{sl}(3,\mathbb{R}) = \left\{ M_a^{(i)}, P_a^{(\bar{\imath})}, M_{ab}^{(i)}, P_{ab}^{(\bar{\imath})} \right\} ,$$

and are related to the original ones through

$$\ell^i M_a^{(i)} = J_{(a,i)} = \lambda_i J_a , \quad \ell^i M_{ab}^{(i)} = T_{(ab,i)} = \lambda_i T_{ab} , \tag{5.35}$$

$$\ell^{\bar{\imath}} P_a^{(\bar{\imath})} = J_{(a,\bar{\imath})} = \lambda_{\bar{\imath}} J_a , \quad \ell^{\bar{\imath}} P_{ab}^{(\bar{\imath})} = T_{(ab,\bar{\imath})} = \lambda_{\bar{\imath}} T_{ab} . \tag{5.36}$$

As in the previous case, i takes even values, while $\bar{\imath}$ takes odd values. However, the $S_{\mathcal{M}}$ semigroup has no zero element, implying that the $\mathfrak{C}_m^{s_3}$ algebra has the form

$$\left[M_a^{(i)}, M_b^{(j)} \right] = \frac{\ell^{\{i+j\}}}{\ell^{i+j}} \epsilon_{abc} M^{c,\{i+j\}}, \quad \left[M_a^{(i)}, P_b^{(\bar{\imath})} \right] = \frac{\ell^{\{i+\bar{\imath}\}}}{\ell^{i+\bar{\imath}}} \epsilon_{abc} P^{c,\{i+\bar{\imath}\}}, \tag{5.37}$$

$$\left[P_a^{(\bar{\imath})}, P_b^{(\bar{j})} \right] = \frac{\ell^{\{\bar{\imath}+\bar{j}\}}}{\ell^{\bar{\imath}+\bar{j}}} \epsilon_{abc} M^{c,\{\bar{\imath}+\bar{j}\}}, \tag{5.38}$$

$$\left[M_a^{(i)}, M_{bc}^{(j)} \right] = \frac{\ell^{\{i+j\}}}{\ell^{i+j}} \epsilon^m{}_{a(b} M_{c)m}^{\{i+j\}}, \quad \left[M_a^{(i)}, P_{bc}^{(\bar{\imath})} \right] = \frac{\ell^{\{i+\bar{\imath}\}}}{\ell^{i+\bar{\imath}}} \epsilon^m{}_{a(b} P_{c)m}^{\{i+\bar{\imath}\}}, \tag{5.39}$$

$$\left[P_a^{(\bar{\imath})}, M_{bc}^{(j)} \right] = \frac{\ell^{\{\bar{\imath}+j\}}}{\ell^{\bar{\imath}+j}} \epsilon^m{}_{a(b} P_{c)m}^{\{\bar{\imath}+j\}}, \quad \left[P_a^{(\bar{\imath})}, P_{bc}^{(\bar{j})} \right] = \frac{\ell^{\{\bar{\imath}+\bar{j}\}}}{\ell^{\bar{\imath}+\bar{j}}} \epsilon^m{}_{a(b} M_{c)m}^{\{\bar{\imath}+\bar{j}\}}, \tag{5.40}$$

$$\left[M_{ab}^{(i)}, M_{cd}^{(j)} \right] = \frac{\ell^{\{i+j\}}}{\ell^{i+j}} \sigma \left(\eta_{a(c} \epsilon_{d)bm} + \eta_{b(c} \epsilon_{d)am} \right) M^{m,\{i+j\}}, \tag{5.41}$$

$$\left[M_{ab}^{(i)}, P_{cd}^{(\bar{\imath})} \right] = \frac{\ell^{\{i+\bar{\imath}\}}}{\ell^{i+\bar{\imath}}} \sigma \left(\eta_{a(c} \epsilon_{d)bm} + \eta_{b(c} \epsilon_{d)am} \right) P^{m,\{i+\bar{\imath}\}}, \tag{5.42}$$

$$\left[P_{ab}^{(i)}, P_{cd}^{(j)} \right] = \frac{\ell^{\{\bar{\imath}+\bar{j}\}}}{\ell^{\bar{\imath}+\bar{j}}} \sigma \left(\eta_{a(c} \epsilon_{d)bm} + \eta_{b(c} \epsilon_{d)am} \right) M^{m,\{\bar{\imath}+\bar{j}\}}, \tag{5.43}$$

where $\{\cdots\}$ means

$$\{i+j\} = \begin{cases} i+j & \text{if } i+j \leq m-2 \\ i+j-2\left[\frac{m-1}{2}\right] & \text{if } i+j > m-2 \end{cases} . \tag{5.44}$$

The $\mathfrak{C}_m^{s_3}$ algebra describes the coupling of spin-3 generators $\left\{ M_{ab}^{(i)}, P_{ab}^{(\bar{\imath})} \right\}$ to the spin-2 generators $\left\{ M_a^{(i)}, P_a^{(\bar{\imath})} \right\}$ of the $\mathfrak{C}_m$ Lie algebra. Let us note that the $\mathfrak{sl}(3,\mathbb{R}) \times \mathfrak{sl}(3,\mathbb{R})$ algebra

(3.5)–(3.10) is recovered for $m=3$, and $m=4$ reproduces the AdS-Lorentz Lie algebra coupled to spin-3 generators, which in turn can be rewritten as three copies of the $\mathfrak{sl}(3,\mathbb{R})$ algebra. This fact might lead one to think that the $\mathfrak{C}_m^{s3}$ algebra could be isomorphic to $m-1$ copies of $\mathfrak{sl}(3,\mathbb{R})$. However, this is an accident that occurs only for $m=3,4$, and it is not true in the generic case. The $\mathfrak{C}_m^{s3}$ algebra reduces to the $\mathfrak{B}_m^{s3}$ algebra described in the previous section when the limit $\ell\to\infty$ is considered. In such limit, $\ell^{\{p+q\}}/\ell^{p+q}\to 0$ for $p+q>m-2$, which abelianizes some commutation relations. The quadratic Casimir in this case reads

$$\begin{aligned} C = & \frac{\mu_{\{i+j\}}}{\ell^{i+j}}\left(\sum_{i,j}^{m-2} M_a^{(i)} M^{a,(j)} - \frac{1}{2\sigma} M_{ab}^{(i)} M^{ab,(j)}\right) \\ & + \frac{\mu_{\{i+\bar{\imath}\}}}{\ell^{i+\bar{\imath}}}\left(\sum_{i,\bar{\imath}}^{m-2} M_a^{(i)} P^{a,(\bar{\imath})} - \frac{1}{2\sigma} M_{ab}^{(i)} P^{ab,(\bar{\imath})}\right) \\ & + \frac{\mu_{\{\bar{\imath}+\bar{\jmath}\}}}{\ell^{\bar{\imath}+\bar{\jmath}}}\left(\sum_{\bar{\imath},\bar{\jmath}}^{m-2} P_a^{(\bar{\imath})} P^{a,(\bar{\jmath})} - \frac{1}{2\sigma} P_{ab}^{(\bar{\imath})} P^{ab,(\bar{\jmath})}\right). \end{aligned} \tag{5.45}$$

The limit $\ell\to\infty$ can also be applied at the level of the $\mathfrak{C}_m^{s3}$ Casimir operator. Nevertheless, this requires the following redefinition in the arbitrary constants,

$$\mu_{\{i+j\}} \to \ell^{\{i+j\}}\mu_{\{i+j\}}\,. \tag{5.46}$$

Then, as in the commutation relations, $\ell^{\{p+q\}}/\ell^{p+q}$ vanishes for $p+q>m-2$, which reproduces the $\mathfrak{B}_m^{s3}$ quadratic Casimir.

Let us consider the $\mathfrak{C}_m^{s3}$ connection one-form

$$A = \omega^{a,(i)} M_a^{(i)} + e^{a,(\bar{\imath})} P_a^{(\bar{\imath})} + \omega^{ab,(i)} M_{ab}^{(i)} + e^{ab,(\bar{\imath})} P_{ab}^{(\bar{\imath})}\,, \tag{5.47}$$

and non-vanishing components of the invariant tensor for the $\mathfrak{C}_m^{s3}$ algebra

$$\left\langle M_a^{(i)} M_b^{(j)}\right\rangle = \frac{\mu_{\{i+j\}}}{2\ell^{i+j}}\eta_{ab}, \quad \left\langle M_a^{(i)} P_b^{(\bar{\imath})}\right\rangle = \frac{\mu_{\{i+\bar{\imath}\}}}{2\ell^{i+\bar{\imath}}}\eta_{ab}, \quad \left\langle P_a^{(\bar{\imath})} P_b^{(\bar{\jmath})}\right\rangle = \frac{\mu_{\{\bar{\imath}+\bar{\jmath}\}}}{2\ell^{\bar{\imath}+\bar{\jmath}}}\eta_{ab}, \tag{5.48}$$

$$\left\langle M_{ab}^{(i)} M_{cd}^{(j)}\right\rangle = -\frac{\sigma}{2\ell^{i+j}}\mu_{\{i+j\}}\left(\eta_{a(c}\,\epsilon_{d)b} - \frac{2}{3}\eta_{ab}\eta_{cd}\right), \tag{5.49}$$

$$\left\langle M_{ab}^{(i)} P_{cd}^{(\bar{\imath})}\right\rangle = -\frac{\sigma}{2\ell^{i+\bar{\imath}}}\mu_{\{i+\bar{\imath}\}}\left(\eta_{a(c}\,\epsilon_{d)b} - \frac{2}{3}\eta_{ab}\eta_{cd}\right), \tag{5.50}$$

$$\left\langle P_{ab}^{(\bar{\imath})} P_{cd}^{(\bar{\jmath})}\right\rangle = -\frac{\sigma}{2\ell^{\bar{\imath}+\bar{\jmath}}}\mu_{\{\bar{\imath}+\bar{\jmath}\}}\left(\eta_{a(c}\,\epsilon_{d)b} - \frac{2}{3}\eta_{ab}\eta_{cd}\right), \tag{5.51}$$

where $\{\cdots\}$ is defined as

$$\{i+j\} = \begin{cases} i+j & \text{if } i+j \le m-2 \\ i+j-2\left[\frac{m-1}{2}\right] & \text{if } i+j > m-2 \end{cases}\,. \tag{5.52}$$

The corresponding three-dimensional CS action (2.10) is

$$S = \kappa\int \frac{\mu_{\{i+j\}}}{\ell^{i+j}}\left[\frac{1}{2}\left(\omega^{a,(i)} d\omega_a^{(j)} + \frac{\ell^{\{i+j\}}}{3\ell^{i+j}}\epsilon_{abc}\,\omega^{a,(l)}\omega^{b,(m)}\omega^{c,(n)}\,\delta_{l+m+n}^{\{i+j\}}\right)\right.$$

$$
\begin{aligned}
&-\sigma\left(\omega_{b}^{a,(i)} d\omega_{a}^{b,(j)}+\frac{2\ell^{\{i+j\}}}{\ell^{i+j}}\epsilon_{abc}\omega^{a,(l)}\omega^{bd,(m)}\omega_{d}^{c,(n)}\,\delta_{l+m+n}^{\{i+j\}}\right)\Bigg]\\
&+\frac{\mu_{\{i+\bar{\imath}\}}}{\ell^{i+\bar{\imath}}}\Bigg[e^{a,(\bar{\imath})}\Big(d\omega_{a}^{(i)}+\frac{\ell^{\{i+\bar{\imath}\}}}{2\ell^{i+\bar{\imath}}}\epsilon_{abc}\omega^{b,(m)}\omega^{c,(n)}\,\delta_{m+n}^{i}\\
&-\frac{2\ell^{\{i+\bar{\imath}\}}}{\ell^{i+\bar{\imath}}}\sigma\epsilon_{abc}\omega^{bd,(m)}\omega_{d}^{c,(n)}\delta_{m+n}^{i}\Big)\\
&-2\sigma e^{ab,(\bar{\imath})}\left(d\omega_{a}^{b,(i)}+\frac{2\ell^{\{i+\bar{\imath}\}}}{\ell^{i+\bar{\imath}}}\epsilon_{acd}\omega^{c,(m)}\omega_{b}^{d,(n)}\,\delta_{m+n}^{i}\right)\Bigg]\\
&+\frac{\mu_{\{\bar{\imath}+\bar{j}\}}}{\ell^{\bar{\imath}+\bar{j}}}\Bigg[\frac{1}{2}e^{a,(\bar{\imath})}\left(de_{a}^{(\bar{j})}+\frac{\ell^{\{\bar{\imath}+\bar{j}\}}}{\ell^{\bar{\imath}+\bar{j}}}\epsilon_{abc}\omega^{b,(m)}e^{c,(\bar{n})}\delta_{m+\bar{n}}^{\bar{j}}\right)\\
&-\sigma e^{ab,(\bar{\imath})}\Bigg(de_{ab}^{(\bar{j})}+\frac{2\ell^{\{\bar{\imath}+\bar{j}\}}}{\ell^{\bar{\imath}+\bar{j}}}\epsilon_{acd}\omega^{c,(m)}e_{b}^{d,(\bar{n})}\delta_{m+\bar{n}}^{\bar{j}}\\
&+\frac{4\ell^{\{\bar{\imath}+\bar{j}\}}}{\ell^{\bar{\imath}+\bar{j}}}\epsilon_{acd}e^{c,(\bar{m})}\omega_{b}^{d,(n)}\delta_{\bar{m}+n}^{\bar{j}}\Bigg)\Bigg].
\end{aligned}
\tag{5.53}
$$

This action describes a spin-3 extension of $\mathfrak{C}_m$ gravity and is divided in two parts. The term proportional to $\mu_{\{i+\bar{\imath}\}}$ corresponds to an Euler type CS action while the terms proportional to $\mu_{\{i+j\}}$ and $\mu_{\{\bar{\imath}+\bar{j}\}}$ describe a Pontryagin type CS action. Note that $\ell^{\{p+q\}}/\ell^{p+q}$ is trivially the identity for $p+q\leq m-2$. Then the parameter ℓ appears in the action for $p+q>m-2$. In other to apply the limit $\ell\rightarrow\infty$ in the CS action, it is necessary to use the redefinition (5.46), leading to the $\mathfrak{B}_m^{s_3}$ action (5.24). The field equations coming from the action (5.53) are

$$
\mathcal{T}^{a,(\bar{\imath})}\equiv de^{a,(\bar{\imath})}+\delta_{\{j+\bar{k}\}}^{\bar{\imath}}\left(\epsilon^{abc}\omega_{b}^{(j)}e_{c}^{(\bar{k})}-4\sigma\epsilon^{abc}e_{bd}^{(\bar{k})}\omega_{c}^{d,(j)}\right)=0\,,
\tag{5.54}
$$

$$
\begin{aligned}
\mathcal{R}^{a,(i)}\equiv{}& d\omega^{a,(i)}+\frac{\delta_{\{j+k\}}^{i}}{2}\epsilon^{abc}\left(\omega_{b}^{(j)}\omega_{c}^{(k)}-4\sigma\omega_{bd}^{(j)}\omega_{c}^{d,(k)}\right)\\
&+\frac{\delta_{\bar{j}+\bar{k}}^{i}}{2\ell^{2}}\epsilon^{abc}\left(e_{b}^{(\bar{j})}e_{c}^{(\bar{k})}-4\sigma e_{bd}^{(\bar{j})}e_{c}^{d,(\bar{k})}\right)=0\,,
\end{aligned}
\tag{5.55}
$$

$$
\mathcal{T}^{ab,(\bar{\imath})}\equiv de^{ab,(\bar{\imath})}+\delta_{\{j+\bar{k}\}}^{\bar{\imath}}\epsilon^{cd(a|}\left(\omega_{c}^{(j)}e_{d}^{\;|b),(\bar{k})}+e_{c}^{(\bar{k})}\omega_{d}^{\;|b),(j)}\right)=0\,,
\tag{5.56}
$$

$$
\mathcal{R}^{ab,(i)}\equiv d\omega^{ab,(i)}+\delta_{\{j+k\}}^{i}\epsilon^{cd(a|}\left(\omega_{c}^{(j)}\omega_{d}^{\;|b),(k)}\right)+\frac{\delta_{\bar{j}+\bar{k}}^{i}}{\ell^{2}}\epsilon^{cd(a|}e_{c}^{(\bar{j})}e_{d}^{\;|b),(\bar{k})}=0\,.
\tag{5.57}
$$

We see that, although the action (5.53) is quite similar to the $\mathfrak{B}_m^{s_3}$ one, the absence of abelian generators in the $\mathfrak{C}_m^{s_3}$ symmetry radically modifies the gravity action and therefore its dynamics. Regarding gauge transformations, they differ from the ones of the $\mathfrak{B}_m^{s_3}$ case, (5.26)–(5.29), in the Kroenecker delta δ_{p+q}^{i}, which is no more restricted to particular values of p and q.

6. Conclusions

We have constructed new spin-3 extensions of Einstein gravity in three dimensions. This has been achieved by expanding the $\mathfrak{sl}(3,\mathbb{R})$ algebra. As a warm up, we first showed how the known spin-3 extensions of the AdS and Poincaré algebras can be obtained as expansions of $\mathfrak{sl}(3,\mathbb{R})$. Using this technique, we have constructed most general invariant tensor in each case and constructed the corresponding CS actions. This reproduces the known HS gravity theories in each case plus exotic terms.

By considering more general semigroups we have constructed the spin-3 extensions of the Maxwell and AdS-Lorentz algebras. In addition, the CS actions invariant under these new HS symmetries have been studied. In the case of the Maxwell algebra, it allows us to introduce a new spin-3 theory with vanishing cosmological constant. This is a novel extension of HS three-dimensional gravity in flat space including topological HS matter. The field equations of the HS field resemble the form of the matter field equations introduced in [89,90]. Therefore the formulation here presented could be a way to realize that kind of matter couplings from an algebraic point of view. At the same time we have showed how this model can be obtained as a flat limit of the CS theory corresponding the spin-3 extension of the AdS-Lorentz algebra. Moreover, the flat limit relating such new spin-3 gravity theories works at the level of the symmetries, Casimir operators, invariant tensors and field equations.

Finally, we have generalized these results to construct two families of spin-3 algebras, denoted as $\mathfrak{C}_m^{s3}$ and $\mathfrak{B}_m^{s3}$, which contain all the previously obtained results as particular cases. These symmetries are related through the limit $\ell \to \infty$ for every value of m. The corresponding CS theories have been studied in each case, yielding new gravity models coupled to HS topological matter in three dimensions.

It would be interesting to go further and analyze the asymptotic symmetries of these new HS theories, as well as their classical duals. This could lead to enlarged versions of known $\mathcal{W}$-type algebras. In particular, in the Maxwell spin-3 case, one could expect an enlargement of the HS extension of the $\mathfrak{bms}_3$ algebra (work in progress). Another appealing aspect of these theories is their solution space. It would be interesting to study, for instance, black holes, cosmologies and conical singularities supporting HS spin matter and analyze their thermodynamics. On the other hand, in order to describe more general models, it would be worth to extend our analysis to an infinite number of interacting HS gauge fields. Another important aspect along this direction that deserves further investigation is the generalization of our results to include fermions.

Acknowledgements

The authors would like to thank to X. Bekaert and A. Campoleoni for enlightening discussions and comments. This work was supported by the Chilean FONDECYT Projects N°3170437 (P.C.), N°3170438 (E.R.) and N°3160581 (P.S-R.). R.C. and O.F. would like to thank to the Dirección de Investigación and Vice-rectoría de Investigación of the Universidad Católica de la Santísima Concepción, Chile, for their constant support.

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