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A quantum model of option pricing: When Black–Scholes meets Schrödinger and its semi-classical limit

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ABSTRACT

The Black–Scholes equation can be interpreted from the point of view of quantum mechanics, as the imaginary time Schrödinger equation of a free particle. When deviations of this state of equilibrium are considered, as a product of some market imperfection, such as: Transaction cost, asymmetric information issues, short-term volatility, extreme discontinuities, or serial correlations; the classical non-arbitrage assumption of the Black–Scholes model is violated, implying a non-risk-free portfolio. From Haven (2002) [1] we know that an arbitrage environment is a necessary condition to embedding the Black–Scholes option pricing model in a more general quantum physics setting. The aim of this paper is to propose a new Black–Scholes–Schrödinger model based on the endogenous arbitrage option pricing formulation introduced by Contreras et al. (2010) [2]. Hence, we derive a more general quantum model of option pricing, that incorporates arbitrage as an external time dependent force, which has an associated potential related to the random dynamic of the underlying asset price. This new resultant model can be interpreted as a Schrödinger equation in imaginary time for a particle of mass $1/\sigma^2$ with a wave function in an external field force generated by the arbitrage potential. As pointed out above, this new model can be seen as a more general formulation, where the perfect market equilibrium state postulated by the Black–Scholes model represent a particular case. Finally, since the Schrödinger equation is in place, we can apply semiclassical methods, of common use in theoretical physics, to find an approximate analytical solution of the Black–Scholes equation in the presence of market imperfections, as it is the case of an arbitrage bubble. Here, as a numerical illustration of the potential of this Schrödinger equation analogy, the semiclassical approximation is performed for different arbitrage bubble forms (step, linear and parabolic) and compare with the exact solution of our general quantum model of option pricing.

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> "The purpose of my lecture today is to explain why, in my view, the prevailing theory of value – what I called, in a shorthand way, "equilibrium economics" – is barren and irrelevant as an apparatus of thought to deal with the manner of operation of economics forces, or as an instrument for non-trivial predictions concerning the effects of economic changes, whether induced by political action or by other causes. I should go further and say that the powerful attraction of the habits of thought engendered by "equilibrium economics" has became a major obstacle to the development of economics as a *science*-meaning by the term "science" a body of theorems based on assumptions that are *empirically* derived (from observations) and which embody hypotheses that are capable of verification both in regard to the assumptions and the predictions."
>
> Kaldor, N (1972) "The Irrelevance of Equilibrium Economics", [3]

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1. Introduction

Since this visionary statement of Kaldor in the early 1970s, many of the branches of economics have moved away from this narrow concept of equilibrium. Examples of this could be found in international economics, endogenous growth theory, game theory, labor economics, environmental economics, among others, see for a review Ref. [4]. The field of industrial organization has summarized this disequilibrium view in economics through a model of the firm, that considers the conjectural-variations approach, see Ref. [5]. This model allows economists to empirically estimate the structure of an industry, or in other words, its level of competition and hence efficiency. In this context, the equilibrium view of perfect markets is now an empirical issue instead of a dogmatic one, and more importantly the perfect market assumption is just one of the potential states of several others, like monopoly and oligopoly.

The main question that arises from this methodological change in economics is why the notion of equilibrium is still dominant in finance, specially regarding the recent financial crisis, where the assumptions of a perfect informed and competitive market were clearly and systematically violated during many years. Indeed, since the 80's economists have realized that, in a real market, futures contracts are not always traded at the price predicted by the simple no-arbitrage relation. Strong empirical evidence have supported this point many times and in different settings, see for example: Refs. [6–8] among others. However, economists have tended to develop several alternative explanations for the variability of the arbitrage, such as: differential tax treatment for spots and futures [9], and marking-to-market requirements for futures, [10]. It was also noted that there are certain factors that influence the arbitrage strategies and slow down the market's reaction on the arbitrage. The factors include constrained capital requirements [11], position limits, and transaction costs [12].

In this paper, we are not interested in explaining why these *market imperfections* are produced, but which are the effects of arbitrage on the option pricing dynamics. We know that the Black–Scholes equation can be interpreted from the point of view of quantum mechanics, as the imaginary time Schrödinger equation of a free particle. When deviations of this state of equilibrium are considered, as a product of some market imperfection, such as: transaction cost, asymmetric information issues, short-term volatility, extreme discontinuities, or serial correlations; the classical non-arbitrage assumption of the Black–Scholes model is violated, implying a non-risk-free portfolio. From Ref. [1] we know that an arbitrage environment is a necessary condition to embedding the Black–Scholes option pricing model in a more general quantum physics setting.

In this context, the main aim of this paper is to propose a new Black–Scholes–Schrödinger model based on the endogenous arbitrage option pricing formulation introduced by Ref. [2]. Thus, we will derive a more general quantum model of option pricing, that incorporates arbitrage as an external time dependent force, which has an associated potential related to the random dynamic of the underlying asset price. This new model could be seen as a more general formulation, where the perfect market equilibrium state postulated by the Black–Scholes model represents a particular case. The second aim of this research is to apply a semiclassical approximation of our Black–Scholes–Schrödinger model. Finally, given our semiclassical solution, we will test several arbitrage bubble functional forms, or in physical terms different potentials.

This paper is structured as follow. First, our disequilibrium (arbitrage) model is presented and discussed in the light of quantum physics. Second, the new Black–Scholes–Schrödinger model is developed and interpreted as a Schrödinger equation in imaginary time for a particle with a wave function in an external time dependent field force generated by the potential. This new model has, as particular case, the perfect market equilibrium of the Black–Scholes model. Third, since the Schrödinger equation is in place, we will apply semiclassical methods of common use in theoretical physics to find an approximate analytical solution of the Black–Scholes equation in the presence of market imperfections, as in the case of an arbitrage bubble. From this approximation, it is quite obvious that the perfect market equilibrium state postulated by the Black–Scholes model represents a particular case of a more general quantum model. As in the case of industrial organization mentioned above, we will have a parameter that will allow us to model the size (importance) of the disequilibrium in the market. Finally, some numerical illustrations of the potential of this Schrödinger equation analogy, the semiclassical approximations are compared for different arbitrage bubble forms (step, linear and parabolic) and with the exact solution of our general quantum model of option pricing.

2. The basic disequilibrium financial model[1]

The Black–Scholes (**B–S**) model gives the dynamic of the option prices in financial markets. This model has a long life in finance and has been widely used since the seventies. In its usual arbitrage free version, the **B–S** equation is given by

$$\frac{\partial \pi}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \pi}{\partial S^2} + r\left(S\frac{\partial \pi}{\partial S} - \pi\right) = 0 \tag{1}$$

where $\pi = \pi(S, t)$ represents the option price as a function of the underlying asset price S and time t. Changing coordinates, as we will see later, the **B–S** equation can be mapped into the heat equation

$$\frac{\partial \pi}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 \pi}{\partial x^2} = 0. \tag{2}$$

1 This section relies strongly on Ref. [2].

By doing a Wick rotation in time $t = -i\tau$, the heat equation maps into

$$i\frac{\partial \pi}{\partial \tau} = -\frac{1}{2}\sigma^2 \frac{\partial^2 \pi}{\partial x^2}. \quad (3)$$

This last equation is the Schrödinger equation of a free particle with $\hbar = 1$ and mass $m = \frac{1}{\sigma^2}$. From this perspective, the **B–S** model can be interpreted from the point of view of quantum mechanics as a dynamic model without interactions. In what follows we will use both approaches indistinctly.

As it is well known, the **B–S** model for an equity rests on several very restrictive assumptions, such as: (i) the price of the underlying instrument follows a geometric Brownian motion, with constant drift μ and volatility σ (a lognormal random walk), (ii) it is possible to short sell the underlying stock, (iii) there are no dividends on the underlying, (iv) trading in the stock is continuous, in other words delta hedging is done continuously, (v) there are no transaction costs or taxes, (vi) all securities are perfectly divisible (it is possible to buy any fraction of a share), (vii) it is possible to borrow and lend at a constant risk-free interest rate r, (viii) there are no arbitrage opportunities.

In analytic terms, if $B(t)$ and $S(t)$ are the risk-free asset and underlying stock prices, the price dynamics of the bond and the stock in this model are given by the following equations:

$$\begin{aligned} dB(t) &= rB(t)dt \\ dS(t) &= \alpha S(t)dt + \sigma S(t)dW(t) \end{aligned} \quad (4)$$

where r, α and σ are constants and $W(t)$ is a Wiener process.

In order to price the financial derivative, it is assumed that it can be traded, so we can form a portfolio based on the derivative and the underlying stock (no bonds are included). Considering only non-dividend paying assets and no consumption portfolios, the purchase of a new portfolio must be financed only by selling from the current portfolio.

Calling $\vec{h}(t) = (h_S, h_\Pi)$ the portfolio, $\vec{P}(t) = (S, \Pi)$ the price vector of shares and $V(t)$ the value of the portfolio at time t; the dynamic of a self-financing portfolio with no consumption is given by

$$dV(t) = \vec{h}(t)d\vec{P}(t). \quad (5)$$

In other words, in a model without exogenous incomes or withdrawals, any change of value is due to changes in asset prices.

Another important assumption for deriving the **B–S** equation is that the market is efficient in the sense that is free of arbitrage possibilities. This is equivalent with the fact that there exists a self-financed portfolio with value process $V(t)$ satisfying the dynamic:

$$dV(t) = rV(t)dt \quad (6)$$

which means that any locally risk-less portfolio has the same rate of return as the bond.

Some efforts have been done to relax some of the above assumptions, in particular, the free arbitrage condition. A model that includes arbitrage possibilities in a explicit way, is the gauge field theoretical model developed by Ilinski [13], Ilinski–Stepanenko [14,15]. Another distinct effort in this direction is Otto [16], who reformulated the original **B–S** model through a stochastic interest rate. Finally, Panayides [17] and Fedotov and Panayides [18] follow an approach suggested by Papanicolaou and Sircar [19], where option pricing with stochastic volatility is modeled.

In Ref. [2] we proposed a generalization (called the **CPV**-model) of the Black–Scholes model including arbitrage possibilities, but in a endogenous way. For this **CPV**-model, the usual equation used in the arbitrage-free hypothesis is changed into

$$dV(t) = rV(t)dt + f(S,t)V(t)dW(t) \quad (7)$$

where f is any function in (S,t) and W is the same Wiener process driving the random dynamic of the underlying asset price S. The validity of this maintained hypothesis has been tested and validated empirically, see for instance Ref. [8].

The function $f = f(S,t)$ can be interpreted as the amplitude of an arbitrage bubble. This bubble could be produced by several reasons, such as: transaction cost, asymmetric information issues, short-term volatility, extreme discontinuities, or serial correlations; but these *market imperfections* and their causes are not the main interest of this research, but their effects on the option pricing dynamics are.

The Black–Scholes equation in the presence of this bubble holds to be

$$\frac{\partial \pi}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \pi}{\partial S^2} + r\frac{(\sigma - \frac{\alpha f}{r})}{(\sigma - f)}\left(S\frac{\partial \pi}{\partial S} - \pi\right) = 0 \quad (8)$$

which is

$$\check{L}_{BS}\pi + \frac{(r-\alpha)f(S,t)}{\sigma - f(S,t)}\left(S\frac{\partial \pi}{\partial S} - \pi\right) = 0 \quad (9)$$

where

$$\check{L}_{BS}\pi = \frac{\partial \pi}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \pi}{\partial S^2} + r\left(S\frac{\partial \pi}{\partial S} - \pi\right) \tag{10}$$

is the usual arbitrage free **BS** operator. The factor

$$\mathcal{V}(S,t) = \frac{(r-\alpha)f(S,t)}{\sigma - f(S,t)} \tag{11}$$

can be interpreted as an effective potential induced by the arbitrage bubble $f(S,t)$. In this way, the presence of arbitrage generates an external time dependent force, which have an associated potential $\mathcal{V}(S,t)$. Then the **CPV**-model developed in Ref. [2] corresponds, from a physics point of view, to an interacting particle with an external field force. It is important to mention that, as the **BS** one, this equation belongs to the Backward Kolmogorov PDE family. Obviously, when arbitrage disappears, the external potential is zero and we recover the usual **BS** dynamic. We can see too, that the option price dynamic $\pi(S,t)$ depends explicitly on the arbitrage bubble form $f(S,t)$. From a financial optic, the arbitrage bubbles should be time-finite lapses and they should have a characteristic amplitude. So, in general, arbitrage bubbles can be defined by three parameters: the born-time, dead-time and the maximum amplitude between these two times. In this work we find an approximate analytical solution for the **BS** equation with arbitrage, for an arbitrary arbitrage bubble form and compare this solution with a numerical simulation (via the Crank–Nicolson method) of the exact interacting solution for three cases: step, linear and parabolic time bubble forms.

3. The Black–Scholes model and the Schrödinger equation revisited

There has been only a few papers on the issue of applying more general elements of quantum physics on finance, specifically in applying a more general version of the Schrödinger differential equation. For some early and very important attempts see Refs. [20,21], which primarily through information and uncertainty issues trying to obtain a Black–Scholes–Schrödinger model. It is important to mention also Ref. [8], that used Gauge theory to explain non-equilibrium pricing, deriving a Schrödinger equation to deal with the money dynamic flow for a single investor. Finally, as pointed out above, [1] is the paper that sets the necessary conditions to embedding the **BS** option pricing model in a more general quantum physics setting. Perhaps one of the reasons of this few attempts has been the lack of consistent disequilibrium (arbitrage) models of option pricing. In the next section, a new Black–Scholes–Schrödinger model based on the endogenous arbitrage option pricing formulation introduced by Contreras et al. [2] is proposed. We will show also that this model satisfies the conditions stated by Haven, Which in turn complies with the Schrödinger differential equation. Finally, in order to develop a semiclassical approximation we need to study first the classical mechanics of our Black–Scholes–Schrödinger model.

3.1. The CPV model as a Black–Scholes–Schrödinger model

Consider again the **CPV** equation with arbitrage

$$\frac{\partial \pi}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \pi}{\partial S^2} + r\frac{\left(\sigma - \frac{\alpha f}{r}\right)}{(\sigma - f)}\left(S\frac{\partial \pi}{\partial S} - \pi\right) = 0 \tag{12}$$

taking the variable change $\xi = \ln S$, we obtain

$$\frac{\partial \pi}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 \pi}{\partial \xi^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial \pi}{\partial \xi} + \frac{(r-\alpha)f}{\sigma - f}\left(\frac{\partial \pi}{\partial \xi} - \pi\right) = 0 \tag{13}$$

and if we make a second (time dependent) change of variables

$$x = \xi - \left(r - \frac{1}{2}\sigma^2\right)t \tag{14}$$

we arrive to

$$\frac{\partial \pi}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 \pi}{\partial x^2} - r\pi + \frac{(r-\alpha)\check{f}}{\sigma - \check{f}}\left(\frac{\partial \pi}{\partial x} - \pi\right) = 0 \tag{15}$$

where

$$\check{f}(x,t) = f\left(e^{x + (r - \frac{1}{2}\sigma^2)t}, t\right) \tag{16}$$

Proposition 3.1. *Given the **CPV**-model in (12) for the price of an option with arbitrage, if we define*

$$\pi(x,t) = e^{-r(T-t)}\psi(x,t) \tag{17}$$

the ψ dynamic is given by

$$\frac{\partial\psi(x,t)}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2\psi(x,t)}{\partial x^2} + \upsilon(x,t)\left(\frac{\partial\psi(x,t)}{\partial x} - \psi(x,t)\right) = 0 \tag{18}$$

where

$$\upsilon(x,t) = \frac{(r-\alpha)\check{f}(x,t)}{\sigma - \check{f}(x,t)}. \tag{19}$$

The last two equations can be interpreted as a Schrödinger equation in imaginary time for a particle of mass $1/\sigma^2$ with wave function $\psi(x,t)$ in an external time dependent field force generated by $\upsilon(x,t)$. Note that for high-volatility markets the particle has less inertia and can accelerate more. If we write the Schrödinger equation as

$$\frac{\partial\psi(x,t)}{\partial t} = \check{H}\psi(x,t) \tag{20}$$

we can read the Hamiltonian operator as

$$\check{H} = -\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} - \upsilon(x,t)\left(\frac{\partial}{\partial x} - I\right). \tag{21}$$

But momentum operator in imaginary time is

$$\check{P} = -\frac{\partial}{\partial x}, \qquad \check{P}^2 = \frac{\partial^2}{\partial x^2} \tag{22}$$

so the Hamiltonian as a function of the momentum operator becomes

$$\check{H} = -\frac{1}{2}\sigma^2\check{P}^2 + \upsilon(x,t)(\check{P} + I). \tag{23}$$

3.2. *On Haven's interpretation of the arbitrage state function*

We state here that the "cartesian" (x,t) space is the natural and correct one to interpret the Black–Scholes as a Schrödinger equation. Some characteristic behavior naturally appears in this coordinate system, as that the interaction potential is retarded (in analogous way with the retarded potentials of classical electrodynamics) as Ref. [1] pointed out, according to the idea that information in the financial markets propagate with finite velocity. In fact we can see from the potential

$$\upsilon(x,t) = \frac{(r-\alpha)f\left(e^{x+(r-\frac{1}{2}\sigma^2)t}, t\right)}{\sigma - f\left(e^{x+(r-\frac{1}{2}\sigma^2)t}, t\right)} \tag{24}$$

that the interaction propagates with speed given by $r - \frac{\sigma^2}{2}$ when the bubble depends on S explicitly. In this "cartesian" system the **BS** model is the model for a quantum free particle (a static model, no forces are present), and when arbitrage appears, it produces a force that changes the usual free dynamic of the **BS** particle. In fact in Ref. [1] the author developed an embedding of the **BS** option price model in a quantum physic setting. As a result, this author found several claims that a general model for option pricing would satisfy. We show here that our interacting theory satisfies all of them if we associate the **BS** equation with the Schrödinger equation in the (x,t) space. See Ref. [1] for the claims.

Claim 3: when the state function is the solution of the **BS** model there are no arbitrage. Obviously only when the bubble amplitude f is null (no arbitrage), the interaction potential is zero too, and our model is the usual non-interacting **BS** equation.

Claim 4: When the state function exists as non-collapsed state function, it is not the solution of the **BS** model and there is arbitrage. Again, if the bubble amplitude $f \neq 0$ (non-null arbitrage), the interaction potential in non-zero too, and our model is the interacting **BS** equation, whose solution is not the free **BS** solution.

Claim 5: We interpret this claim in the (x,t) space. For a time-independent arbitrage bubble $f = f(S)$ the potential $\upsilon(x,t)$ depends on the combination $x + (r - \frac{1}{2}\sigma^2)t$, so we identify the arbitrage parameter $\aleph$ in Ref. [1] as $\aleph = (r - \frac{1}{2}\sigma^2)t$. For a market with higher volatility σ, the arbitrage parameter $\aleph$ becomes higher too.

Claim 6: We interpret this claim with the asymptotic behavior of the potential $\upsilon(x,t)$ for higher bubble amplitudes. In fact, when f goes to infinity the potential goes to α, the constant mean value of the underlying asset.

Claim 7: The interpretation of this claim is obscure, because strictly in the **BS** model there is not a Planck constant at all. In fact the Schrödinger equation obtained by our analogy in the (x,t) space has Planck constant equal to 1. The parameter that can play the role of the Planck constant is the volatility, because this appears in the potential, so we must expand the propagator in $\frac{1}{\sigma}$ in analogous way that the expansion of the propagator in $\frac{1}{\hbar}$ in the interaction picture of quantum mechanics.

3.3. Classical mechanics of this new Black–Scholes–Schrödinger model

If we take the classical limit "$\hbar \longrightarrow 0$" the quantum Hamiltonian becomes the classical hamiltonian function

$$\mathcal{H}(x,P) = -\frac{1}{2}\sigma^2 P^2 + \upsilon(x,t)(P+1). \tag{25}$$

The classical Hamiltonian equations

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial P}, \qquad \dot{P} = -\frac{\partial \mathcal{H}}{\partial x} \tag{26}$$

reduces in this case to

$$\dot{x} = -\sigma^2 P + \upsilon(x,t) \tag{27}$$

$$\dot{P} = -(r-\alpha)(P+1)\frac{\sigma \frac{\partial \check{f}}{\partial x}}{(\sigma - \check{f})^2}. \tag{28}$$

The corresponding Lagrangian

$$\mathcal{L} = P\dot{x} - \mathcal{H}(x,P) \tag{29}$$

becomes

$$\mathcal{L} = -\frac{1}{2\sigma^2}\left(\dot{x} - \upsilon(x,t)\right)^2 - \upsilon(x,t). \tag{30}$$

The Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0 \tag{31}$$

gives for this system, the following Newton equation

$$\ddot{x} - \frac{\partial \upsilon}{\partial t} - \left[\upsilon(x,t) + \sigma^2\right]\frac{\partial \upsilon}{\partial x} = 0. \tag{32}$$

We can consider here some special cases in detail.

The time-independent arbitrage model

First, if the bubble depends only on S, that is $f = f(S)$, this implies that

$$\check{f}(x,t) = f\left(\mathrm{e}^{x + (r - \frac{1}{2}\sigma^2)t}\right) \tag{33}$$

and we have in this case the identity

$$\frac{\partial \upsilon}{\partial t} = \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial \upsilon}{\partial x} \tag{34}$$

so the equivalent Newton equation reads

$$\ddot{x} - \frac{\partial}{\partial x}\left[\frac{\upsilon^2(x,t)}{2}\right] - \sigma^2\left(\frac{\partial \upsilon}{\partial x}\right) = \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial \upsilon}{\partial x} \tag{35}$$

or

$$\ddot{x} - \frac{\partial}{\partial x}\left[U(x,t)\right] = 0 \tag{36}$$

where

$$U(x,t) = \frac{\upsilon^2(x,t)}{2} + \left(\frac{1}{2}\sigma^2 + r\right)\upsilon(x,t). \tag{37}$$

The time-dependent arbitrage model

In the second case, the arbitrage bubble depends only on the time coordinate $f(s,t) = f(t)$ so

$$\check{f}(x,t) = f\left(e^{x-(r-\frac{1}{2}\sigma^2)t}, t\right) = f(t) \tag{38}$$

and

$$\upsilon(x,t) = \frac{(r-\alpha)f(t)}{\sigma - f(t)} = \upsilon(t) \tag{39}$$

so

$$\frac{\partial \upsilon}{\partial x} = 0. \tag{40}$$

The Euler–Lagrange equation now reads

$$\frac{d}{dt}(\dot{x} - \upsilon(t)) = 0 \tag{41}$$

that is

$$\dot{x} = C + \upsilon(t) \tag{42}$$

which can be easily integrated as

$$x(t) = Ct + \int \frac{(r-\alpha)f(t)}{\sigma - f(t)} dt + D \tag{43}$$

where C and D are arbitrary constants. In what follows, we consider an arbitrage bubble that is only time dependent, $f(S,t) = f(t)$. Note that in this case and for a time step function bubble

$$f(t) = \begin{cases} 0 & t < T_1 \\ f_0 & T_1 < t < T_2 \\ 0 & T_2 < t \end{cases} \tag{44}$$

the classical **BS** particle changes discontinuously its velocity when it reaches the bubble and recovers the original one once it leaves the bubble, in analogous way to the trajectory of a ray of light in a glass. So the bubble behaves as a glass but in time (not in space as usual).

4. The Black–Scholes–Schrödinger semiclassical limit

Semiclassical methods have been used to find approximate solutions of the Schrödinger equation in different areas of theoretical physics, such as nuclear physics [22], quantum gravity [23], chemical reactions [24], quantum field theory [25] and path integrals [26]. When the system has interactions, the semiclassical approach gives an approximate solution for the wave function of the system, while for the free interaction case, the semiclassical approximation can give exact results [27]. In this section, we develop the first financial application, based on our Black–Scholes–Schrödinger model.

4.1. The semiclassical approximation

The solution of the Schrödinger equation can be written as

$$\psi(x,t) = \int_{-\infty}^{\infty} \left(xt|x'T\right) \Phi(x') dx' \tag{45}$$

where $\Phi(x)$ is a specific contract (Call, Put, Binary Call...) in the x space, and $(xt|x'T)$ is the propagator which admits the path integral representation

$$\left(xt|x'T\right) = \int Dx(t) e^{S[x(t)]}. \tag{46}$$

The last integral is done over all paths connecting the points $x(t) = x$ and $x(T) = x'$. If we write $x(t)$ as $x(t) = x_{class}(t) + \eta(t)$, the propagator becomes

$$\left(xt|x'T\right) = \int D\eta(t) e^{S[x_{class}(t)+\eta(t)]} \tag{47}$$

if we expand the action around the classical path, we have

$$S\left[x_{class}(t)+\eta(t)\right]=S\left[x_{class}(t)\right]+\frac{\delta S[\eta]}{\delta\eta}\eta+\frac{1}{2}\frac{\delta^2 S[\eta]}{\delta\eta^2}\eta^2+\cdots \tag{48}$$

where all functional derivatives are evaluated on the classical path. Due to the action which is extremal for the classical path, we get for the propagator

$$\left(xt|x'T\right)=\mathrm{e}^{S[x_{class}(t)]}\int D\eta(t)\mathrm{e}^{\frac{1}{2}\frac{\delta^2 S[\eta]}{\delta\eta^2}\eta^2+\cdots} \tag{49}$$

and the contribution of the second order terms is (see for example Ref. [26])

$$\frac{1}{\sqrt{2\pi\sigma^2 H(t)}} \tag{50}$$

where the function $H(t)$ satisfies the differential equation

$$\frac{\partial^2 H(t)}{\partial t^2}=\frac{1}{2}\mathcal{V}''\left(x_{class}(t),t\right)H(t) \tag{51}$$

with the initial conditions

$$H(x,t_0)=0,\qquad \left.\frac{\mathrm{d}H(t)}{\mathrm{d}t}\right|_{t_0}=1. \tag{52}$$

When the arbitrage amplitude does not depend on $S, f(S,t)=f(t)$, then $\mathcal{V}''(t)=0$ and $H(t)$ is given by

$$H(t)=t-t_0 \tag{53}$$

so the second order contribution gives in this case

$$\frac{1}{\sqrt{2\pi\sigma^2(T-t)}}. \tag{54}$$

Finally the semiclassical approximation for the propagator of $\psi(x,t)$ up to second order terms is

$$\left(xt|x'T\right)=\frac{\mathrm{e}^{S[x_{class}(t)]}}{\sqrt{2\pi\sigma^2(T-t)}}. \tag{55}$$

The solution for the option price in the x space is then

$$\pi(x,t)=\mathrm{e}^{(-r(T-t))}\psi(x,t)=\int_{-\infty}^{\infty}\mathrm{e}^{(-r(T-t))}\left(xt|x'T\right)\Phi(x')\mathrm{d}x' \tag{56}$$

so the propagator for the option price is, in the semiclassical approximation

$$\left(xt|x'T\right)_{SC}=\mathrm{e}^{(-r(T-t))}\frac{\mathrm{e}^{S[x_{class}(t)]}}{\sqrt{2\pi\sigma^2(T-t)}}. \tag{57}$$

4.2. The semiclassical solution

In order to find the semiclassical approximation for the option price, in the presence of a time dependent arbitrage bubble $f=f(t)$, we must integrate the equation of motion

$$\dot{x}=C+v(t) \tag{58}$$

with the initial condition $x(t)=x$ and final condition $x(T)=x'$. This gives the equation

$$x'-x=C(T-t)+\int_t^T v(u)\mathrm{d}u \tag{59}$$

where the constant C can be calculated as

$$C=\frac{x'-x}{T-t}-\frac{1}{T-t}\int_t^T v(u)\mathrm{d}u. \tag{60}$$

If we replace the classical solution in the Lagrangian we obtain

$$L = -\frac{1}{2\sigma^2}C^2 - v(t) \tag{61}$$

and the action evaluated over the classical path is then

$$S[x_{class}] = -\frac{1}{2\sigma^2}C^2(T-t) - \int_t^T v(u)\mathrm{d}u \tag{62}$$

which is equal to

$$S[x_{class}] = -\frac{1}{2\sigma^2(T-t)}\left[(x'-x) - \rho(t,T)\right]^2 - \rho(t,T) \tag{63}$$

where

$$\rho(t,T) = \int_t^T v(u)\mathrm{d}u. \tag{64}$$

The semiclassical propagator in the x space is then

$$\left(xt|x'T\right)_{SC} = \frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}}\mathrm{e}^{-\frac{1}{2\sigma^2(T-t)}\left[(x'-x)-\rho(t,T)\right]^2-\rho(t,T)} \tag{65}$$

and the semiclassical solution is, in an explicit way

$$\pi_{SC}(x,t) = \int_{-\infty}^{\infty}\frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}}\mathrm{e}^{-\frac{1}{2\sigma^2(T-t)}\left[(x'-x)-\rho(t,T)\right]^2-\rho(t,T)}\Phi(x')\mathrm{d}x'. \tag{66}$$

We can now write the propagator in the $S - t$ space. Because

$$x = \ln(S) - \left(r - \frac{1}{2}\sigma^2\right)t \tag{67}$$

and

$$x' = \ln(S') - \left(r - \frac{1}{2}\sigma^2\right)T \tag{68}$$

we have

$$x' - x = -\left[\ln(S/S') + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right] \tag{69}$$

and $\mathrm{d}x = \mathrm{d}S/S$ the semiclassical propagator in the S space is

$$\left(St|S'T\right)_{SC} = \frac{1}{S'}\frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}}\mathrm{e}^{-\frac{1}{2\sigma^2(T-t)}\left[\ln(S/S')+(r-\frac{1}{2}\sigma^2)(T-t)+\rho(t,T)\right]^2-\rho(t,T)} \tag{70}$$

which can be written as

$$\left(St|S'T\right)_{SC} = \frac{1}{S'}\frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}}\frac{1}{\mathrm{e}^{\rho}}\mathrm{e}^{-\frac{1}{2\sigma^2(T-t)}\left[\ln(\mathrm{e}^{\rho}S/S')+(r-\frac{1}{2}\sigma^2)(T-t)\right]^2} \tag{71}$$

but the Black–Scholes propagator is ($\rho = 0$)

$$\left(St|S'T\right)_{BS} = \frac{1}{S'}\frac{\mathrm{e}^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}}\mathrm{e}^{-\frac{1}{2\sigma^2(T-t)}\left[\ln(S/S')+(r-\frac{1}{2}\sigma^2)(T-t)\right]^2} \tag{72}$$

so we see that

$$\left(St|S'T\right)_{SC} = \frac{1}{\mathrm{e}^{\rho}}\left(\mathrm{e}^{\rho}St|S'T\right)_{BS} \tag{73}$$

the semiclassical solution is given then by

$$\pi_{SC}(S,t) = \int_{-\infty}^{\infty}\left(St|S'T\right)_{SC}\Phi(S')\mathrm{d}S' \tag{74}$$

then

$$\pi_{SC}(S,t) = \frac{1}{\mathrm{e}^{\rho}}\int_{-\infty}^{\infty}\left(\mathrm{e}^{\rho}St|S'T\right)_{BS}\Phi(S')\mathrm{d}S' \tag{75}$$

so we arrive at the main result of this section.

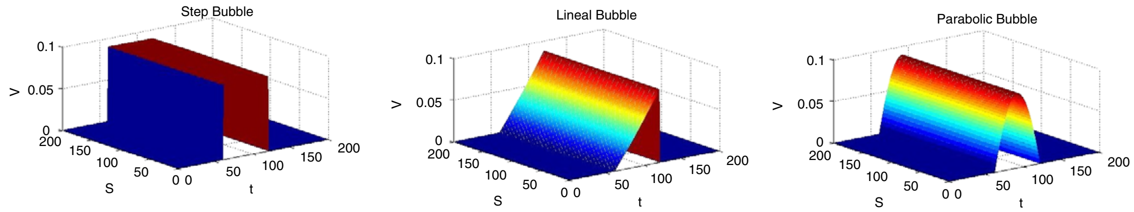

Fig. 1. Step, linear and parabolic bubbles.

Proposition 4.1. *The semiclassical approximation for the **CPV**-model for the option price, in presence of a time dependent arbitrage bubble $f = f(t)$ can be computed as*

$$\pi_{SC}(S,t) = \frac{1}{e^{\rho(t,T)}}\pi_{BS}\left(e^{\rho(t,T)}S, t\right) \tag{76}$$

where $\pi_{BS}(S,t)$ is the arbitrage-free Black–Scholes solution for the specific option with contract $\Phi(S)$.

In this way, the function $\rho(t,T)$ renormalizes the bare arbitrage-free **BS** solution. One important fact of this last equation is that it permits one to obtain an approximation of our Black–Scholes–Schrödinger model from the classical **BS** model, by means of a rescaling of the price variable, so the usual computational codes can be easily modified to obtain an approximation for the interacting model. On the other hand, the economic equilibrium represented by the traditional **BS** model has proven over the years to be a very good starting point of the price trajectory, especially in the deepest and more developed future markets.

5. Some numerical illustrations

In order to explore the behavior of the semiclassical solution we make an analysis for three different bubble forms $f(t)$: step, linear and parabolic functions as shown in Fig. 1.

We use as contract function $\Phi(S)$ those of a binary put

$$\Phi(S) = \begin{cases} 0 & 0 < S < K \\ 1 & K < S \end{cases} \tag{77}$$

for which the pure **BS** solution $\pi_{BS}(S,t)$ is given by

$$\pi_{BS}(S,t) = e^{-r(T-t)}\left[1 - N(d_2(S,t))\right] \tag{78}$$

where

$$d_2(S,t) = \frac{\ln\frac{S}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{(T-t)}} \tag{79}$$

and $N(x)$ is the normal distribution function $N(0,1)$.

5.1. Time step arbitrage bubble

In the case of a time step bubble, given by

$$f(t) = \begin{cases} 0 & 0 < t < T_1 \\ H & T_1 < t < T_2 \\ 0 & T_2 < t < T \end{cases} \tag{80}$$

we obtain for the ρ factor

$$\rho(t,T) = \begin{cases} (T_2 - T_1)\,\dfrac{(r-\alpha)H}{\sigma - H} & 0 < t < T_1 \\[2ex] (T_2 - t)\,\dfrac{(r-\alpha)H}{\sigma - H} & T_1 < t < T_2 \\[2ex] 0 & T_2 < t < T. \end{cases}$$

We analyze a binary put for $\alpha = -0.6$, $\sigma = 0.5$. The figures show, from left to right, the semiclassical approximation for the option price, the option price dynamics as given by a numerical integration of **BS** equation, and finally their difference, for several values of the arbitrage bubble amplitude H. We use $T_1 = 0.3$ and $T_2 = 0.6$ for numerical purposes (Figs. 2–5).

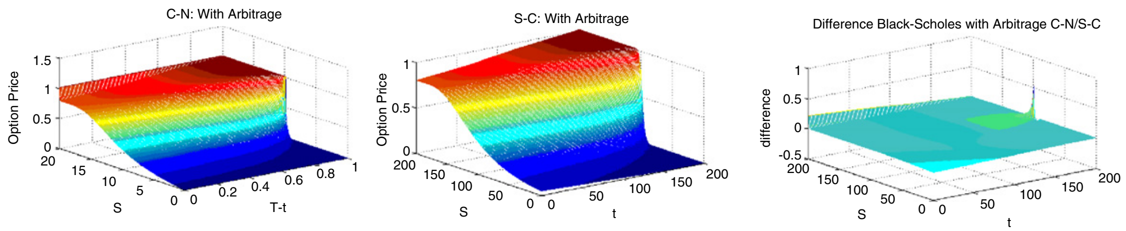

Fig. 2. $H = 0.1\,\sigma$.

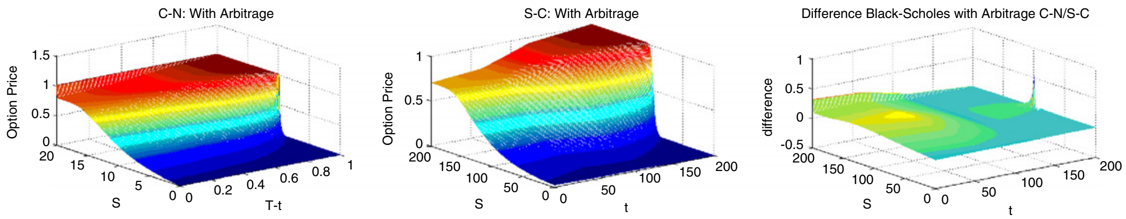

Fig. 3. $H = 0.4\,\sigma$.

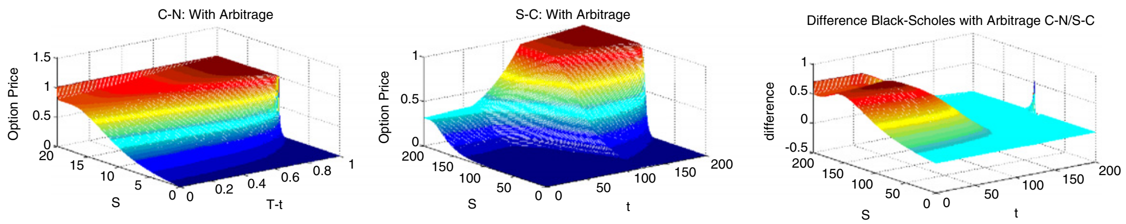

Fig. 4. $H = 0.8\,\sigma$.

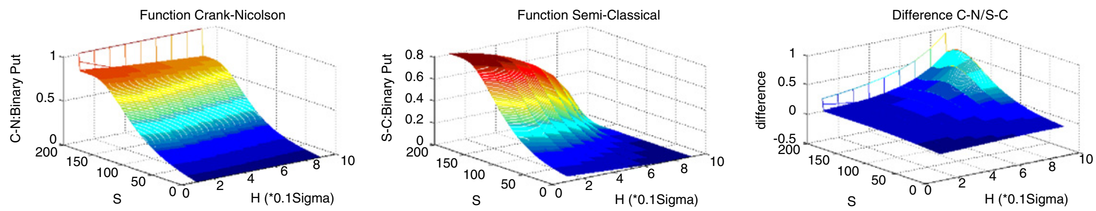

Fig. 5. Difference between exact and semi-classical option prices vs. H for the step bubble.

5.2. Time linear arbitrage bubble

In the case of the linear function

$$f(t) = \begin{cases} 0 & 0 < t < T_1 \\ \dfrac{H}{(T_2 - T_1)}(t - T_1) & T_1 < t < T_2 \\ 0 & T_2 < t < T \end{cases} \tag{81}$$

the $\rho(t, T)$ factor is given in this case by

$$\rho(t) = \begin{cases} \dfrac{(r-\alpha)(T_2 - T_1)}{H}\left[\sigma \ln\left(\dfrac{\sigma}{\sigma - H}\right) - H\right] & 0 < t < T_1 \\ \dfrac{(r-\alpha)(T_2 - T_1)}{H}\left[\sigma \ln\left(\dfrac{H(t - T_1) - \sigma(T_2 - T_1)}{(T_2 - T_1)(H - \sigma)}\right) - \dfrac{H(T_2 - t)}{T_2 - T_1}\right] & T_1 < t < T_2 \\ 0 & T_2 < t < T. \end{cases} \tag{82}$$

For this case the semiclassical approximation and the numerical integration are given by the figures below. Again we use $T_1 = 0.3$ and $T_2 = 0.6$ (Fig. 6).

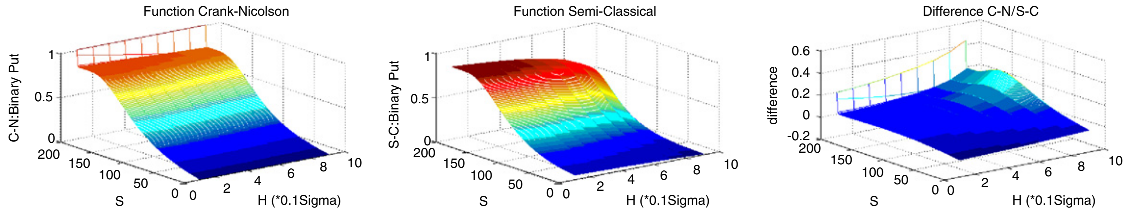

Fig. 6. Difference between exact and semi-classical option prices vs. H for the linear bubble.

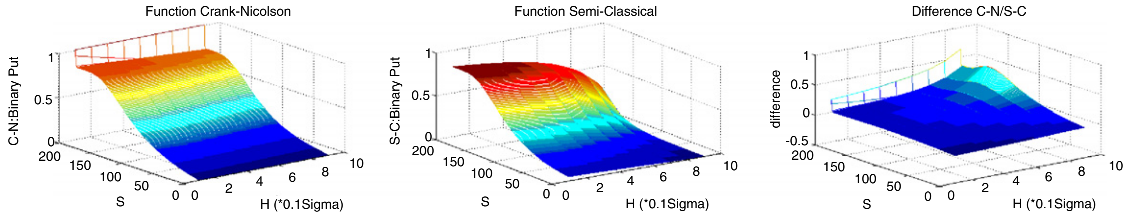

Fig. 7. Difference between exact and semi-classical option prices vs. H for the parabolic bubble.

5.3. Time parabolic arbitrage bubble

In the case of the parabolic arbitrage given by

$$f(t) = \begin{cases} 0 & 0 < t < T_1 \\ -\dfrac{4H}{(T_2 - T_1)^2}(t - T_1)(t - T_2) & T_1 < t < T_2 \\ 0 & T_2 < t < T \end{cases} \tag{83}$$

we can compute also the $\rho(t, T)$ factor and in this case, the semiclassical approximation and the numerical integration are given by the figures below. Again we use $T_1 = 0.3$ and $T_2 = 0.6$ (Fig. 7).

From our numerical illustrations we can conclude that for the different time dependent arbitrage bubble forms, the semiclassical solution is a good approximation. In general, if the bubble amplitudes remains below the 40% of the market volatility the approximation could be considered quite robust.

6. Conclusions and further research

We have developed an interacting model for option pricing that generalizes the usual Black–Scholes formulation to include a more general case of quantum interactions, defined by arbitrage possibilities, and triggered by market imperfections. Specifically, we have proposed a new Black–Scholes–Schrödinger model based on the endogenous arbitrage option pricing formulation introduced by Contreras et al. [2].

Indeed, this new quantum model of option pricing incorporates arbitrage as an external time dependent force, explicitly defining an associated potential, related to the random dynamic of the underlying asset price. The resultant formulation can be interpreted as a Schrödinger equation in imaginary time for a particle of mass $1/\sigma^2$ with a wave function in an external time dependent field force generated by the arbitrage potential.

This new model has as a particular case the perfect market equilibrium postulated by the Black–Scholes model. Given the fact that perfect competition can be measured empirically in financial markets, our quantum model could be calibrated on a case by case basis. Besides, considering the wide knowledge of the Schrödinger equation by physicists, it is expected that numerous and different new applications could be developed in econophysics from our interacting model.

Since the Schrödinger equation is in place, we can apply semiclassical methods of common use in theoretical physics to find an approximate analytical solution of the Black–Scholes equation in the presence of market imperfections, as is the case of an arbitrage bubble. From our numerical illustrations we concluded that for the different time dependent arbitrage bubble forms, the semiclassical solution is a good approximation, if the bubble amplitudes remain below 40% of the market's volatility. We suppose here that the arbitrage bubble is given, but the model [2] can be thought as an inverse scattering problem, that is, from the real financial data, the "nuclear potential" $V(S, t)$ and the arbitrage bubble $f(S, t)$, can be obtained. Still, we can go further, and see model [2] as an evolutionary dynamic problem, where the arbitrage bubble should be determined maximizing or minimizing a certain functional in each time step. These matters will be developed in future works.

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